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ON SOME VECTOR VALUED SEQUENCE SPACE USING ORLICZ FUNCTION

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ABSTRACT. In this paper, we introduce some new sequence space using Orlicz function and study some properties of this space.

1. INTRODUCTION

J.Lindenstrauss and L.Tzafriri [7] used the idea of Orlicz function M (see definition below) to construct the sequence space l_M of all sequences of scalars (x_n) such that

$$\sum_{k=1}^{\infty} M(|x_k|/\rho) < \infty, \text{ for some } \rho > 0.$$

The space l_M (for instance see [7]) becomes a Banach space which is called an Orlicz sequence space.

DEFINITION 1.1. *Let $M\colon [0,\infty) \to [0,\infty)$. Then M is called an Orlicz function if*

i) $M(0) = 0$;
ii) $M(x) > 0$*, for all* x;
iii) M *is continuous, non-decreasing and convex;*
iv) $M(x) \to \infty$ *as* $x \to \infty$.

Obviously Orlicz function generalizes the function

$$M(x) = x^p \quad (\text{where } p \geq 1).$$

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An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x p(t)dt$$

where p known as the kernel of M, is right differential for $t \geq 0, p(0) = 0, p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$, as $t \to \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that

$$M(2u) \leq KM(u) \quad (u \geq 0)$$

The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \leq lKM(u)$ for all values of u and $l > 1$ (Krasnoselkii and Ruticsky [**5**]).

2. Space $F(E_k, M)$

Let E_k be Banach spaces over the field of complex numbers C with norms $\|\cdot\|_{E_k}$, $k = 1, 2, 3, \ldots$, and F be a normal sequence space with monotone norm $\|\cdot\|_F$ and having a Schauder basis e_k where $e_k = (0, 0, \ldots , 1, 0, \ldots)$, with 1 in k-th place. We denote the linear space of all sequences $x = (x_k)$ with $x_k \in E_k$ for each k under the usual coordinatewise operations :

$$\alpha x = (\alpha x_k) \text{ and } x + y = (x_k + y_k)$$

for each $\alpha \in C$ by $S(E_k)$. If $x \in S(E_k)$ and $\lambda = (\lambda_k)$ is a scalar sequence and then we shall write $\lambda x = (\lambda_k x_k)$. Further let M be an Orlicz function. We define

$$\begin{aligned} F(E_k, M) = \{x = (x_k) \in S(E_k)\colon & x_k \in E_k \text{ for each } k \text{ and} \\ & (M(\|x_k\|_{E_k}/\rho)) \in F, \text{ for some } \rho > 0\}. \end{aligned} \tag{1}$$

For $x = (x_k) \in F(E_k, M)$, we define

$$\|x\| = \inf\{\rho > 0\colon \|(M(\|x_k\|_{E_k}/\rho))\|_F \leq 1\}. \tag{2}$$

It is shown that $F(E_k, M)$ turns out to be a complete normed space under the norm defined by (2). Inclusion relations separability, convergence criteria etc. are discussed in the subsequent section of this paper.

It can be seen that for suitable choice of the sequence space F, E_k's and M the space $F(E_k, M)$ includes many of the known scalars as well as vector valued sequence spaces as particular cases.

For example, choosing F to be l_M and $E_k = C$, $k = 1, 2, 3, \ldots$ in $F(E_k, M)$ one gets the scalar valued sequence space l_M, known as Orlicz sequence space defined by Lindenstrauss & Tzafriri [**7**].

If $E_k = X$, a vector space over C , $M(t) = t$ $(t > 0)$ then the class $F(E_k, M)$ gives the class $F(X)$ of X-valued sequences which includes as particular case, the many known sequence spaces introduced by Leonard [**6**], Maddox [**9**] and others.

Thus the generalized sequence space $F(E_k, M)$ unifies several spaces studied by various authors.

3.

In this section we study algebraic and topological properties of the sequence space $F(E_k, M)$.

THEOREM 3.1. *$F(E_k, M)$ is a linear space over the field of complex numbers C.*

PROOF. Let $x = (x_k)$, $y = (y_k) \in F(E_k, M)$ and $\alpha, \beta \in C$. So there exists $\rho_1, \rho_2 > 0$ such that

$$(M(\|x_k\|_{E_k}/\rho_1)) \in F \text{ and } (M(\|y_k\|_{E_k}/\rho_2)) \in F.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex, so

$$\begin{aligned} M(\|\alpha x_k + \beta y_k\|_{E_k}/\rho_3) &\le M(\|\alpha x_k\|_{E_k}/\rho_3 + \|\beta y_k\|_{E_k}/\rho_3) \\ &\le 1/2M(\|x_k\|_{E_k}/\rho_1) + 1/2M(\|y_k\|_{E_k}/\rho_2) \\ &\le M(\|x_k\|_{E_k}/\rho_1 + M(\|y_k\|_{E_k}/\rho_2). \end{aligned}$$

Since F is a normal space, so

$$(M(\|\alpha x_k + \beta y_k\|_{E_k}/\rho_3)) \in F$$

which shows $\alpha x + \beta y \in F(E_k, M)$. Hence $F(E_k, M)$ is a linear space. □

THEOREM 3.2. *$F(E_k, M)$ is a normed space under the norm defined by* (2).

PROOF. Let $x = (x_k)$, $y = (y_k)$ be the elements of $F(E_k, M)$. It is easy to verify that $\|x\| > 0$ and for $x = \theta = (\theta_1, \theta_2, \ldots)$ the null element of $F(E_k, M)$ (where θ_i is the zero element of E_i for each i) we have $\|\theta\| = 0$. Further, from the previous theorem it follows that $x + y \in F(E_k, M)$. To show $\|x + y\| \le \|x\| + \|y\|$, consider

$$\|x + y\| = \inf\{\rho > 0\colon \|(M(\|x_k + y_k\|_{E_k}/\rho))\|_F \le 1\}.$$

There exists $\rho_1 > 0$, $\rho_2 > 0$ such that

$$(M(\|x_k\|_{E_k}/\rho_1)),\ (M(\|y_k\|_{E_k}/\rho_2)) \in F.$$

Let $\rho_3 = \max(2\rho_1, 2\rho_2)$. Since M is non-decreasing and convex, so

$$\begin{aligned} M(\|x_k + y_k\|_{E_k}/\rho_3) &\le 1/2M(\|x_k\|_{E_k}/\rho_1) + 1/2M(\|y_k\|_{E_k}/\rho_2\|) \\ &\le M(\|x_k\|_{E_k}/\rho_1) + M(\|y_k\|_{E_k}/\rho_2). \end{aligned}$$

From the above inequality we can conclude that $||x+y|| \leq ||x|| + ||y||$. To show $||\lambda x|| = |\lambda| ||x||$ for $\lambda \in C$, consider

$$\begin{aligned} ||\lambda x|| &= \inf\{\rho > 0 \colon ||(M(||\lambda x_k||_{E_k}/\rho))||_F \leq 1\} \\ &= \inf\{|\lambda|\rho/|\lambda| > 0 \colon ||(M(||x_k||_{E_k}/\rho/|\lambda|))||_F \leq 1\} \\ &= |\lambda| ||x||. \end{aligned}$$

Now it is left to prove that $||x|| = 0$ implies $x = \theta$. Suppose this is not true i.e. suppose $||x|| = 0$ but $x \neq \theta$. So from the given assumption that $||x|| = 0$ we have

$$\inf\{\rho > 0 \colon ||(M(||x_k||_{E_k}/\rho))||_F \leq 1\} = 0.$$

This implies that there exists $0 < \rho_\varepsilon < \varepsilon$ such that

$$||(M(||x_k||_{E_k}/\rho_\varepsilon))||_F \leq 1.$$

Since (e_k) is a Schauder basis for F and F is normal space so we have

$$M(||x_k||_{E_k}/\rho_\varepsilon)||e_k||_F \leq ||(M(||x_k||_{E_k}/\rho_\varepsilon))||_F \leq 1. \tag{3}$$

Suppose $x_{k_m} \neq \theta_{k_m}$ for some m. Letting $\varepsilon \to 0$, then

$$M(||x_{k_m}||_{E_{k_m}}/\varepsilon)||e_{k_m}||_F \to \infty,$$

which is contradiction to (3). Therefore $x_{k_m} = \theta_{k_m}$ for each m. So $x = \theta$. This completes the proof. □

THEOREM 3.3. *$F(E_k, M)$ is complete normed space under the norm given by* (2).

PROOF. It is sufficient to prove that every Cauchy sequence

$$(x^i) = ((x^i_k)) \text{ in } F(E_k, M)$$

is convergent. Let (x_i) be any Cauchy sequence in $F(E_k, M)$. Using the definition of norm (2) we get

$$||(M(||x^i_k - x^j_k||_{E_k}/||x^i - x^j||))||_F \leq 1.$$

Since F is a normal space and (e_k) is a Schauder basis of F, it follows that

$$M(||x^i_k - x^j_k||_{E_k}/||x^i - x^j||)||e_k||_F \leq ||(M(||x^i_k - x^j_k||_{E_k}/||x^i - x^j||))||_F \leq 1$$

We choose γ with $\gamma||e_k||_F > 1$ and $x_0 > 0$, such that

$$\gamma||e_k||_F(x_0/2)p(x_0/2) \geq 1,$$

where p is the kernel associated with M. Hence,

$$M(||x^i_k - x^j_k||_{E_k}/||x^i - x^j||)||e_k||_F \leq \gamma||e_k||_F(x_0/2)p(x_0/2).$$

Using the integral representation of Orlicz function M, we get

$$||x^i_k - x^j_k||_{E_k} \leq \gamma x_0||x^i - x^j||. \tag{4}$$

For given $\varepsilon > 0$, we choose an integer i_0 such that

$$(5) \qquad \|x^i - x^j\| < \varepsilon/\gamma x_0, \text{ for all } i, j > i_0.$$

From (4) & (5) we get

$$\|x_k^i - x_k^j\|_{E_k} < \varepsilon \text{ for all } i, j > i_0.$$

So, there exists a sequence $x = (x_k)$ such that $x_k \in E_k$ for each k and

$$\|x_k^i - x_k\|_{E_k} \to 0, \text{ as } i \to \infty,$$

for each fixed k. For given $\varepsilon > 0$, we choose an integer $n > 1$, such that $\|x^i - x^j\| < \varepsilon/2$, for all $i, j > n$ and a $\rho > 0$, such that $\|x^i - x^j\| < \rho < \varepsilon/2$. Since, F is a normal space and (e_k) is Schauder basis of F, so

$$(6) \qquad \|\sum_{k=1}^{n} M(\|x_k^i - x_k^j\|_{E_k}/\rho)e_k\|_F \leq \|(M(\|x_k^i - x_k^j\|_{E_k}/\rho))\|_F \leq 1.$$

Since M is continuous, so by taking $j \to \infty$ and $i, j > n$ in (6) we get $\|\sum_{k=1}^{n} M(\|x_k^i - x_k\|_{E_k}/2\rho)e_k\|_F < 1$. Letting $n \to \infty$, we get $\|x^i - x\| < 2\rho < \varepsilon$, for all $i > n$. So (x^i) converges to x in the norm of $F(E_k, M)$. Now we show that $x \in F(E_k, M)$. Since, $x^i = (x_k^i) \in F(E_k, M)$, so there exists a $\rho > 0$ such that $(M(\|x_k^i\|_{E_k}/\rho)) \in F$. Since $\|x_k^i - x_k\|_{E_k} \to 0$ as $i \to \infty$, for each fixed k so we can choose a positive number $\delta_k^i, 0 < \delta_k^i < 1$, such that

$$M(\|x_k^i - x_k\|_{E_k}/\rho) < \delta_k^i M(\|x_k^i\|_{E_k}/\rho).$$

Now consider

$$\begin{aligned} M(\|x_k\|_{E_k}/2\rho) &= M(\|x_k^i + x_k - x_k^i\|_{E_k}/2\rho) \\ &\leq 1/2M(\|x_k^i - x_k\|_{E_k}/\rho) + 1/2M(\|x_k^i\|_{E_k}/\rho) \end{aligned}$$

(because M is convex)

$$< (1/2\delta_k^i + 1/2)M(\|x_k^i\|_{E_k}/\rho).$$

But F is normal so $(M(\|x_k\|_{E_k}/2\rho)) \in F$. Hence $x = (x_k) \in F(E_k, M)$. This completes the proof. □

THEOREM 3.4. *$F(E_k) \subset F(E_k, M)$, if M satisfies the Δ_2-condition where $F(E_k) = \{(x_k)\colon x_k \in E_k,\ \forall k,$ and $(\|x_k\|_{E_k}/\rho) \in F,$ for some $\rho > 0\}$.*

PROOF. Let $x = (x_k) \in F(E_k)$. So for some $\rho > 0$; $(\|x_k\|_{E_k}/\rho) \in F$. We define the two sequences $y = (y_k)$ and $z = (z_k)$ such that

$$\|y_k\|_{E_k} = \begin{cases} \|x_k\|_{E_k}/\rho & \text{if } \|x_k\|_{E_k}/\rho > 1; \\ 0 & \text{if } \|x_k\|_{E_k}/\rho \leq 1; \end{cases}$$

and

$$\|z_k\|_{E_k} = \begin{cases} 0 & \text{if } \|x_k\|_{E_k}/\rho > 1; \\ \|x_k\|_{E_k}/\rho & \text{if } \|x_k\|_{E_k}/\rho \leq 1. \end{cases}$$

Hence $\|x_k\|_{E_k}/\rho = \|y_k\|_{E_k} + \|z_k\|_{E_k}$. Obviously $y = (y_k),\ z = (z_k) \in F(E_k)$. Now

$$\begin{aligned} M(\|x_k\|_{E_k}/\rho) &= M(\|y_k\|_{E_k} + \|z_k\|_{E_k}) \\ &\leq 1/2M(2\|y_k\|_{E_k}) + 1/2M(2\|z_k\|_{E_k}) \\ &< 1/2K_1\|y_k\|_{E_k}M(2) + 1/2\|z_k\|_{E_k}M(2), \end{aligned}$$

where K_1 is a constant. Since, F is a normal space, so

$$(M(\|x_k\|_{E_k}/\rho)) \in F \text{ i.e. } x = (x_k) \in F(E_k, M).$$

Hence $F(E_k) \subset F(E_k, M)$. □

4. Some inclusion relation is derived for the space of multiplier of $F(E_k, M)$

Suppose E_k are normed algebras. Define: $V = \Pi E_k$ product of Banach spaces E_k, and $S[F(E_k, M)]$, the space of multiplier of $F(E_k, M)$

$$\begin{aligned} S[F(E_k, M)] = \{a = (a_k) \in V : (M(\|a_k x_k\|_{E_k}/\rho)) \in F, \\ \text{for all } x = (x_k) \in F(E_k, M)\}, \end{aligned}$$

$$l_\infty(E_k) = \{x = (x_k \in V : \sup_k \|x_k\|_{E_k} < \infty\}.$$

THEOREM 4.1. *$l_\infty(E_k) \subseteq S[F(E_k, M)]$ if M satisfies the Δ_2-condition.*

PROOF. Let $a = (a_k) \in l_\infty(E_k)$, $H = \sup_{k\geq 1} \|a_k\|_{E_k}$ and $x = (x_k) \in F(E_k, M)$. Then for some $\rho > 0$, $(M(\|x_k\|_{E_k}/\rho)) \in F$ and

$$\begin{aligned} M(\|a_k x_k\|_{E_k}/\rho) &\leq M(\|a_k\|_{E_k}\|x_k\|_{E_k}/\rho) \\ &\leq M[(1 + [H])\|x_k\|_{E_k}/\rho] \end{aligned}$$

(where $[H]$ denotes the integer part of H)

$$\leq K_1(1 + [H])M(\|x_k\|_{E_k}/\rho),$$

(where K_1 is a constant). Since F is normal space, so

$$(M(\|a_k x_k\|_{E_k}/\rho)) \in F \text{ i.e. } ax = (a_k x_k) \in F(E_k, M).$$

Hence $l_\infty(E_k) \subseteq S[F(E_k, M)]$

5. This section deals with some properties of a space $[F(E_k, M)]$ introduced here as a subspace of $F(E_k, M)$

We define

$$[F(E_k, M)] = \{x = (x_k) \colon x_k \in E_k \text{ for each } k \text{ and for every } \rho > 0, (M(\|x_k\|_{E_k}/\rho)) \in F\}.$$

F is the same as in section 2 and the topology of $[F(E_k, M)]$ is introduced by the norm of $F(E_k, M)$ given by (2).

THEOREM 5.1. *$[F(E, M)]$ is a complete normed space under the norm given by* (2).

PROOF. Since $[F(E_k, M)]$ is already shown as a complete normed space under the norm (2) and $[F(E_k, M)]$ is a subspace of $F(E_k, M)$, so to show that $[F(E_k, M)]$ is complete under the norm (2), it is sufficient to show that it is closed. For this let us consider $(x^i) = ((x_k^i))$ as sequence in $[F(E_k, M)]$ such that $\|x^i - x\| \to 0 \quad (i \to \infty)$, where $x = (x_k) \in F(E_k, M)$. So for given $\xi > 0$, we can choose and integer i_0 such that

$$\|x^i - x\| < \xi/2, \ \forall i > i_0.$$

Consider

$$\begin{aligned} M(\|x_k\|_{E_k}/\xi) &\leq 1/2M(2\|x_k^i - x_k\|_{E_k}/\xi) + 1/2M(2\|x_k^i\|_{E_k}/\xi) \\ &\leq 1/2M(\|x_k^i - x_k\|_{E_k}/\|x^i - x\|_{E_k}) + 1/2M(2\|x_k^i\|_{E_k}/\xi). \end{aligned}$$

Since

$$(M(\|x_k^i - x_k\|_{E_k}/\|x^i - x\|)), (M(2\|x_k^i\|_{E_k}/\xi)) \in F$$

and F is normal space so $(M(\|x_k\|_{E_k}/\xi)) \in F$. This implies $x = (x_k) \in [F(E_k, M)]$. Hence $[F(E_k, M)]$ is complete. □

PROPOSITION 5.2. *$[F(E_k, M)]$ is an AK space.*

PROOF. Let $x = (x_k) \in [F(E_k, M)]$. Therefore, for every $\rho > 0$, $(M(\|x_k\|_{E_k}/\rho)) \in F$. Since (e_k) is a Schauder basis of F, so for given $\varepsilon (0 < \varepsilon < 1)$, we can find an integer n_0 such that

$$\|\sum_{k \geq n_0}^{\infty} M(\|x_k\|/\varepsilon) e_k\|_F < 1. \tag{7}$$

Using the definition of norm, we have

$$\|x - x^{[n]}\| = \inf\{\xi > 0 \colon \|\sum_{k \geq n+1}^{\infty} M(\|x_k\|_{E_k}/\xi) e_k\|_F \leq 1\}, \tag{8}$$

where $x^{[n]} = n$-th section of x. From (7) and (8), it is obvious that

$$\|x - x^{[n]}\| < \varepsilon, \text{ for all } n > n_0.$$

Therefore $[F(E_k, M)]$ is an AK space. □

THEOREM 5.3. *Let* $(x^i) = ((x_k^i))$ *be a sequence of elements of* $[F(E_k, M)]$ *and* $x = (x_k) \in [F(E_k, M)]$. *Then* $x^i \to x$ *in* $[F(E_k, M)]$ *if and only if*

(i) $x_k^i \to x_k$ *in* E_k *for each* $k \geq 1$;

(ii) $||x^i|| \to ||x||$ *as* $i \to \infty$.

PROOF. Necessary part is obvious.

Sufficient part. Suppose that (i) & (ii) hold and let n be an arbitrary positive integer, then

$$||x^i - x|| \leq ||x^i - x^{i[n]}|| + ||x^{i[n]} - x^{[n]}|| + ||x^{[n]} - x||,$$

where $x^{i[n]}, x^{[n]}$ denote n-th sections of x^i & x respectively. Letting $i \to \infty$, we get

$$\begin{aligned}\limsup_{i\to\infty} ||x^i - x|| &\leq \limsup_{i\to\infty} ||x^i - x^{i[n]}|| + \limsup_{i\to\infty} ||x^{i[n]} - x^{[n]}|| + ||x^{[n]} - x|| \\ &\leq 2||x^{[n]} - x||.\end{aligned}$$

Since n is arbitrary, so taking $n \to \infty$, we get $\limsup_{i\to\infty} ||x^i - x|| = 0$ i.e. $||x^i - x|| \to 0$ as $i \to \infty$. □

THEOREM 5.4. $[F(E_k, M)]$ *is separable if* E_k *is separable for each* k.

PROOF. Suppose E_k is separable for each k. Then, there exists a countable dense subset H_k of E_k. Let Z denote set of finite sequences $z = (z_k)$ where $z_k \in H$ for each k and

$$(z_k) = (z_1, z_2, \ldots, z_n, \theta_{n+1}, \theta_{n+2}, \ldots)$$

for arbitrary integer n. Obviously Z is a countable subset of $[F(E_k, M)]$. We shall prove that Z is dense in $[F(E_k, M)]$. Let $x = (x_k) \in [F(E_k, M)]$. Since $[F(E_k, M)]$ is an AK space, so $||x - x^{[n]}|| \to 0$ as $n \to \infty$, where $x^{[n]} = n$-th section of x. So for given $\varepsilon > 0$, there exists an integer $n_1 > 1$ such that

$$||x - x^{[n]}|| < \varepsilon/2 \text{ for all } n \geq n_1.$$

We take $n = n_1$. Therefore

$$||x - x^{[n_1]}|| < \varepsilon/2.$$

We choose $y = (y_k) = (y_1, \ldots, y_{n_1}, \theta_{n_1+1}, \theta_{n_1+2}, \ldots) \in Z$ such that

$$||x_k^{[n_1]} - y_k||_{E_k} < \varepsilon/(M(1)2n_1||e_k||_F) \text{ for each } k.$$

Now

$$\begin{aligned}||x - y|| &= ||x - x^{[n_1]} + x^{[n_1]} - y|| \\ &\leq ||x - x^{[n_1]}|| + ||x^{[n_1]} - y|| < \varepsilon.\end{aligned}$$

This implies that Z is dense in $[F(E_k, M)]$. Hence $[F(E_k, M)]$ is separable.

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