GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 127 – 138

FUZZIFICATIONS OF IDEALS IN BCC-ALGEBRAS

Wiesław A. Dudek and Young Bae Jun
Technical University, Wrocław, Poland
Gyeongsang National University, Chinju, Korea

Abstract. In this paper we consider the fuzzification of ideals in the sense of W. A. Dudek in BCC-algebras. We discuss the relations among fuzzy BCK-ideal, fuzzy BCC-ideal and fuzzy g-ideal. We state fuzzy characteristic g-ideals, and also discuss fuzzy relations on BCC-algebras.

1. Introduction

In 1966, Y. Imai and K. Iséki ([12]) defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra ([14]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [17]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori ([15]) introduced a notion of BCC-algebras, and W. A. Dudek ([3, 4]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [9], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun ([6]) considered the fuzzification of BCC-ideals in BCC-algebras. They showed that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing an example. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra, and in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. W. A. Dudek, Y. B. Jun and Z. Stojaković ([7]) described several properties of fuzzy BCC-ideals in BCC-algebras, and

2000 *Mathematics Subject Classification.* 06F35, 03G25, 94D05.
Key words and phrases. BCK-algebra, BCC-algebra, fuzzy BCC-ideal, fuzzy g-ideal, level, fuzzy characteristic ideal.

discussed an extension of fuzzy BCC-ideals. In [5], W. A. Dudek introduced a new notion of ideals in BCC-algebras, and gave its characterizations.

In this paper we consider the fuzzification of ideals in the sense of W. A. Dudek in BCC-algebras. We discuss the relations among fuzzy BCK-ideal, fuzzy BCC-ideal and fuzzy g-ideal. We state fuzzy characteristic g-ideals, and also discuss fuzzy relations on BCC-algebras.

2. Preliminaries

By a *BCK-algebra* we mean an algebra $(G, *, 0)$ of type (2,0) satisfying the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in G$.

In what follows, a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

Definition 2.1. *A non-empty set G with a constant 0 and a binary operation denoted by juxtaposition is called a BCC-algebra if for all $x, y, z \in G$ the following axioms hold:*

(1) $(xy \cdot zy) \cdot xz = 0$,
(2) $xx = 0$,
(3) $0x = 0$,
(4) $x0 = x$,
(5) $xy = 0$ *and* $yx = 0$ *imply* $x = y$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [4]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies: (6) $xy \cdot z = xz \cdot y$.

On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural order "$\leq$" by putting (7) $x \leq y \Longleftrightarrow xy = 0$.

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, in any BCC-algebra (also in BCK-algebra), the following are true:

(8) $xy \cdot zy \leq xz$,
(9) $x \leq y$ *implies* $xz \leq yz$ *and* $zy \leq zx$.

A non-empty subset A of a BCK-algebra G is called an *ideal* if $0 \in A$ and $y, xy \in A$ imply $x \in A$. In the sequel this ideal will be called a *BCK-ideal* and will be considered also in BCC-algebras.

A non-empty subset A of a BCC-algebra G is called a *BCC-ideal* if $0 \in A$ and $y, xy \cdot z \in A$ imply $xz \in A$.

Definition 2.2. *A fuzzy set μ in a BCK-algebra G is called a fuzzy BCK-ideal of G if*
(FK1) $\mu(0) \geq \mu(x), \quad \forall x \in G,$
(FK2) $\mu(x) \geq \min\{\mu(xy),\, \mu(y)\}, \quad \forall x, y \in G.$

Definition 2.3. ([6]). *A fuzzy set μ in a BCC-algebra G is called a fuzzy BCC-ideal of G if*

(FK1) $\mu(0) \geq \mu(x), \quad \forall x \in G,$
(FC1) $\mu(xy) \geq \min\{\mu(xa \cdot y),\, \mu(a)\}, \quad \forall a, x, y \in G.$

3. Fuzzy g-ideals in BCC-algebras

Definition 3.1. ([5]). *A subset A of a BCC-algebra G is called an ideal if it satisfies*
(I1) $0 \in A,$
(I2) $ab \in A$ *for* $a \in A$ *and* $b \in G,$
(I3) $b(ba_1 \cdot a_2) \in A$ *for* $a_1, a_2 \in A$ *and* $b \in G.$

Here we call this ideal A a *g-ideal* to avoid the confusion. We begin with the fuzzification of the above g-ideal.

Definition 3.2. *A fuzzy set μ in a BCC-algebra G is called a fuzzy g-ideal if it satisfies*
(FK1) $\mu(0) \geq \mu(a), \quad \forall a \in G,$
(FI1) $\mu(ab) \geq \mu(a), \quad \forall a, b \in G,$
(FI2) $\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1),\, \mu(a_2)\}, \quad \forall b, a_1, a_2 \in G.$

Observe that (FK1) follows from (FI1) and (2). Using (FI1) we know that every fuzzy g-ideal is a fuzzy subalgebra. Moreover, putting $a_1 = a$ and $a_2 = 0$ in (FI2) we obtain the following proposition.

Proposition 3.3. If μ is a fuzzy g-ideal of a BCC-algebra G, then

$$\mu(b \cdot ba) \geq \mu(a), \;\; \forall a, b \in G.$$

Corollary 3.4. Every fuzzy g-ideal μ of a BCC-algebra G is order reversing, i.e., if $x \leq a$ then $\mu(x) \geq \mu(a)$ for all $a, x \in G$.

Proof. If $x, a \in G$ are such that $x \leq a$, then $\mu(x) = \mu(x0) = \mu(x \cdot xa) \geq \mu(a)$, which completes the proof. ☐

Theorem 3.5. A fuzzy set μ in a BCC-algebra G is a fuzzy g-ideal if and only if it is a fuzzy BCC-ideal.

Proof. Let μ be a fuzzy g-ideal and let $a, x, y \in G$. Then

$$\begin{aligned} \mu(xy) &= \mu(xy \cdot 0) \\ &= \mu(xy \cdot ((xy \cdot (xa \cdot y))(x \cdot xa))) \\ &\geq \min\{\mu(xa \cdot y), \mu(x \cdot xa)\} \\ &\geq \min\{\mu(xa \cdot y), \mu(a)\}, \end{aligned}$$

which shows that μ satisfies (FC1). Hence μ is a fuzzy BCC-ideal.

Conversely, let μ be a fuzzy BCC-ideal. Then $\mu(y) \leq \mu(x)$ for all $x \leq y$. Indeed,

$$\begin{aligned} \mu(x) &= \mu(x0) = \mu(x \cdot xy) \geq \min\{\mu(xy \cdot xy),\, \mu(y)\} \\ &= \min\{\mu(0),\, \mu(y)\} = \mu(y). \end{aligned}$$

Moreover, for all $a, x \in G$, we have

$$\begin{aligned} \mu(ax) &\geq \min\{\mu(aa \cdot x), \mu(a)\} \\ &= \min\{\mu(0x), \mu(a)\} \\ &= \min\{\mu(0), \mu(a)\} = \mu(a), \end{aligned}$$

which proves (FI1). To prove (FI2), let $x, a_1, a_2 \in G$. Note that

$$\begin{aligned} \mu(x \cdot xa_1) &\geq \min\{\mu(xa_1 \cdot xa_1), \mu(a_1)\} \\ &= \min\{\mu(0), \mu(a_1)\} = \mu(a_1). \end{aligned}$$

Since $xa_2 \cdot (xa_1 \cdot a_2) \leq x \cdot xa_1$ by (8), then

$$\mu(xa_2 \cdot (xa_1 \cdot a_2)) \geq \mu(x \cdot xa_1) \geq \mu(a_1).$$

By using (FC1), we see that

$$\begin{aligned} \mu(x(xa_1 \cdot a_2) &\geq \min\{\mu(xa_2 \cdot (xa_1 \cdot a_2)), \mu(a_2)\} \\ &\geq \min\{\mu(a_1), \mu(a_2)\}, \end{aligned}$$

which proves (FI2). Hence μ is a fuzzy g-ideal. □

Theorem 3.6. Let μ be a fuzzy set in a BCK-algebra G. Then μ is a fuzzy g-ideal if and only if μ is a fuzzy BCK-ideal.

Proof. Since every BCK-algebra is a BCC-algebra, every fuzzy g-ideal is a fuzzy BCC-ideal (see Theorem 3.5) and hence a fuzzy BCK-ideal. Let μ be a fuzzy BCK-ideal. Then

$$\begin{aligned} \mu(ax) &\geq \min\{\mu(ax \cdot a), \mu(a)\} \\ &= \min\{\mu(aa \cdot x), \mu(a)\} \\ &= \min\{\mu(0x), \mu(a)\} \\ &= \min\{\mu(0), \mu(a)\} \\ &= \mu(a), \end{aligned}$$

which shows (FI1). Now let $x, a_1, a_2 \in G$. Using (6), (8) and (II), we have

$$x(xa_1 \cdot a_2) \cdot a_2 = xa_2 \cdot (xa_1 \cdot a_2) \leq x \cdot xa_1 \leq a_1.$$

Since every fuzzy BCK-ideal of a BCK-algebra is order reversing, it follows that $\mu(x(xa_1 \cdot a_2) \cdot a_2) \geq \mu(a_1)$, and hence using (FK2) we obtain

$$\begin{aligned}\mu(x(xa_1 \cdot a_2)) &\geq \min\{\mu(x(xa_1 \cdot a_2) \cdot a_2), \mu(a_2)\} \\ &\geq \min\{\mu(a_1), \mu(a_2)\},\end{aligned}$$

which proves that μ satisfies (FI2). This completes the proof. □

The following example shows that a fuzzy BCK-ideal of a BCC-algebra may not be a fuzzy g-ideal.

EXAMPLE 3.7. Consider a BCC-algebra $G = \{0, a, b, c, d\}$ with Cayley table as follows (cf. [9]):

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Let μ be a fuzzy set in G defined by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in \{0, a\}, \\ t_2 & \text{otherwise}, \end{cases}$$

where $t_1 > t_2$ in $[0, 1]$. It is easy to verify that μ is a fuzzy BCK-ideal of G, but it is not a fuzzy g-ideal since

$$\mu(d(da \cdot a)) = t_2 < t_1 = \min\{\mu(a), \mu(a)\}.$$

PROPOSITION 3.8. Let A be a non-empty subset of a BCC-algebra G and let μ be a fuzzy set in G defined by

$$\mu(a) := \begin{cases} t_1 & \text{if } a \in A, \\ t_2 & \text{otherwise}, \end{cases}$$

where $t_1 > t_2$ in $[0, 1]$. Then μ is a fuzzy g-ideal of G if and only if A is a g-ideal of G.

PROOF. Assume that μ is a fuzzy g-ideal of G. Since $\mu(0) \geq \mu(a)$ for all $a \in G$, we have $\mu(0) = t_1$ and so $0 \in A$. Let $a \in A$ and $b \in G$. Then $\mu(ab) \geq \mu(a) = t_1$ and thus $\mu(ab) = t_1$. Hence $ab \in A$. For any $a_1, a_2 \in A$ and $b \in G$, we get $\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1), \mu(a_2)\} = t_1$ which implies that $\mu(b(ba_1 \cdot a_2)) = t_1$. It follows that $b(ba_1 \cdot a_2) \in A$. Therefore A is a g-ideal of G.

Conversely suppose that A is a g-ideal of G. Since $0 \in A$, it follows that $\mu(0) = t_1 \geq \mu(a)$ for all $a \in G$. Let $a, b \in G$. If $a \in A$, then $ab \in A$ and so $\mu(ab) = t_1 = \mu(a)$. If $a \in G \setminus A$, then $\mu(a) = t_2$ and hence $\mu(ab) \geq t_2 = \mu(a)$.

Finally let $a_1, a_2, b \in G$. If $a_1 \in G \setminus A$ or $a_2 \in G \setminus A$, then $\mu(a_1) = t_2$ or $\mu(a_2) = t_2$. It follows that

$$\mu(b(ba_1 \cdot a_2)) \geq t_2 = \min\{\mu(a_1), \mu(a_2)\}.$$

Assume that $a_1, a_2 \in A$. Then $b(ba_1 \cdot a_2) \in A$ and thus

$$\mu(b(ba_1 \cdot a_2)) = t_1 = \min\{\mu(a_1), \mu(a_2)\}.$$

Hence μ is a fuzzy g-ideal of G. □

LEMMA 3.9. ([5]). An initial segment $[0, c] := \{x \in G : 0 \leq x \leq c\}$ of a BCC-algebra G is a g-ideal if and only if the inequality $x(xc \cdot c) \leq c$ holds for all $x \in G$.

If we combine Proposition 3.8 with Lemma 3.9, then we have the following theorem.

THEOREM 3.10. Let μ be a fuzzy set in a BCC-algebra G defined by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in [0, c], \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0, 1]$. Then μ is a fuzzy g-ideal if and only if the inequality $x(xc \cdot c) \leq c$ holds for all $x \in G$.

As a simple consequence of the above Theorem and [10, Proposition 2.7] we obtain

COROLLARY 3.11. Let μ be as in Theorem 3.10. Then
(i) μ is a fuzzy g-ideal if and only if $xc \cdot y \leq c$ implies $xy \leq c$ for all $x, y \in G$.
(ii) μ is a fuzzy g-ideal if and only if $xc \leq c$ implies $x \leq c$ for all $x \in G$.

4. FUZZY CHARACTERISTIC g-IDEALS

For an endomorphism f of a BCC-algebra G and a fuzzy set μ in G, we define a new fuzzy set μ^f in G by $\mu^f(x) = \mu(f(x))$ for all $x \in G$.

PROPOSITION 4.1. Let f be an endomorphism of a BCC-algebra G. If μ is a fuzzy g-ideal of G, then so is μ^f.

PROOF. We first have that $\mu^f(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = \mu^f(0)$ for all $x \in G$. Let $a, b \in G$. Then

$$\mu^f(ab) = \mu(f(ab)) = \mu(f(a)f(b)) \geq \mu(f(a)) = \mu^f(a),$$

proving the condition (FI1). Finally for any $b, a_1, a_2 \in G$ we get

$$\begin{aligned} \mu^f(b(ba_1 \cdot a_2)) &= \mu(f(b(ba_1 \cdot a_2))) \\ &= \mu(f(b)(f(b)f(a_1) \cdot f(a_2))) \\ &\geq \min\{\mu(f(a_1)), \mu(f(a_2))\} \\ &= \min\{\mu^f(a_1), \mu^f(a_2)\}, \end{aligned}$$

ending the proof. □

DEFINITION 4.2. *A g-ideal A of a BCC-algebra G is said to be* characteristic *if $f(A) = A$ for all $f \in \mathrm{Aut}(G)$, where Aut(G) is the set of all automorphisms of G.*

DEFINITION 4.3. *A fuzzy g-ideal μ of a BCC-algebra G is said to be* fuzzy characteristic *if $\mu^f(x) = \mu(x)$ for all $x \in G$ and $f \in \mathrm{Aut}(G)$.*

LEMMA 4.4. Let μ be a fuzzy set in a BCC-algebra G and let $t \in \mathrm{Im}(\mu)$. Then μ is a fuzzy g-ideal of G if and only if the level subset

$$\mu_t := \{x \in G | \mu(x) \geq t\}$$

is a g-ideal of G, which is called a *level g-ideal* of μ.

PROOF. Assume that μ is a fuzzy g-ideal of G. Clearly $0 \in \mu_t$. Let $a \in \mu_t$ and $b \in G$. Then $\mu(a) \geq t$ and so $\mu(ab) \geq \mu(a) \geq t$, which implies that $ab \in \mu_t$. Now let $a_1, a_2 \in \mu_t$ and $b \in G$. Then

$$\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1), \mu(a_2)\} \geq t$$

and thus $b(ba_1 \cdot a_2) \in \mu_t$. Hence μ_t is a g-ideal of G.

Conversely suppose that μ_t is a g-ideal of G. If there exists $a_0 \in G$ such that $\mu(0) < \mu(a_0)$, then $\mu(0) < \frac{1}{2}(\mu(0) + \mu(a_0)) < \mu(a_0)$ and hence $a_0 \in \mu_s$ where $s = \frac{1}{2}(\mu(0) + \mu(a_0))$. Since $0 \in \mu_s$, we have $\mu(0) \geq s$, a contradiction. Assume that $\mu(a_0b_0) < \mu(a_0)$ for some $a_0, b_0 \in G$. Taking $u = \frac{1}{2}(\mu(a_0b_0) + \mu(a_0))$, then $\mu(a_0b_0) < u < \mu(a_0)$ and thus $a_0 \in \mu_u$ and $a_0b_0 \notin \mu_u$. This is a contradiction. Finally suppose that there exist $a_1, a_2, b \in G$ such that

$$\mu(b(ba_1 \cdot a_2)) < \min\{\mu(a_1), \mu(a_2)\}.$$

If we take $v = \frac{1}{2}(\mu(b(ba_1 \cdot a_2)) + \min\{\mu(a_1), \mu(a_2)\})$, then $\mu(b(ba_1 \cdot a_2)) < v < \min\{\mu(a_1), \mu(a_2)\}$ and so $a_1, a_2 \in \mu_v$ and $b(ba_1 \cdot a_2) \notin \mu_v$, a contradiction. This completes the proof. □

LEMMA 4.5. Let μ be a fuzzy g-ideal of a BCC-algebra G and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.

PROOF. Straightforward. □

THEOREM 4.6. For a fuzzy g-ideal μ of a BCC-algebra G, the following are equivalent:
(i) μ is fuzzy characteristic.
(ii) Each level g-ideal of μ is characteristic.

PROOF. Assume that μ is a fuzzy characteristic and let $t \in \mathrm{Im}(\mu)$, $f \in \mathrm{Aut}(G)$ and $x \in \mu_t$. Then $\mu^f(x) = \mu(x) \geq t$, i.e., $\mu(f(x)) \geq t$, and so $f(x) \in \mu_t$, i.e., $f(\mu_t) \subset \mu_t$. Now let $x \in \mu_t$ and let $y \in G$ be such that $f(y) = x$. Then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t$, whence $y \in \mu_t$, so that $x = f(y) \in f(\mu_t)$. Consequently $\mu_t \subset f(\mu_t)$. Hence $f(\mu_t) = \mu_t$ and μ_t is characteristic.

Conversely suppose that each level g-ideal of μ is characteristic and let $x \in G$, $f \in \mathrm{Aut}(G)$ and $\mu(x) = t$. Then, by virtue of Lemma 4.5, $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. It follows from hypothesis that $f(x) \in f(\mu_t) = \mu_t$, so that $\mu^f(x) = \mu(f(x)) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$, which implies from the injectivity of f that $x \in \mu_s$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) = t = \mu(x)$ showing that μ is fuzzy characteristic. □

5. Cartesian product of fuzzy g-ideals

Definition 5.1. ([1]). *A* fuzzy relation *on any set S is a fuzzy set*

$$\mu : S \times S \to [0,1].$$

Definition 5.2. ([1]). *If μ is a fuzzy relation on a set S and ν is a fuzzy set in S, then μ is a* fuzzy relation *on ν if*

$$\mu(x,y) \leq \min\{\nu(x), \nu(y)\}, \quad \forall x, y \in S.$$

Definition 5.3. ([1]). *Let μ and ν be fuzzy sets in a set S. The* Cartesian product *of μ and ν is defined by*

$$(\mu \times \nu)(x,y) = \min\{\mu(x), \nu(y)\}, \quad \forall x, y \in S.$$

Lemma 5.4. ([1]). Let μ and ν be fuzzy sets in a set S. Then
(i) $\mu \times \nu$ is a fuzzy relation on S,
(ii) $(\mu \times \nu)_t = \mu_t \times \nu_t$ for all $t \in [0,1]$.

Definition 5.5. ([1]). *If ν is a fuzzy set in a set S, the* strongest fuzzy relation *on S that is a fuzzy relation on ν is μ_ν, given by*

$$\mu_\nu(x,y) = \min\{\nu(x), \nu(y)\}, \quad \forall x, y \in S.$$

Lemma 5.6. ([1]). For a given fuzzy set ν in a set S, let μ_ν be the strongest fuzzy relation on S. Then for $t \in [0,1]$, we have that $(\mu_\nu)_t = \nu_t \times \nu_t$.

Proposition 5.7. For a given fuzzy set ν in a BCC-algebra G, let μ_ν be the strongest fuzzy relation on G. If μ_ν is a fuzzy g-ideal of $G \times G$, then $\nu(a) \leq \nu(0)$ for all $a \in G$.

Proof. From the fact that μ_ν is a fuzzy g-ideal of $G \times G$, it follows from (FK1) that $\mu_\nu(a,a) \leq \mu_\nu(0,0)$ for all $a \in G$, where $(0,0)$ is the zero element of $G \times G$. But this means that $\min\{\nu(0), \nu(0)\} \geq \min\{\nu(a), \nu(a)\}$, which implies that $\nu(0) \geq \nu(a)$. □

The following proposition is an immediate consequence of Lemma 5.6, and we omit the proof.

Proposition 5.8. If ν is a fuzzy g-ideal of a BCC-algebra G, then the level g-ideals of μ_ν are given by $(\mu_\nu)_t = \nu_t \times \nu_t$ for all $t \in [0,1]$.

THEOREM 5.9. Let μ and ν be fuzzy g-ideals of a BCC-algebra G. Then $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$.

PROOF. Note first that for every $(x, y) \in G \times G$,

$$(\mu \times \nu)(0,0) = \min\{\mu(0), \nu(0)\} \geq \min\{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y).$$

Let $(a_1, a_2), (b_1, b_2) \in G \times G$. Then

$$\begin{aligned} &(\mu \times \nu)((a_1, a_2) * (b_1, b_2)) = (\mu \times \nu)(a_1 b_1, a_2 b_2) \\ &= \min\{\mu(a_1 b_1), \nu(a_2 b_2)\} \geq \min\{\mu(a_1), \nu(a_2)\} = (\mu \times \nu)(a_1, a_2). \end{aligned}$$

For any $(b_1, b_2), (x_1, x_2), (y_1, y_2) \in G \times G$, we have

$$\begin{aligned} &(\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (x_1, x_2)) * (y_1, y_2))) \\ &= (\mu \times \nu)((b_1, b_2) * ((b_1 x_1, b_2 x_2) * (y_1, y_2))) \\ &= (\mu \times \nu)((b_1, b_2) * (b_1 x_1 \cdot y_1, b_2 x_2 \cdot y_2)) \\ &= (\mu \times \nu)(b_1(b_1 x_1 \cdot y_1), b_2(b_2 x_2 \cdot y_2)) \\ &= \min\{\mu(b_1(b_1 x_1 \cdot y_1)), \nu(b_2(b_2 x_2 \cdot y_2))\} \\ &\geq \min\{\min\{\mu(x_1), \mu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} \\ &= \min\{\min\{\mu(x_1), \nu(x_2)\}, \min\{\mu(y_1), \nu(y_2)\}\} \\ &= \min\{(\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)\}. \end{aligned}$$

Hence $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$. □

THEOREM 5.10. Let μ and ν be fuzzy sets in a BCC-algebra G such that $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$. Then
(i) either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \nu(0)$ for all $x \in G$.
(ii) if $\mu(x) \leq \mu(0)$ for all $x \in G$, then either $\mu(x) \leq \nu(0)$ or $\nu(x) \leq \nu(0)$.
(iii) if $\nu(x) \leq \nu(0)$ for all $x \in G$, then either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \mu(0)$.
(iv) either μ or ν is a fuzzy g-ideal of G.

PROOF. (i) Suppose that $\mu(x) > \mu(0)$ and $\nu(y) > \nu(0)$ for some $x, y \in G$. Then $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \min\{\mu(0), \nu(0)\} = (\mu \times \nu)(0, 0)$, which is a contradiction and we obtain (i).

(ii) Assume that there exist $x, y \in G$ such that $\mu(x) > \nu(0)$ and $\nu(y) > \nu(0)$. Then $(\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\} = \nu(0)$ and hence

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(0) = (\mu \times \nu)(0, 0).$$

This is a contradiction. Hence (ii) holds.

(iii) is by similar method to part (ii).

(iv) Since, by (i), either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \nu(0)$ for all $x \in G$, without loss of generality we may assume that $\nu(x) \leq \nu(0)$ for all $x \in G$. It follows from (iii) that either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \mu(0)$. If $\nu(x) \leq \mu(0)$ for any $x \in G$, then

$$\begin{aligned} \nu(x) &= \min\{\mu(0), \nu(x)\} = (\mu \times \nu)(0, x) \\ &\leq (\mu \times \nu)((0, x) * (y_1, y_2)) = (\mu \times \nu)(0 y_1, x y_2) \\ &= (\mu \times \nu)(0, x y_2) = \nu(x y_2) \end{aligned}$$

for all $x, y_1, y_2 \in G$, which proves that ν satisfies the condition (FI1). Now

$$\begin{aligned}
&\min\{\nu(a_1), \nu(a_2)\} \\
&= \min\{\min\{\mu(0), \nu(a_1)\}, \min\{\mu(0), \nu(a_2)\}\} \\
&= \min\{(\mu \times \nu)(0, a_1), (\mu \times \nu)(0, a_2)\} \\
&\le (\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (0, a_1)) * (0, a_2))) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1 0 \cdot 0, b_2 a_1 \cdot a_2)) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1, b_2 a_1 \cdot a_2)) \\
&= (\mu \times \nu)(b_1 b_1, b_2(b_2 a_1 \cdot a_2)) \\
&= (\mu \times \nu)(0, b_2(b_2 a_1 \cdot a_2)) \\
&= \min\{\mu(0), \nu(b_2(b_2 a_1 \cdot a_2))\} \\
&= \nu(b_2(b_2 a_1 \cdot a_2))
\end{aligned}$$

for all $a_i, b_j \in G$, $i = 1, 2$; $j = 1, 2$. Hence ν is a fuzzy g-ideal of G. Now we consider the case $\mu(x) \le \mu(0)$ for all $x \in G$. Suppose that $\nu(y) > \mu(0)$ for some $y \in G$. Then $\nu(0) \ge \nu(y) > \mu(0)$. Since $\mu(0) \ge \mu(x)$ for all $x \in G$, it follows that $\nu(0) > \mu(x)$ for any $x \in G$. Hence $(\mu \times \nu)(x, 0) = \min\{\mu(x), \nu(0)\} = \mu(x)$ for all $x \in G$. Thus

$$\begin{aligned}
&\mu(x) = (\mu \times \nu)(x, 0) \le (\mu \times \nu)((x, 0) * (y_1, y_2)) \\
&= (\mu \times \nu)(xy_1, 0y_2) = (\mu \times \nu)(xy_1, 0) = \mu(xy_1)
\end{aligned}$$

for all $x, y_1, y_2 \in G$. Moreover

$$\begin{aligned}
&\min\{\mu(a_1), \mu(a_2)\} \\
&= \min\{(\mu \times \nu)(a_1, 0), (\mu \times \nu)(a_2, 0)\} \\
&\le (\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (a_1, 0)) * (a_2, 0))) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1 a_1 \cdot a_2, b_2 0 \cdot 0)) \\
&= (\mu \times \nu)(b_1(b_1 a_1 \cdot a_2), b_2 b_2) \\
&= (\mu \times \nu)(b_1(b_1 a_1 \cdot a_2), 0) \\
&= \mu(b_1(b_1 a_1 \cdot a_2))
\end{aligned}$$

for all $a_i, b_j \in G$, $i = 1, 2$; $j = 1, 2$, which proves that μ is a fuzzy g-ideal of G. This completes the proof. □

Now we give an example to show that if $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$, then μ and ν both need not be fuzzy g-ideals of G.

Example 5.11. Let G be a BCC-algebra with $|G| \ge 2$ and let $s, t \in [0, 1)$ be such that $s \le t$. Define fuzzy sets μ and ν in G by $\mu(x) = s$ and

$$\nu(x) = \begin{cases} t & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in G$, respectively. Then $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} = s$ for all $(x, y) \in G \times G$, that is, $\mu \times \nu$ is a constant function and so $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$. Now μ is a fuzzy g-ideal of G, but ν is not a fuzzy g-ideal of G since for $x \ne 0$ we have $\nu(0) = t < 1 = \nu(x)$.

THEOREM 5.12. Let ν be a fuzzy set in a BCC-algebra G and let μ_ν be the strongest fuzzy relation on G. Then ν is a fuzzy g-ideal of G if and only if μ_ν is a fuzzy g-ideal of $G \times G$.

PROOF. Assume that ν is a fuzzy g-ideal of G. Clearly $\mu_\nu(0,0) \geq \mu_\nu(x,y)$ for any $(x,y) \in G \times G$. Now

$$\begin{aligned}\mu_\nu(a_1,a_2) &= \min\{\nu(a_1),\nu(a_2)\} \leq \min\{\nu(a_1b_1),\nu(a_2b_2)\} \\ &= \mu_\nu(a_1b_1,a_2b_2) = \mu_\nu((a_1,a_2)*(b_1,b_2))\end{aligned}$$

for all $(a_1,a_2),(b_1,b_2) \in G \times G$, and

$$\begin{aligned}&\min\{\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)\} \\ &= \min\{\min\{\nu(a_1),\nu(a_2)\},\min\{\nu(b_1),\nu(b_2)\}\} \\ &= \min\{\min\{\nu(a_1),\nu(b_1)\},\min\{\nu(a_2),\nu(b_2)\}\} \\ &\leq \min\{\nu(x(xa_1\cdot b_1)),\nu(y(ya_2\cdot b_2))\} \\ &= \mu_\nu(x(xa_1\cdot b_1),y(ya_2\cdot b_2)) \\ &= \mu_\nu((x,y)*(((x,y)*(a_1,a_2))*(b_1,b_2)))\end{aligned}$$

for all $(x,y),(a_1,a_2),(b_1,b_2) \in G \times G$. Hence μ_ν is a fuzzy g-ideal of $G \times G$.

Conversely suppose that μ_ν is a fuzzy g-ideal of $G \times G$. Then

$$\min\{\nu(0),\nu(0)\} = \mu_\nu(0,0) \geq \mu_\nu(x,y) = \min\{\nu(x),\nu(y)\}$$

for all $(x,y) \in G \times G$. It follows that $\nu(x) \leq \nu(0)$ for all $x \in G$. Now we have

$$\begin{aligned}\nu(a) &= \min\{\nu(a),\nu(0)\} = \mu_\nu(a,0) \leq \mu_\nu((a,0)*(b_1,b_2)) \\ &= \mu_\nu(ab_1,0b_2) = \mu_\nu(ab_1,0) = \min\{\nu(ab_1),\nu(0)\} = \nu(ab_1)\end{aligned}$$

for all $a, b_1 \in G$, and

$$\begin{aligned}&\min\{\min\{\nu(a_1),\nu(a_2)\},\min\{\nu(b_1),\nu(b_2)\}\} \\ &= \min\{\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)\} \\ &\leq \mu_\nu((x,y)*(((x,y)*(a_1,a_2))*(b_1,b_2))) \\ &= \mu_\nu(x(xa_1\cdot b_1),y(ya_2\cdot b_2)) \\ &= \min\{\nu(x(xa_1\cdot b_1)),\nu(y(ya_2\cdot b_2))\}\end{aligned}$$

for all $(x,y),(a_1,a_2),(b_1,b_2) \in G \times G$. Taking $a_2 = b_2 = 0$ (resp. $a_1 = b_1 = 0$) and using (2) and (4), then

$$\min\{\nu(a_1),\nu(b_1)\} \leq \nu(x(xa_1\cdot b_1))$$

$$(\text{resp. } \min\{\nu(a_2),\nu(b_2)\} \leq \nu(x(xa_2\cdot b_2))).$$

Hence ν is a fuzzy g-ideal of G. ∎

REFERENCES

[1] P. Bhattacharya and N. P. Mukherjee, *Fuzzy relations and fuzzy groups*, Inform. Sci. **36** (1985), 267 – 282.

[2] P. S. Das, *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84** (1981), 264 – 269.

[3] W. A. Dudek, *The number of subalgebras of finite BCC-algebras*, Bull. Inst. Math. Academia Sinica **20** (1992), 129 – 136.

[4] W. A. Dudek, *On proper BCC-algebras*, Bull. Inst. Math. Academia Sinica **20** (1992), 137 − 150.
[5] W. A. Dudek, *A new characterization of ideals in BCC-algebras*, Novi Sad J. Math. **29** (1999), 139 − 145.
[6] W. A. Dudek and Y. B. Jun, *Fuzzy BCC-ideals in BCC-algebras*, Math. Montisnigri vol. **10** (1999), 21 − 30.
[7] W. A. Dudek, Y. B. Jun and Z. Stojaković, *On fuzzy BCC-algebras*, Fuzzy Sets and Systems (in print)
[8] W. A. Dudek, Y. B. Jun, *Normalizations of fuzzy BCC-ideals in BCC-algebras*, Math. Moravica **3** (1999) 17 − 24.
[9] W. A. Dudek and X. H. Zhang, *On ideals and congruences in BCC-algebras* , Czechoslovak Math. J. **48(123)** (1998), 21 − 29.
[10] W. A. Dudek and X. H. Zhang, *Initial segments in BCC-algebras*, Math. Moravica **4** (2000), (in print)
[11] D. H. Foster, *Fuzzy topological groups*, J. Math. Anal. Appl. **67** (1979), 549 − 564.
[12] Y. Imai and K. Iséki, *On axiom system of propositional calculi XIV*, Proc. Japan Academy **42** (1966), 19 − 22.
[13] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japon. **21** (1976), 351 − 366.
[14] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1978), 1 − 26.
[15] Y. Komori, *The class of BCC-algebras is not a variety*, Math. Japon. **29** (1984), 391 − 394.
[16] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa, Seoul, Korea, 1994.
[17] A. Wroński, *BCK-algebras do not form a variety*, Math. Japon. **28** (1983), 211−213.
[18] O. G. Xi, *Fuzzy BCK-algebras*, Math. Japon. **36** (1991), 935 − 942.
[19] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338 − 353.

Institute of Mathematics,
Technical University,
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: dudek@im.pwr.wroc.pl

Department of Mathematics Education,
Gyeongsang National University
Chinju 660-701, Korea
E-mail: ybjun@nongae.gsnu.ac.kr

Received: 01.10.99.
Revised: 13.05.2000.