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FUZZY TOPOLOGICAL GAMES, α-METACOMPACTNESS AND α-PERFECT MAPS

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Abstract. The behavior of Fuzzy Topological Games and α-metacompactness under α-perfect maps are studied. Also an attempt is made to bring out some close relationships between Fuzzy Topological Games and α-Metacompactness.

1. Introduction

As a generalization of the topological game $G(\mathbf{K}, X)$ introduced by Telgarsky [9], the author [7] introduced the fuzzy topological game $G'(\mathbf{K}, X)$ and studied some properties of $G'(\mathbf{K}, X)$. In this paper some nice properties such as preservation of winning strategies under various kinds of mapping such as *F-continuous*, *F-closed*, *F-open*, *α-perfect* are discussed. Also the behavior of α-metacompactness under the above mentioned mappings and some results connecting $G'(\mathbf{K}, X)$ and α-metacompactness are also discussed.

2. Fuzzy topological games and mappings

2.1 Notation By $\mathbf{K}$ we denote a non-empty family of fuzzy topological spaces, where all are assumed to be T_1. That is all fuzzy singletons are fuzzy closed. $\underline{I}^x$ denotes the family of all fuzzy closed subsets of X. Also $X \in \mathbf{K}$ implies $\underline{I}^x \subseteq \mathbf{K}$. $\mathbf{DK}(\mathbf{FK})$ denote the class of all fuzzy topological spaces which have a discrete (finite) fuzzy closed α-shading by members of $\mathbf{K}$. (A family $\cup$ of fuzzy sets in a fts X is said to be an α-shading if for each $x \in X$, there is a $U \in \cup$ with $U(x) > \alpha$ [3])

2.2 Definition [7] Let $\mathbf{K}$ be a class of fuzzy topological spaces and let $X \in \mathbf{K}$. Then the fuzzy topological game $G'(\mathbf{K}, X)$ is defined as follows. There are

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two players Player I and Player II. They alternatively choose consecutive terms of the sequence $(E_1, F_1, E_2, F_2, \ldots)$ of fuzzy subsets of X. When each player chooses his term he knows $\mathbf{K}$, X and their previous choices. A sequence $(E_1, F_1, E_2, F_2, \ldots)$ is a play for $G'(\mathbf{K}, X)$ if it satisfies the following conditions for each $n \geq 1$.

1. E_n is a choice of Player I
2. F_n is a choice of Player II
3. $E_n \in \underline{I}^x \cap \mathbf{K}$
4. $F_n \in \underline{I}^x$
5. $E_n \vee F_n < F_{n-1}$ where $F_0 = X$
6. $E_n \wedge F_n = 0$

Player I wins the play if $\inf_{n\geq 1} F_n = 0$. Otherwise Player II wins the game.

2.3 Definition [7] A finite sequence $(E_1, F_1, E_2, F_2, \ldots, E_m, F_m)$ is admissible if it satisfies condition (1)–(6) for each $n \leq m$.

2.4 Definition [7] Let S' be a crisp function defined as follows

$$S' \; : \; \bigcup_{n\geq 1} (\underline{I}^x)^n \longrightarrow \underline{I}^x \cap \mathbf{K}$$

Let $S_1 = \{X\}$. $S_2 = \{F \in \underline{I}^x : (S'(X), F)$ is admissible for $G'(\mathbf{K}, X)\}$. Continuing like this inductively we get $S_n = \{(F_1, F_2, \ldots, F_n) : (E_1, F_1, E_2, F_2, \ldots, E_n, F_n)$ is admissible for $G'(\mathbf{K}, X)$ where $F_0 = X$ and $E_1 = S'(E_1, F_1, E_2, F_2, \ldots, F_{i-1})$ for each $i \leq n\}$. Then the restriction S of S' to $\cup_{n\geq 1} S_n$ is called a fuzzy strategy for Player I in $G'(\mathbf{K}, X)$.

2.5 Definition [7] If Player I wins every play $(E_1, F_1, E_2, F_2, \ldots, E_n, F_n, \ldots)$ such that $E_n = S(F_1, F_2, \ldots, F_{n-1})$, then we say that S is a fuzzy winning strategy.

2.6 Definition [7] A function $S : \underline{I}^x \xrightarrow{\text{into}} \underline{I}^x \cap \mathbf{K}$ is called a fuzzy stationary strategy for Player I in $G'(\mathbf{K}, X)$ if $S(F) < F$ for each $F \in \underline{I}^x$. We say that S is a fuzzy stationary winning strategy if he wins every play $(S(X), F_1, S(F_1), F_2, \ldots)$

2.7 Result [7] A function $S : \underline{I}^x \xrightarrow{\text{into}} \underline{I}^x \cap \mathbf{K}$ is a fuzzy stationary winning strategy if and only if it satisfies

1. For each $F \in \underline{I}^x$, $S(F) < F$
2. If $\{F_n : n \geq 1\}$ satisfies $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$ for each $n \geq 1$ then $\inf_{n\geq 1} F_n = 0$.

2.8 Theorem [7] Player I has a fuzzy winning strategy in $G'(\mathbf{K}, X)$ if and only if he has a fuzzy stationary winning strategy in it.

2.9 Definition [2] Let f be function from a fts (X, T) to a fts (Y, S). Then f is said to be F-continuous if for each $b \in S$, $f^{-1}(b) \in T$ or equivalently for each closed fuzzy set h in (Y, S), $f^{-1}(h)$ is closed in (X, T).

2.10 DEFINITION [2] Let f be function from a fts (X,T) to a fts (Y,S). Then f is F-*open* (F-*closed*) iff for each open (closed) fuzzy set a in (X,T), $f(a)$ is open (closed) fuzzy set in (Y,S).

2.11 THEOREM Let X and Y be two fuzzy topological spaces and $\mathbf{K}_1$ and $\mathbf{K}_2$ be two classes of fts such that $X \in \mathbf{K}_1$ and $Y \in \mathbf{K}_2$. If f is an F-continuous function from X on to Y which maps all $E \in \underline{I^x} \cap \mathbf{K}_1$ to $f(E) \in \underline{I^x} \cap \mathbf{K}_2$ and if Player I has a fuzzy winning strategy in $G'(\mathbf{K}_1, X)$, then Player I has a fuzzy winning strategy in $G'(\mathbf{K}_2, Y)$.

PROOF Let S be a fuzzy stationary winning strategy for Player I in $G'(\mathbf{K}_1, X)$. Thus Player I wins every play of the form $(S(X), F_1, S(F_1), \ldots)$. Now we will define a stationary winning strategy t for Player I in $G'(\mathbf{K}_2, Y)$. Now consider the play $(t(Y), P_1, t(P_1), P_2, \ldots)$ where $P_n = t(F_n)$ and $t : \underline{I^Y} \overset{\text{into}}{\longrightarrow} \underline{I^Y} \cap \mathbf{K}_2$ is defined by $t(P_n) = f[S(F_n)]$. Now t is a stationary winning strategy for $G'(\mathbf{K}_2, Y)$.

$$\begin{aligned}
\text{For} \quad t(F_n) &= f[S(F_n)] \\
&< f(F_n) \\
&= P_n \text{ Therefore } t \text{ is a fuzzy stationary strategy.} \\
\text{Now} \quad t(P_n) \wedge P_{n+1} &= f[S(F_n)] \wedge f(F_{n+1}) \\
&= f[S(F_n) \wedge F_{n+1}] \\
&= f(0) \\
&= 0 \\
\text{Also} \quad t(Y) \wedge P_1 &= f[S(X)] \wedge P_1 \\
&= f[S(X)] \wedge f(F_1) \\
&= f[S(X) \wedge F_1] \\
&= f(0) \\
&= 0
\end{aligned}$$

Therefore it follows from Result 2.7 that $\inf_{n \geq 1} F_n = 0$ and hence t is a stationary winning strategy for Player I in $G'(\mathbf{K}_2, Y)$.

2.12 THEOREM Let $f : X \overset{\text{onto}}{\longrightarrow} Y$ be an F-continuous F-closed mapping such that $f^{-1}(E) \in \underline{I^x} \cap \mathbf{K}_1$, whenever $E \in \underline{I^x} \cap \mathbf{K}_2$. If Player I has fuzzy winning strategy in $G'(\mathbf{K}_2, Y)$, then Player I has a fuzzy winning strategy in $G'(\mathbf{K}_1, X)$.

PROOF Let S be a fuzzy stationary winning strategy for Player I in $G'(\mathbf{K}_2, Y)$. Therefore Player I wins every play of the form $(S(Y), F_1, S(F_1), \ldots)$. Now we will define a function $t : \underline{I^x} \longrightarrow \underline{I^x} \cap \mathbf{K}_1$ as follows. Now $f : X \overset{\text{onto}}{\longrightarrow} Y$

is F-closed and hence we take $P_n = f^{-1}(F_n)$ where $P_n \in \underline{I^x}$ and $t(P_n) = f^{-1}[S(F_n)]$ for all $P_n \in \underline{I^x}$.

$$\begin{aligned} \text{Now} \quad t(P_n) &= f^{-1}[S(F_n)] \\ &< f^{-1}(F_n) \\ &= P_n \text{ Thus } t \text{ is a fuzzy stationary strategy.} \end{aligned}$$

Now consider the play $(t(X), P_1, t(P_1), \ldots)$

$$\begin{aligned} t(P_n) \wedge P_{n+1} &= f^{-1}[S(F_n)] \wedge P_n \\ &= f^{-1}[S(F_n)] \wedge f^{-1}(F_{n+1}) \\ &= f^{-1}[S(F_n) \wedge F_{n+1}] \\ &= f^{-1}(0) \\ &= 0 \\ \text{Also} \quad t(X) \wedge P_1 &= f^{-1}[S(X)] \wedge P_1 \\ &= f^{-1}[S(X)] \wedge f^{-1}(F_1) \\ &= f^{-1}[S(X) \wedge F_1] \\ &= f^{-1}(0) \\ &= 0 \end{aligned}$$

Therefore from Result 2.7 it follows that $\inf P_n = 0$ and hence t is a winning strategy also. Thus t is a fuzzy winning strategy for Player I in $G'(\mathbf{K}_1, X)$. This completes the proof.

As an immediate consequence of Theorem 2.11 and Theorem 2.12 we get the following two theorems.

2.13 Theorem Let X and Y are two fts and let $f : X \overset{\text{onto}}{\longrightarrow} Y$ be an F-continuous function and $f^{-1}(E) \in \underline{I^x} \cap \mathbf{K}_1$ whenever $E \in \underline{I^x} \cap \mathbf{K}_2$. If Player II has a fuzzy winning strategy in $G'(\mathbf{K}_1, X)$, then Player II has a fuzzy winning strategy in $G'(\mathbf{K}_2, Y)$.

2.14 Theorem Let $f : X \overset{\text{onto}}{\longrightarrow} Y$ be an F-continuous F-closed mapping such that $f^{-1}(E) \in \underline{I^Y} \cap \mathbf{K}_2$ whenever $E \in \underline{I^x} \cap \mathbf{K}_1$. If Player II has a fuzzy winning strategy in $G'(\mathbf{K}_2, Y)$, then Player II has a fuzzy winning strategy in $G'(\mathbf{K}_1, X)$.

2.15 Definition [5] Let $0 \leq \alpha < 1$ (resp. $0 < \alpha \leq 1$). An F-closed F-continuous function f from a fts X to a fts Y is said to be α-perfect (resp α^*-perfect) if and only if $f^{-1}(y)$ is α-compact (resp α^*-compact) for each $y \in Y$. (See [3])

2.16 Definition A class $\mathbf{K}$ of fuzzy topological spaces is said to be α-perfect if $X \in \mathbf{K}$ is equivalent to $Y \in \mathbf{K}$, provided that there exists an α-perfect map from X onto Y.
From Theorems 2.11, 2.12, 2.13 and 2.14 the next theorem follows immediately.

2.17 Theorem Let $\mathbf{K}$ be an α-perfect class of fts. If there is an α- perfect map from X onto Y. Then

1. If Player I has a fuzzy winning strategy in $G'(\mathbf{K}, X)$ then he has the same in $G'(\mathbf{K}, Y)$.
2. If Player II has a fuzzy winning strategy in $G'(\mathbf{K}, X)$, then he has the same in $G'(\mathbf{K}, Y)$.

3. Metacompactness and Mappings

An approach to fuzzy paracompactness using the concept of α-shading was introduced by Malghan and Benchalli [4]. The author [6] extended this concept to metacompact spaces and characterization for the same was also obtained.
3.1 Definition [4] A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X, T) is said to be locally finite if for each x in X there exists an open fuzzy set g with $g(x) = 1$ such that $a_s \leq 1 - g$ holds for all but atmost finitely many s in S.
3.2 Definition [4] A family $\{a_s : s \in S\}$ of fuzzy sets in a fts (X, T) is said to be point finite if for each x in X, $a_s(x) = 0$ for all but atmost finitely many s in S (or equivalently as $a_s(x) > 0$ for atmost finitely many s in S).
3.3 Definition [3] Let (X, T) be a fts and $\alpha \in [0, 1)$. A collection $\cup$ of fuzzy sets is called an α-shading (resp. $\alpha*$-shading) of X if for each $x \in X$, there exists $g \in \cup$ with $g(x) > \alpha$ (resp. $g(x) \geq \alpha$).
3.4 Definition [4] Let (X, T) be a fts and $\alpha \in [0, 1)$. Let $\cup$ and $\vee$ be any two α-shadings (resp. $\alpha*$-shading) of X. Then $\cup$ is a refinement of $\vee(\cup < \vee)$ if for each $g \in \cup$ there is an $h \in \vee$ such that $g \leq h$.
3.5 Definition [4] A fts (X, T) is said to be α-paracompact (resp. $\alpha*$-paracompact) if each α-shading (resp. $\alpha*$-shading) of X by open fuzzy sets has a locally finite α-shading (resp. $\alpha*$-shading) refinement by open fuzzy sets.
3.6 Definition [6] A fuzzy topological space (X, T) is said to be α-metacompact (resp. $\alpha*$-metacompact) if each α-shading (resp. $\alpha*$-shading) of X by open fuzzy sets has a point finite α-shading (resp. $\alpha*$-shading) refinement by open fuzzy sets.
3.7 Definition [8] A collection $\{A_i : i \in I\}$ of fuzzy subsets of a fts X is said to be closure preserving if for each $J \subseteq I$, $cl_X[\vee A_i : i \in J] = \vee[cl_X A_i : i \in J]$

3.8 Result Let $f : X \xrightarrow{\text{onto}} Y$ be an F-closed F-continuous mapping, where X and Y are fts. If $\{U_\alpha : \alpha \in \Lambda\}$ is a closure preserving family of fuzzy sets in X then so is $\{f(U_\alpha) : \alpha \in \Lambda\}$.

Proof Since f is F-continuous, it follows clearly that $f(cl\ U_\alpha) \leq cl\ f(U_\alpha)$ for every $\alpha \in \Lambda$. Now we have $U_\alpha \leq cl\ U_\alpha$ for every $\alpha \in \Lambda$.
Therefore $f(U_\alpha) \leq f(cl\ U_\alpha)$.

$$\begin{aligned} \text{That is } cl[f(U_\alpha)] &\leq cl[f(cl\ U_\alpha)]. \\ &= f(cl\ U_\alpha) \text{ since } f \text{ is } F\text{-closed} \end{aligned}$$

Therefore we get $cl\ [f(U_\alpha)] = f(cl\ U_\alpha)$ for every $\alpha \in \Lambda$.
Now for any collection $\{f(U_\alpha) : \alpha \in \Lambda\}$, clearly we have

$$\vee_{\alpha\in\Lambda} cl\ [f(U_\alpha)] \leq cl[\vee\{f(U_\alpha) : \alpha \in \Lambda\}]$$

$$\begin{aligned} \text{Again} \qquad f(U_\alpha) &\leq cl\ [f(U_\alpha)] \\ &= f(cl\ U_\alpha) \end{aligned}$$

Therefore we have $\vee\{f(U_\alpha) : \alpha \in \Lambda\} \leq \vee\{f(cl\ U_\alpha) : \alpha \in \Lambda\}$.

$$\begin{aligned} \text{That is, } cl[\vee\{f(U_\alpha) : \alpha \in \Lambda\}] &\leq cl[\vee\{f(cl\ U_\alpha) : \alpha \in \Lambda\}] \\ &= cl\ [f[\vee(cl\ U_\alpha) : \alpha \in \Lambda\}]] \\ &= cl\ [f(cl[\vee\{U_\alpha : \alpha \in \Lambda\})] \text{ since } \{U_\alpha : \alpha \in \Lambda\} \text{ is closure preserving} \\ &= f(cl\ [\vee\{U_\alpha : \alpha \in \Lambda\}]) \text{ since } F \text{ is } F\text{-closed} \\ &= f(\vee\{cl\ U_\alpha : \alpha \in \Lambda\}) \\ &= \vee\{f(cl\ U_\alpha) : \alpha \in \Lambda\} \\ &= \vee\{cl\ [f(U_\alpha) : \alpha \in \Lambda]\} \end{aligned}$$

Thus we get, $\vee_{\alpha\in\Lambda} cl[f(U_\alpha)] \geq cl\ [\vee\{f(U_\alpha) : \alpha \in \Lambda\}]$
And hence we have $\vee_{\alpha\in\Lambda} cl[f(U_\alpha)] = cl\ [\vee\{f(U_\alpha) : \alpha \in \Lambda\}]$. This completes the proof.

3.9 Result. Let X and Y be two fts and let $f : X \xrightarrow{\text{onto}} Y$ be finite to one. If $\mathbf{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a point finite collection of fuzzy sets in X, then $\{f(u_\alpha) : \alpha \in \Lambda\}$ is also a point finite collection in Y.

Proof Given that f is onto and finite to one, it follows that for every $y \in Y$, we have a finite (support) fuzzy subset $f^{-1}(y)$ in X. Let $x \in f^{-1}(y)$. Then since $\{U_\alpha : \alpha \in \Lambda\}$ is a point finite collection in X, $U_\alpha(x) > 0$ for at most finitely many $\alpha \in \Lambda$. Now since $f^{-1}(y)$ is finite, we get a finite sub-collection $\mathbf{U}_F$ of $\mathbf{U}$. Now consider the collection $\{f(u_F) : u_F \in \mathbf{U}_F\}$. This is finite and $f(u_F)(y) > 0$ for all $u_F \in \mathbf{U}_F$. Thus $\{f(U_\alpha) : \alpha \in \Lambda\}$ is a point finite collection in Y.

3.10 Theorem Let X and Y be two fts and let $f : X \xrightarrow{\text{onto}} Y$ be a finite to one F-open F-continuous mapping. If X is α-metacompact then so is Y.

Proof Given that X is α-metacompact, let $\mathbf{U}$ be an α-shading of Y by open fuzzy sets. Since f is F-continuous, it follows that $\mathbf{U}' = \{f^{-1}(U) : U \in \mathbf{U}\}$ is an α-shading of X by open fuzzy sets. Since X is α-metacompact, it follows that $\mathbf{U}'$ has a point finite α-shading refinement by open fuzzy sets say $\mathbf{V}$. Now clearly $\{f(V) : V \in \mathbf{V}\}$ is a point finite α-shading of Y and it refines $\mathbf{U}$ also. Since f is F-open, $f(V)$ is open also. Hence Y is α-metacompact.

3.11 Theorem Let $f : X \longrightarrow Y$ be an F-continuous, F-closed function. If X is α-metacompact, then Y is also α-metacompact.

Proof Let $\mathbf{U}$ be an α-shading of Y by open fuzzy sets. Then by a characterization of α-metacompactness [6], it is enough to prove $\mathbf{U}^F$ has a closure preserving α-shading refinement by closed fuzzy sets, where $\mathbf{U}^F$ is the collection of all unions of finite sub-collections from $\mathbf{U}$. Now since f is F-continuous $\mathbf{W} = \{f^{-1}(U) : U \in \mathbf{U}\}$ is an α-shading of X by open fuzzy sets. Since X is α-metacompact, it follows that $\mathbf{W}^F$ has a closure preserving α-shading refinement $\mathbf{F}$ by closed fuzzy sets. Since f is F-closed it follows that $f(F)$ is closed for each $F \in \mathbf{F}$. Thus $\{f(F) : F \in \mathbf{F}\}$ is the required closure preserving α-shading refinement of $\mathbf{U}^F$ by closed fuzzy sets.
3.12 Definition [3] A fts X is said to be α-compact if every α-shading of X by open fuzzy sets has a finite α-sub-shading by open fuzzy sets.
3.13 Definition Let X and Y be two fts. Then $f : X \to Y$ is F-open α-compact if f is F-open with α-compact fibers (where fibers of a mapping $f : X \longrightarrow Y$ are the sets $f^{-1}(y)$ for $y \in Y$).
3.14 Definition Let X and Y be two fts. If $y \in Int(f(y))$ whenever $f^{-1}(y) < U$ where $y \in Y$ and U is an open fuzzy set in X, then $f : X \longrightarrow Y$ is pseudo F-open
3.15 Definition Let $\mathbf{U}$ be a collection of fuzzy subsets of a fts X. We say that $\mathbf{U}$ is α-compact finite if $\{U \in \mathbf{U} : U \wedge K \neq 0\}$ is finite for any α-compact subset K of X.

3.16 Lemma Locally finite families of fuzzy sets are α-compact finite.

Proof Let $\mathbf{U}$ be a locally finite family of fuzzy subsets of a fts X. Let K be α-compact. Since $\mathbf{U}$ is locally finite, for any $x \in K$, we can find an open fuzzy set w_x such that $w_x(x) = l$ and $U_s \leq 1 \backslash w_x$ holds for all but atmost finitely many s. Now clearly $\{w_x : x \in X\}$ is a 1^*-shading of K and since K is α-compact we get a finite subshading say $\{w_{x1}, w_{x2}, \cdots, w_{xk}\}$ for some finite k, where each of w_{xi} has non-empty meet with at most finitely many $U \in \mathbf{U}$. Hence it follows that $\{U \in \mathbf{U} : U \wedge K \neq 0\}$ is finite.

3.17 THEOREM If $f : X \longrightarrow Y$ is F-continuous pseudo F-open α-compact with X α-paracompact, then X is α-metacompact.

PROOF Consider an α-shading $\mathbf{U}$ of Y by open fuzzy sets. Now since f is F-continuous and X is α-paracompact it follows that $\mathbf{U}' = \{f^{-1}(U) : U \in \mathbf{U}\}$ is an α-shading of X by open fuzzy sets. So $\mathbf{U}'$ has a locally finite α-shading refinement by open fuzzy sets, say $\mathbf{V}$. Now consider $\mathbf{K} = \{f(V) : V \in \mathbf{V}\}$. Since f is F-open α-compact and for every $y \in Y$, $f^{-1}(y)$ is α-compact, it follows from lemma 3.16 that $f^{-1}(y)$ has non-empty meet with atmost finitely many members of $\mathbf{V}$. Also since every locally finite family is point finite, it follows that $\mathbf{V}$ is point finite and hence $\mathbf{K}$ is also point finite. Since f is pseudo F-open it follows clearly that $y \in Int(st(y, \mathbf{K}))$ for every $y \in Y$. [where $st(x, \mathbf{U}) = \vee\{U \in \mathbf{U} : U(x) > 0\}$] Now from the characterization of α-metacompactness in [6] the proof is complete.

3.18 DEFINITION Let X be a fts and $\mathbf{U}$ be any α-shading of X, then for any $x \in X$, we define $\alpha - Ord(x, \mathbf{U}) = \text{Card}\{U \in \mathbf{U} : U(x) > \alpha\}$.

3.19 LEMMA Let X be a fts and $\mathbf{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a point finite α-shading by open fuzzy sets. Let $B_n = \{x \in X : \alpha - \text{Ord}(x, \mathbf{U}) \leq n\}$. Then $\{B_n : n \geq 0\}$ is an α-shading of X by closed fuzzy sets. If $n > 0$ and F is a closed fuzzy set with $F < B_n$ and $F \wedge B_{n-1} = 0$, then F has a discrete α-shading by closed fuzzy sets where each member is contained in some $U \in \mathbf{U}$.

PROOF For any $x \in X$ with $B_n(x) = 0$ for some n, it follows from the definition of B_n that there some $\Lambda' \subset \Lambda$ with $n+1$ numbers such that $U_\lambda(x) > \alpha$ for all $\lambda \in \Lambda'$. Now since each U_λ is fuzzy open, so is $\wedge\{U_\lambda : \lambda \in \Lambda'\}$. This is an open fuzzy neighbourhood of x disjoint from B_n. Therefore it follows that $1 \backslash B_n$ is fuzzy open and so the B_n are closed fuzzy sets.

Also given that $\mathbf{U}$ is a point finite α-shading of X, there exists atmost finitely many $U \in \mathbf{U}$ with $U(x) > \alpha$ for any $x \in X$. Then clearly $B_n(x) > \alpha$ for some n. Thus $\{B_n : n \geq 0\}$ is an α-shading of X.

Take F as in the statement of the Lemma. Let Ω be the set of all subsets of Λ which have n elements and for each $\gamma \in \Omega$ define $V_\gamma = \wedge\{U_\lambda : \lambda \in \gamma\}$. Now clearly $V_\gamma \wedge F < U_\lambda$ for each λ in γ and the collection $\{V_\gamma \wedge F : \gamma \in \Omega\}$ is disjoint and hence a discrete α-shading of X by closed fuzzy sets.

3.20 COROLLARY Let $\mathbf{U} = \{U_\lambda : \lambda < \eta\}$ be a point finite α-shading of an fts X by open fuzzy sets and $X_n = \{x \in X : \alpha - Ord(x, \mathbf{U}) \leq n\}$ for each $n \geq 1$. Then $\{X_n : n \geq 1\}$ is a countable α-shading of X by closed fuzzy sets and $\mathbf{B}_n = \{B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) : \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \eta\}$ is a discrete clopen α-shading of $X_n \backslash X_{n-1}$ for each $n \geq 1$ where $B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = (\bigwedge_{i \leq n} U_{\lambda i}) \wedge (X_n \backslash X_{n-1})$.

PROOF Take $F = X_n \backslash X_{n-1}$ in Lemma 3.19 and the corollary follows.

3.21 Definition A class of fts $\mathbf{K}$ is said to be finitely additive if every space in $X_n \backslash X_{n-1}$ with a finite α-shading by members of $\mathbf{K}$ belong to $\mathbf{K}$.
3.22 Definition [8] A fts X is $\mathbf{K}$-scattered if for every $0 \neq F \in \underline{I^x}$, there exists a point $x \in F$ and a fuzzy neighbourhood N of x with $N(x) > 0$ where $N < F$ and $N \in \mathbf{K}$.
3.23 Definition [8] An α-disjoint α-shading $\{L_\lambda : \lambda < \eta\}$ of a fts is a $\mathbf{K}$-scattered partition if for some $N \in \mathbf{K}$, $L_\lambda(x) \leq N(x)$ for all $x \in X$ and $\vee\{L_\mu : \mu < \eta\}$ is fuzzy open in X for each $\lambda < \mu$.

3.24 Theorem Let $\mathbf{K}$ be a finitely additive class of fts. If a hereditarily α-metacompact space X is $\mathbf{K}$-scattered then Player I has a winning strategy in $G'(D\mathbf{K}, X)$.

Proof Since X is fuzzy $\mathbf{K}$-scattered, X has a fuzzy $\mathbf{K}$-scattered partition, say $\mathbf{V} = \{V_\lambda : \lambda < \eta\}$. Now from proposition 3.11 of [8] it follows that there exists a point finite fuzzy open expansion $\mathbf{U} = \{U_\lambda : \lambda < \eta\}$ of $\mathbf{V}$. Now $\mathbf{V}$ is a α-shading of X, it follows that $\mathbf{U}$ is also an α-shading of X. Let X_n and B_n, $n \geq 1$ be taken as in corollary 3.20. For each $F \in \underline{I^X}$, take $k(F) = Min\{k \geq 1 : F \wedge X_k \neq 0\}$ and $\mathbf{B}(F) = \{B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \wedge F : B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) \in \mathbf{B}_k$ and $k = k(F)\}$ and $\mathbf{B}(0) = \{0\}$. Now by corollary 3.20 it follows that each member of $\mathbf{B}(F)$ is fuzzy closed in X and $\mathbf{B}(F)$ is discrete in X.

We have $B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) = \wedge_{i \leq k} U_{\lambda i} \wedge (X_k \backslash X_{k-1})$. Thus $B(\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_k) < \vee_{i \leq k} U_{\lambda i} < \vee_{i \leq k} V_{\lambda i}$. Also since each $\mathbf{B}(F)$ is fuzzy closed and $\mathbf{K}$ is finitely additive.

$$\bigcup \mathbf{B}(F) = \bigcup_{B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) \in B_k} (B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) \wedge F) \text{ where } k = k(F)$$

Also by corollary 3.20, $\mathbf{B}_k$ is a discrete α-shading of $X_k \backslash X_{k-1}$ by closed fuzzy sets. Hence $(X_k \backslash X_{k-1}) \wedge F \in \mathbf{DK} \cap \underline{I^X}$ where $k = k(F)$

Now we define a fuzzy stationary winning strategy S of Player I for $G'(DK, X)$ as follows
$S : \underline{I^X} \to DK \cap \underline{I^X}$, where $S(F) = (X_{k(F)} \backslash X_{k(F)-1}) \wedge F$

Consider the play $(S(X), F_1, S(F_1), F_2, \ldots)$ of $G'(DK, X)$. We have clearly $S(F_n) < Fn$ and hence S is stationary. Now we want to prove S is winning, that is $Inf_{n \geq 1} F_n = 0$. Now since $\{X_n : n \geq 1\}$ is an α-shading of X and $F_n \wedge X_n = 0$ for all $k = 1, 2, \ldots$, it follows that it is enough to prove $F_n \wedge X_n = 0$ for all $n \geq 0$. We will prove this by induction. Let $F_n \wedge X_n = 0$ and assume that $F_n \wedge X_{n+1} \neq 0$. Therefore by definition of $k(Fn)$ we get

$k(Fn) = n + 1$.

$$\text{Now } S(F_n) \wedge F_{n+1} = ((X_{n+1} \backslash X_n) \wedge F_n) \wedge F_{n+1} = (X_{n+1} \backslash X_n) \wedge F_{n+1} = 0$$

Now clearly $X_n \wedge F_n = 0$ and $F_{n+1} < F_n$. Hence $F_n \wedge X_{n+1} = 0$. Therefore it follows that $F_{n+1} \wedge X_{n+1} = 0$. Thus the proof is complete by induction.

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