FUZZIFICATIONS OF IDEALS IN BCC-ALGEBRAS

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Abstract. In this paper we consider the fuzzification of ideals in the sense of W. A. Dudek in BCC-algebras. We discuss the relations among fuzzy BCK-ideal, fuzzy BCC-ideal and fuzzy \(g\)-ideal. We state fuzzy characteristic \(g\)-ideals, and also discuss fuzzy relations on BCC-algebras.

1. Introduction

In 1966, Y. Imai and K. Iséki ([12]) defined a class of algebras of type (2,0) called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra ([14]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [17]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori ([15]) introduced a notion of BCC-algebras, and W. A. Dudek ([3, 4]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [9], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun ([6]) considered the fuzzification of BCC-ideals in BCC-algebras. They showed that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing an example. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra, and in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. W. A. Dudek, Y. B. Jun and Z. Stojaković ([7]) described several properties of fuzzy BCC-ideals in BCC-algebras, and

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discussed an extension of fuzzy BCC-ideals. In [5], W. A. Dudek introduced a new notion of ideals in BCC-algebras, and gave its characterizations.

In this paper we consider the fuzzification of ideals in the sense of W. A. Dudek in BCC-algebras. We discuss the relations among fuzzy BCK-ideal, fuzzy BCC-ideal and fuzzy $g$-ideal. We state fuzzy characteristic $g$-ideals, and also discuss fuzzy relations on BCC-algebras.

2. Preliminaries

By a \textit{BCK-algebra} we mean an algebra $(G, *, 0)$ of type $(2,0)$ satisfying the following axioms:

\begin{enumerate}[(I)]
    \item $((x * y) * (x * z)) * (z * y) = 0$,
    \item $(x * (x * y)) * y = 0$,
    \item $x * x = 0$,
    \item $0 * x = 0$,
    \item $x * y = 0$ and $y * x = 0$ imply $x = y$,
\end{enumerate}

for all $x, y, z \in G$.

In what follows, a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

\textbf{Definition 2.1.} A non-empty set $G$ with a constant $0$ and a binary operation denoted by juxtaposition is called a BCC-algebra if for all $x, y, z \in G$ the following axioms hold:

\begin{enumerate}[(1)]
    \item $(xy \cdot zy) \cdot xz = 0$,
    \item $xx = 0$,
    \item $0x = 0$,
    \item $x0 = x$,
    \item $xy = 0$ and $yx = 0$ imply $x = y$.
\end{enumerate}

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [4]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

\begin{enumerate}[(6)]
    \item $xy \cdot z = xz \cdot y$.
\end{enumerate}

On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural order “$\leq$” by putting \begin{equation}
(7) \quad x \leq y \iff xy = 0.
\end{equation}

It is not difficult to verify that this order is partial and $0$ is its smallest element. Moreover, in any BCC-algebra (also in BCK-algebra), the following are true:

\begin{enumerate}[(8)]
    \item $xy \cdot zy \leq xx$,
    \item $x \leq y$ implies $xz \leq yz$ and $zy \leq zx$.
\end{enumerate}

A non-empty subset $A$ of a BCK-algebra $G$ is called an \textit{ideal} if $0 \in A$ and $y, xy \in A$ imply $x \in A$. In the sequel this ideal will be called a \textit{BCK-ideal} and will be considered also in BCC-algebras.
A non-empty subset $A$ of a BCC-algebra $G$ is called a BCC-ideal if $0 \in A$ and $y, xy \cdot z \in A$ imply $xz \in A$.

**Definition 2.2.** A fuzzy set $\mu$ in a BCK-algebra $G$ is called a fuzzy BCK-ideal of $G$ if

- $(FK1)$ $\mu(0) \geq \mu(x)$, $\forall x \in G$,
- $(FK2)$ $\mu(x) \geq \min\{\mu(xy), \mu(y)\}$, $\forall x, y \in G$.

**Definition 2.3.** ([6]). A fuzzy set $\mu$ in a BCC-algebra $G$ is called a fuzzy BCC-ideal of $G$ if

- $(FK1)$ $\mu(0) \geq \mu(x)$, $\forall x \in G$,
- $(FC1)$ $\mu(xy) \geq \min\{\mu(xa \cdot y), \mu(a)\}$, $\forall a, x, y \in G$.

3. Fuzzy $g$-ideals in BCC-algebras

**Definition 3.1.** ([5]). A subset $A$ of a BCC-algebra $G$ is called an ideal if it satisfies

- $(I1)$ $0 \in A$,
- $(I2)$ $ab \in A$ for $a \in A$ and $b \in G$,
- $(I3)$ $b(ba \cdot a) \in A$ for $a_1, a_2 \in A$ and $b \in G$.

Here we call this ideal $A$ a $g$-ideal to avoid the confusion. We begin with the fuzzification of the above $g$-ideal.

**Definition 3.2.** A fuzzy set $\mu$ in a BCC-algebra $G$ is called a fuzzy $g$-ideal if it satisfies

- $(FK1)$ $\mu(0) \geq \mu(a)$, $\forall a \in G$,
- $(FI1)$ $\mu(ab) \geq \mu(a)$, $\forall a, b \in G$,
- $(FI2)$ $\mu(b(ba \cdot a)) \geq \min\{\mu(a_1), \mu(a_2)\}$, $\forall b, a_1, a_2 \in G$.

Observe that $(FK1)$ follows from $(FI1)$ and (2). Using $(FI1)$ we know that every fuzzy $g$-ideal is a fuzzy subalgebra. Moreover, putting $a_1 = a$ and $a_2 = 0$ in (FI2) we obtain the following proposition.

**Proposition 3.3.** If $\mu$ is a fuzzy $g$-ideal of a BCC-algebra $G$, then

$$\mu(b \cdot ba) \geq \mu(a), \forall a, b \in G.$$

**Corollary 3.4.** Every fuzzy $g$-ideal $\mu$ of a BCC-algebra $G$ is order reversing, i.e., if $x \leq a$ then $\mu(x) \geq \mu(a)$ for all $a, x \in G$.

**Proof.** If $x, a \in G$ are such that $x \leq a$, then $\mu(x) = \mu(x0) = \mu(x \cdot xa) \geq \mu(a)$, which completes the proof.

**Theorem 3.5.** A fuzzy set $\mu$ in a BCC-algebra $G$ is a fuzzy $g$-ideal if and only if it is a fuzzy BCC-ideal.
Proof. Let \( \mu \) be a fuzzy \( g \)-ideal and let \( a, x, y \in G \). Then
\[
\begin{align*}
\mu(xy) &= \mu(xy \cdot 0) \\
&= \mu(xy \cdot ((xy \cdot (xa \cdot y))(x \cdot xa))) \\
&\geq \min\{\mu(xa \cdot y), \mu(x \cdot xa)\} \\
&\geq \min\{\mu(xa \cdot y), \mu(a)\},
\end{align*}
\]
which shows that \( \mu \) satisfies (FC1). Hence \( \mu \) is a fuzzy BCC-ideal.

Conversely, let \( \mu \) be a fuzzy BCC-ideal. Then \( \mu(y) \leq \mu(x) \) for all \( x \leq y \).
Indeed,
\[
\mu(x) = \mu(x0) = \mu(x \cdot xy) \geq \min\{\mu(xy \cdot xy), \mu(y)\} \\
= \min\{\mu(0), \mu(y)\} = \mu(y).
\]
Moreover, for all \( a, x \in G \), we have
\[
\mu(ax) \geq \min\{\mu(aa \cdot x), \mu(a)\} \\
= \min\{\mu(0x), \mu(a)\} \\
= \min\{\mu(0), \mu(a)\} = \mu(a),
\]
which proves (FI1). To prove (FI2), let \( x, a_1, a_2 \in G \). Note that
\[
\mu(x \cdot xa_1) \geq \min\{\mu(xa_1 \cdot xa_1), \mu(a_1)\} \\
= \min\{\mu(0), \mu(a_1)\} = \mu(a_1).
\]
Since \( xa_2 \cdot (xa_1 \cdot a_2) \leq x \cdot xa_1 \) by (8), then
\[
\mu(xa_2 \cdot (xa_1 \cdot a_2)) \geq \mu(x \cdot xa_1) \geq \mu(a_1).
\]
By using (FC1), we see that
\[
\mu(x(xa_1 \cdot a_2)) \geq \min\{\mu(xa_2 \cdot (xa_1 \cdot a_2)), \mu(a_2)\} \\
\geq \min\{\mu(a_1), \mu(a_2)\},
\]
which proves (FI2). Hence \( \mu \) is a fuzzy \( g \)-ideal.

Theorem 3.6. Let \( \mu \) be a fuzzy set in a BCK-algebra \( G \). Then \( \mu \) is a fuzzy \( g \)-ideal if and only if \( \mu \) is a fuzzy BCK-ideal.

Proof. Since every BCK-algebra is a BCC-algebra, every fuzzy \( g \)-ideal is a fuzzy BCC-ideal (see Theorem 3.5) and hence a fuzzy BCK-ideal. Let \( \mu \) be a fuzzy BCK-ideal. Then
\[
\mu(ax) \geq \min\{\mu(ax \cdot a), \mu(a)\} \\
= \min\{\mu(aa \cdot x), \mu(a)\} \\
= \min\{\mu(0x), \mu(a)\} \\
= \min\{\mu(0), \mu(a)\} \\
= \mu(a),
\]
which shows (FI1). Now let \( x, a_1, a_2 \in G \). Using (6), (8) and (II), we have
\[
x(xa_1 \cdot a_2) \cdot a_2 = xa_2 \cdot (xa_1 \cdot a_2) \leq x \cdot xa_1 \leq a_1.
\]
Since every fuzzy BCK-ideal of a BCK-algebra is order reversing, it follows
that \( \mu(x(xa_1 \cdot a_2) \cdot a_2) \geq \mu(a_1) \), and hence using (FK2) we obtain
\[
\mu(x(xa_1 \cdot a_2)) \geq \min\{\mu(x(xa_1 \cdot a_2) \cdot a_2), \mu(a_2)\} \\
\geq \min\{\mu(a_1), \mu(a_2)\},
\]
which proves that \( \mu \) satisfies (FI2). This completes the proof.

The following example shows that a fuzzy BCK-ideal of a BCC-algebra may not be a fuzzy \( g \)-ideal.

Example 3.7. Consider a BCC-algebra \( G = \{0, a, b, c, d\} \) with Cayley table as follows (cf. [9]):

\[
\begin{array}{cccc}
\cdot & 0 & a & b & c & d \\
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 & 0 \\
b & b & b & 0 & 0 & 0 \\
c & c & c & a & 0 & 0 \\
d & d & c & d & c & 0 \\
\end{array}
\]

Let \( \mu \) be a fuzzy set in \( G \) defined by
\[
\mu(x) := \begin{cases} 
  t_1 & \text{if } x \in \{0, a\}, \\
  t_2 & \text{otherwise}, 
\end{cases}
\]
where \( t_1 > t_2 \) in \([0, 1]\). It is easy to verify that \( \mu \) is a fuzzy BCK-ideal of \( G \), but it is not a fuzzy \( g \)-ideal since
\[
\mu(d(da \cdot a)) = t_2 < t_1 = \min\{\mu(a), \mu(a)\}.
\]

Proposition 3.8. Let \( A \) be a non-empty subset of a BCC-algebra \( G \) and let \( \mu \) be a fuzzy set in \( G \) defined by
\[
\mu(a) := \begin{cases} 
  t_1 & \text{if } a \in A, \\
  t_2 & \text{otherwise}, 
\end{cases}
\]
where \( t_1 > t_2 \) in \([0, 1]\). Then \( \mu \) is a fuzzy \( g \)-ideal of \( G \) if and only if \( A \) is a \( g \)-ideal of \( G \).

Proof. Assume that \( \mu \) is a fuzzy \( g \)-ideal of \( G \). Since \( \mu(0) \geq \mu(a) \) for all \( a \in G \), we have \( \mu(0) = t_1 \) and so \( 0 \in A \). Let \( a \in A \) and \( b \in G \). Then \( \mu(ab) \geq \mu(a) = t_1 \) and thus \( \mu(ab) = t_1 \). Hence \( ab \in A \). For any \( a_1, a_2 \in A \) and \( b \in G \), we get
\[
\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1), \mu(a_2)\} = t_1
\]
which implies that \( \mu(b(ba_1 \cdot a_2)) = t_1 \). It follows that \( b(ba_1 \cdot a_2) \in A \). Therefore \( A \) is a \( g \)-ideal of \( G \).

Conversely suppose that \( A \) is a \( g \)-ideal of \( G \). Since \( 0 \in A \), it follows that
\[
\mu(0) = t_1 \geq \mu(a)
\]
for all \( a \in G \). Let \( a, b \in G \). If \( a \in A \), then \( ab \in A \) and so \( \mu(ab) = t_1 = \mu(a) \). If \( a \in G \setminus A \), then \( \mu(a) = t_2 \) and hence \( \mu(ab) \geq t_2 = \mu(a) \).
Finally let $a_1, a_2, b \in G$. If $a_1 \in G \setminus A$ or $a_2 \in G \setminus A$, then $\mu(a_1) = t_2$ or $\mu(a_2) = t_2$. It follows that
\[
\mu(b(a_1 \cdot a_2)) \geq t_2 = \min\{\mu(a_1), \mu(a_2)\}.
\]
Assume that $a_1, a_2 \in A$. Then $b(a_1 \cdot a_2) \in A$ and thus
\[
\mu(b(a_1 \cdot a_2)) = t_1 = \min\{\mu(a_1), \mu(a_2)\}.
\]
Hence $\mu$ is a fuzzy $g$-ideal of $G$.

**Lemma 3.9.** ([5]). An initial segment $[0, c] := \{x \in G : 0 \leq x \leq c\}$ of a BCC-algebra $G$ is a $g$-ideal if and only if the inequality $xc \cdot c \leq c$ holds for all $x \in G$.

If we combine Proposition 3.8 with Lemma 3.9, then we have the following theorem.

**Theorem 3.10.** Let $\mu$ be a fuzzy set in a BCC-algebra $G$ defined by
\[
\mu(x) := \begin{cases} 
    t_1 & \text{if } x \in [0, c], \\
    t_2 & \text{otherwise},
\end{cases}
\]
where $t_1 > t_2$ in $[0, 1]$. Then $\mu$ is a fuzzy $g$-ideal if and only if the inequality $xc \cdot c \leq c$ holds for all $x \in G$.

As a simple consequence of the above Theorem and [10, Proposition 2.7] we obtain

**Corollary 3.11.** Let $\mu$ be as in Theorem 3.10. Then
(i) $\mu$ is a fuzzy $g$-ideal if and only if $xc \cdot y \leq c$ implies $xy \leq c$ for all $x, y \in G$.
(ii) $\mu$ is a fuzzy $g$-ideal if and only if $xc \leq c$ implies $x \leq c$ for all $x \in G$.

4. **Fuzzy characteristic $g$-ideals**

For an endomorphism $f$ of a BCC-algebra $G$ and a fuzzy set $\mu$ in $G$, we define a new fuzzy set $\mu^f$ in $G$ by $\mu^f(x) = \mu(f(x))$ for all $x \in G$.

**Proposition 4.1.** Let $f$ be an endomorphism of a BCC-algebra $G$. If $\mu$ is a fuzzy $g$-ideal of $G$, then so is $\mu^f$.

**Proof.** We first have that $\mu^f(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = \mu^f(0)$ for all $x \in G$. Let $a, b \in G$. Then
\[
\mu^f(ab) = \mu(f(ab)) = \mu(f(a)f(b)) \geq \mu(f(a)) = \mu^f(a),
\]
proving the condition (FI1). Finally for any $b, a_1, a_2 \in G$ we get
\[
\mu^f(b(a_1 \cdot a_2)) = \mu(f(b(a_1 \cdot a_2))) \\
= \mu(f(b)f(a_1) \cdot f(a_2)) \\
\geq \min\{\mu(f(a_1)), \mu(f(a_2))\} \\
= \min\{\mu^f(a_1), \mu^f(a_2)\},
\]
ending the proof.
Definition 4.2. A g-ideal $A$ of a BCC-algebra $G$ is said to be characteristic if $f(A) = A$ for all $f \in \text{Aut}(G)$, where $\text{Aut}(G)$ is the set of all automorphisms of $G$.

Definition 4.3. A fuzzy g-ideal $\mu$ of a BCC-algebra $G$ is said to be fuzzy characteristic if $\mu f(x) = \mu(x)$ for all $x \in G$ and $f \in \text{Aut}(G)$.

Lemma 4.4. Let $\mu$ be a fuzzy set in a BCC-algebra $G$ and let $t \in \text{Im}(\mu)$. Then $\mu$ is a fuzzy g-ideal of $G$ if and only if the level subset

$$\mu_t := \{x \in G | \mu(x) \geq t\}$$

is a g-ideal of $G$, which is called a level g-ideal of $\mu$.

Proof. Assume that $\mu$ is a fuzzy g-ideal of $G$. Clearly $0 \in \mu_t$. Let $a \in \mu_t$ and $b \in G$. Then $\mu(a) \geq t$ and so $\mu(ab) \geq \mu(a) \geq t$, which implies that $ab \in \mu_t$. Now let $a_1, a_2 \in \mu_t$ and $b \in G$. Then

$$\mu(b(a_1 \cdot a_2)) \geq \min\{\mu(a_1), \mu(a_2)\} \geq t$$

and thus $b(a_1 \cdot a_2) \in \mu_t$. Hence $\mu_t$ is a g-ideal of $G$.

Conversely suppose that $\mu_t$ is a g-ideal of $G$. If there exists $a_0 \in G$ such that $\mu(0) < \mu(a_0)$, then $\mu(0) < \frac{1}{2}(\mu(0) + \mu(a_0)) < \mu(a_0)$ and hence $a_0 \in \mu_t$ where $s = \frac{1}{2}(\mu(0) + \mu(a_0))$. Since $0 \in \mu_t$, we have $\mu(0) \geq s$, a contradiction. Assume that $\mu(ab_0) < \mu(a_0)$ for some $a_0, b_0 \in G$. Taking $u = \frac{1}{2}(\mu(a_0 b_0) + \mu(a_0))$, then $\mu(a_0 b_0) < u < \mu(a_0)$ and thus $a_0 \in \mu_u$ and $a_0 b_0 \not \in \mu_u$. This is a contradiction. Finally suppose that there exist $a_1, a_2, b \in G$ such that

$$\mu(b(a_1 \cdot a_2)) < \min\{\mu(a_1), \mu(a_2)\}.$$ 

If we take $v = \frac{1}{2}(\mu(b(a_1 \cdot a_2)) + \min\{\mu(a_1), \mu(a_2)\})$, then $\mu(b(a_1 \cdot a_2)) < v < \min\{\mu(a_1), \mu(a_2)\}$ and so $a_1, a_2 \in \mu_v$ and $b(a_1 \cdot a_2) \not \in \mu_v$, a contradiction. This completes the proof.

Lemma 4.5. Let $\mu$ be a fuzzy g-ideal of a BCC-algebra $G$ and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \not \in \mu_s$ for all $s > t$.

Proof. Straightforward.

Theorem 4.6. For a fuzzy g-ideal $\mu$ of a BCC-algebra $G$, the following are equivalent:

(i) $\mu$ is fuzzy characteristic.
(ii) Each level g-ideal of $\mu$ is characteristic.

Proof. Assume that $\mu$ is a fuzzy characteristic and let $t \in \text{Im}(\mu)$, $f \in \text{Aut}(G)$ and $x \in \mu_t$. Then $\mu f(x) = \mu(x) \geq t$, i.e., $\mu(f(x)) \geq t$, and so $f(x) \in \mu_t$, i.e., $f(\mu_t) \subset \mu_t$. Now let $x \in \mu_t$ and let $y \in G$ be such that $f(y) = x$. Then $\mu(y) = \mu f(y) = \mu(f(y)) = \mu(x) \geq t$, whence $y \in \mu_t$, so that $x = f(y) \in f(\mu_t)$. Consequently $\mu_t \subset f(\mu_t)$. Hence $f(\mu_t) = \mu_t$ and $\mu_t$ is characteristic.
Conversely suppose that each level $g$-ideal of $\mu$ is characteristic and let $x \in G$, $f \in \text{Aut}(G)$ and $\mu(x) = t$. Then, by virtue of Lemma 4.5, $x \in \mu_t$ and $x \not\in \mu_s$ for all $s > t$. It follows from hypothesis that $f(x) \in f(\mu_t) = \mu_t$, so that $\mu^f(x) = \mu(f(x)) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$, which implies from the injectivity of $f$ that $x \in \mu_s$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) = t = \mu(x)$ showing that $\mu$ is fuzzy characteristic.

5. Cartesian product of fuzzy $g$-ideals

Definition 5.1. ([1]). A fuzzy relation on any set $S$ is a fuzzy set
$$\mu : S \times S \rightarrow [0, 1].$$

Definition 5.2. ([1]). If $\mu$ is a fuzzy relation on a set $S$ and $\nu$ is a fuzzy set in $S$, then $\mu$ is a fuzzy relation on $\nu$ if
$$\mu(x, y) \leq \min\{\nu(x), \nu(y)\}, \, \forall x, y \in S.$$

Definition 5.3. ([1]). Let $\mu$ and $\nu$ be fuzzy sets in a set $S$. The Cartesian product of $\mu$ and $\nu$ is defined by
$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}, \, \forall x, y \in S.$$

Lemma 5.4. ([1]). Let $\mu$ and $\nu$ be fuzzy sets in a set $S$. Then
(i) $\mu \times \nu$ is a fuzzy relation on $S$,
(ii) $(\mu \times \nu)_t = \mu \times \nu_t$ for all $t \in [0, 1]$.

Definition 5.5. ([1]). If $\nu$ is a fuzzy set in a set $S$, the strongest fuzzy relation on $S$ that is a fuzzy relation on $\nu$ is $\mu_\nu$, given by
$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}, \, \forall x, y \in S.$$

Lemma 5.6. ([1]). For a given fuzzy set $\nu$ in a set $S$, let $\mu_\nu$ be the strongest fuzzy relation on $S$. Then for $t \in [0, 1]$, we have that $(\mu_\nu)_t = \nu_t \times \nu_t$.

Proposition 5.7. For a given fuzzy set $\nu$ in a BCC-algebra $G$, let $\mu_\nu$ be the strongest fuzzy relation on $G$. If $\mu_\nu$ is a fuzzy $g$-ideal of $G \times G$, then $\nu(a) \leq \nu(0)$ for all $a \in G$.

Proof. From the fact that $\mu_\nu$ is a fuzzy $g$-ideal of $G \times G$, it follows from (FK1) that $\mu_\nu(a, a) \leq \mu_\nu(0, 0)$ for all $a \in G$, where $(0, 0)$ is the zero element of $G \times G$. But this means that $\min\{\nu(0), \nu(0)\} \geq \min\{\nu(a), \nu(a)\}$, which implies that $\nu(0) \geq \nu(a)$.

The following proposition is an immediate consequence of Lemma 5.6, and we omit the proof.

Proposition 5.8. If $\nu$ is a fuzzy $g$-ideal of a BCC-algebra $G$, then the level $g$-ideals of $\mu_\nu$ are given by $(\mu_\nu)_t = \nu_t \times \nu_t$ for all $t \in [0, 1]$. 
THEOREM 5.9. Let \( \mu \) and \( \nu \) be fuzzy \( g \)-ideals of a BCC-algebra \( G \). Then \( \mu \times \nu \) is a fuzzy \( g \)-ideal of \( G \times G \).

PROOF. Note first that for every \((x, y) \in G \times G\),
\[(\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\} \geq \min\{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y)\.

Let \((a_1, a_2), (b_1, b_2) \in G \times G\). Then
\[(\mu \times \nu)((a_1, a_2) \ast (b_1, b_2)) = (\mu \times \nu)(a_1b_1, a_2b_2) = \min\{\mu(a_1b_1), \nu(a_2b_2)\} \geq \min\{\mu(a_1), \nu(a_2)\} = (\mu \times \nu)(a_1, a_2).

For any \((b_1, b_2), (x_1, x_2), (y_1, y_2) \in G \times G\), we have
\[(\mu \times \nu)((b_1, b_2) \ast ((b_1, b_2) \ast (x_1, x_2)) \ast (y_1, y_2))) = (\mu \times \nu)(b_1, b_2) \ast (b_1, b_2) \ast (b_1, b_2) \ast (y_1, y_2)) = (\mu \times \nu)(b_1, b_2) \ast (b_1x_1, b_2x_2, y_1, y_2)) \geq \min\{\mu(b_1), \nu(b_2)\} \geq \min\{\mu(b_1), \nu(b_2)\} = \min\{\mu(b_1), \nu(b_2)\} = \min\{\mu(x_1), \nu(x_2)\} = \min\{\mu(y_1), \nu(y_2)\} = \min\{(\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)\}.

Hence \( \mu \times \nu \) is a fuzzy \( g \)-ideal of \( G \times G \).

THEOREM 5.10. Let \( \mu \) and \( \nu \) be fuzzy sets in a BCC-algebra \( G \) such that \( \mu \times \nu \) is a fuzzy \( g \)-ideal of \( G \times G \). Then
(i) either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \nu(0) \) for all \( x \in G \).
(ii) if \( \mu(x) \leq \mu(0) \) for all \( x \in G \), then either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \nu(0) \).
(iii) if \( \nu(x) \leq \nu(0) \) for all \( x \in G \), then either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \mu(0) \).
(iv) either \( \mu \) or \( \nu \) is a fuzzy \( g \)-ideal of \( G \).

PROOF. (i) Suppose that \( \mu(x) > \mu(0) \) and \( \nu(y) > \nu(0) \) for some \( x, y \in G \). Then \( (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \min\{\mu(0), \nu(0)\} = (\mu \times \nu)(0, 0) \), which is a contradiction and we obtain (ii).

(ii) Assume that there exist \( x, y \in G \) such that \( \mu(x) > \nu(0) \) and \( \nu(y) > \nu(0) \). Then \( (\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\} = \nu(0) \) and hence \( (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(0) = (\mu \times \nu)(0, 0) \).

This is a contradiction. Hence (ii) holds.

(iii) is by similar method to part (ii).

(iv) Since, by (i), either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \nu(0) \) for all \( x \in G \), without loss of generality we may assume that \( \nu(x) \leq \nu(0) \) for all \( x \in G \). It follows from (iii) that either \( \mu(x) \leq \mu(0) \) or \( \nu(x) \leq \mu(0) \). If \( \nu(x) \leq \mu(0) \) for any \( x \in G \), then
\[
\nu(x) = \min\{\mu(0), \nu(x)\} = (\mu \times \nu)(0, x) \leq (\mu \times \nu)((0, x) \ast (y_1, y_2)) = (\mu \times \nu)(0y_1xy_2) = (\mu \times \nu)(0, y_2) = \nu(xy_2).
\]
for all $x, y_1, y_2 \in G$, which proves that $\nu$ satisfies the condition (FI1). Now
\[
\min\{\nu(a_1), \nu(a_2)\} = \min\{\min\{\mu(0), \nu(a_1)\}, \min\{\mu(0), \nu(a_2)\}\}
\]
\[
= \min\{(\mu \times \nu)(0, a_1), (\mu \times \nu)(0, a_2)\}
\]
\[
\leq (\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (0, a_1)) * (0, a_2)))
\]
\[
= (\mu \times \nu)((b_1, b_2) * (b_10 \cdot 0, b_2 a_1 \cdot a_2))
\]
\[
= (\mu \times \nu)((b_1, b_2) * (b_1 b_2 a_1 \cdot a_2))
\]
\[
= (\mu \times \nu)(b_1 b_2 a_1 \cdot a_2)
\]
\[
= \min\{\mu(0), \nu(b_2(a_1 \cdot a_2))\}
\]
\[
= \nu(b_2(a_1 \cdot a_2))
\]
for all $a_i, b_j \in G$, $i = 1, 2$; $j = 1, 2$. Hence $\nu$ is a fuzzy $g$-ideal of $G$. Now we consider the case $\mu(x) \leq \mu(0)$ for all $x \in G$. Suppose that $\nu(y) > \mu(0)$ for some $y \in G$. Then $\nu(0) \geq \nu(0)$ for all $x \in G$, it follows that $\nu(0) > \mu(x)$ for any $x \in G$. Hence $(\mu \times \nu)(x, 0) = \min\{\mu(x), \nu(0)\} = \mu(x)$ for all $x \in G$. Thus
\[
\mu(x) = (\mu \times \nu)(x, 0) \leq (\mu \times \nu)((x, 0) * (y_1, y_2))
\]
\[
= (\mu \times \nu)(xy_1, 0y_2) = (\mu \times \nu)(xy_1, 0) = \mu(xy_1)
\]
for all $x, y_1, y_2 \in G$. Moreover
\[
\min\{\mu(a_1), \mu(a_2)\} = \min\{(\mu \times \nu)(a_1, 0), (\mu \times \nu)(a_2, 0)\}
\]
\[
\leq (\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (a_1, 0)) * (a_2, 0)))
\]
\[
= (\mu \times \nu)((b_1, b_2) * (b_1 a_1 \cdot a_2, b_20 \cdot 0))
\]
\[
= (\mu \times \nu)(b_1(b_1 a_1 \cdot a_2), b_2 b_2)
\]
\[
= (\mu \times \nu)(b_1(b_1 a_1 \cdot a_2), 0)
\]
\[
= \mu(b_1(b_1 a_1 \cdot a_2))
\]
for all $a_i, b_j \in G$, $i = 1, 2$; $j = 1, 2$, which proves that $\mu$ is a fuzzy $g$-ideal of $G$. This completes the proof. \hfill \blacksquare

Now we give an example to show that if $\mu \times \nu$ is a fuzzy $g$-ideal of $G \times G$, then $\mu$ and $\nu$ both need not be fuzzy $g$-ideals of $G$.

**Example 5.11.** Let $G$ be a BCC-algebra with $|G| \geq 2$ and let $s, t \in [0, 1)$ be such that $s \leq t$. Define fuzzy sets $\mu$ and $\nu$ in $G$ by $\mu(x) = s$ and
\[
\nu(x) = \begin{cases} 
    t & \text{if } x = 0, \\
    1 & \text{otherwise},
\end{cases}
\]
for all $x \in G$, respectively. Then $\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} = s$ for all $(x, y) \in G \times G$, that is, $\mu \times \nu$ is a constant function and so $\mu \times \nu$ is a fuzzy $g$-ideal of $G \times G$. Now $\mu$ is a fuzzy $g$-ideal of $G$, but $\nu$ is not a fuzzy $g$-ideal of $G$ since for $x \neq 0$ we have $\nu(0) = t < 1 = \nu(x)$.\hfill \blacksquare
Theorem 5.12. Let $\nu$ be a fuzzy set in a BCC-algebra $G$ and let $\mu_\nu$ be the strongest fuzzy relation on $G$. Then $\nu$ is a fuzzy $g$-ideal of $G$ if and only if $\mu_\nu$ is a fuzzy $g$-ideal of $G \times G$.

Proof. Assume that $\nu$ is a fuzzy $g$-ideal of $G$. Clearly $\mu_\nu(0,0) \geq \mu_\nu(x,y)$ for any $(x,y) \in G \times G$. Now
\[
\mu_\nu(a_1,a_2) = \min\{\nu(a_1),\nu(a_2)\} \leq \min\{\nu(a_1,b_1),\nu(a_2,b_2)\}
\]
for all $(a_1,a_2),(b_1,b_2) \in G \times G$, and
\[
\min\{\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)\}
= \min\{\min\{\nu(a_1),\nu(a_2)\},\min\{\nu(b_1),\nu(b_2)\}\}
= \min\{\min\{\nu(a_1),\nu(b_1)\},\min\{\nu(a_2),\nu(b_2)\}\}
\leq \min\{\nu(x(xa_1 \cdot b_1)),\nu(y(ya_2 \cdot b_2))\}
= \mu_\nu(x(xa_1 \cdot b_1),y(ya_2 \cdot b_2))
= \mu_\nu((x,y) * ((x,y) * (a_1,a_2)) * (b_1,b_2))
\]
for all $(x,y),(a_1,a_2),(b_1,b_2) \in G \times G$. Hence $\mu_\nu$ is a fuzzy $g$-ideal of $G \times G$.

Conversely suppose that $\mu_\nu$ is a fuzzy $g$-ideal of $G \times G$. Then
\[
\min\{\nu(0),\nu(0)\} = \mu_\nu(0,0) \geq \mu_\nu(x,y) = \min\{\nu(x),\nu(y)\}
\]
for all $(x,y) \in G \times G$. It follows that $\nu(x) \leq \nu(0)$ for all $x \in G$. Now we have
\[
\nu(a) = \min\{\nu(a),\nu(0)\} = \mu_\nu(a,0) \leq \mu_\nu((a,0) * (b_1,b_2))
= \mu_\nu(ab_1,0b_2) = \mu_\nu(ab_1,0) = \min\{\nu(ab_1),\nu(0)\} = \nu(ab_1)
\]
for all $a,b_1 \in G$, and
\[
\min\{\min\{\nu(a_1),\nu(a_2)\},\min\{\nu(b_1),\nu(b_2)\}\}
= \min\{\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)\}
\leq \mu_\nu((x,y) * ((x,y) * (a_1,a_2)) * (b_1,b_2))
= \mu_\nu(x(xa_1 \cdot b_1),y(ya_2 \cdot b_2))
= \min\{\nu(x(xa_1 \cdot b_1)),\nu(y(ya_2 \cdot b_2))\}
\]
for all $(x,y),(a_1,a_2),(b_1,b_2) \in G \times G$. Taking $a_2 = b_2 = 0$ (resp. $a_1 = b_1 = 0$) and using (2) and (4), then
\[
\min\{\nu(a_1),\nu(b_1)\} \leq \nu(x(xa_1 \cdot b_1))
\]
(resp. $\min\{\nu(a_2),\nu(b_2)\} \leq \nu(x(xa_2 \cdot b_2))$).
Hence $\nu$ is a fuzzy $g$-ideal of $G$.

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