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ON SYMMETRIC (36,15,6) DESIGNS

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ABSTRACT. Up to isomorphism there are 4 symmetric (36,15,6) designs with automorphisms of order 7 and 38 symmetric (36,15,6) designs with automorphisms of order 5. For those designs full automorphism groups are determined. Also, all symmetric (36,15,6) designs having automorphisms of order 3 acting with 9 and 6 fixed points, or cyclic automorphism groups of order 4 acting standarly are constructed and orders of their full automorphism groups are determined.

1. INTRODUCTION AND PRELIMINARIES

A symmetric (v,k,λ) design is a finite incidence structure $(\mathcal{P},B,I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P}\times\mathcal{B}$, with following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every pair of elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Let $\mathcal{D} = (P,B,I)$ be a symmetric (v,k,λ) design and $G \leq Aut\mathcal{D}$. Group G has the same number of point and block orbits. Let us denote the number of $G-$orbits by t, point orbits by $\mathcal{P}_1,\dots,\mathcal{P}_t$, block orbits by $\mathcal{B}_1,\dots,\mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$. Further, denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with the representative of the block orbit $\mathcal{B}_i$. For those numbers following equalities hold:

$$(1) \qquad \sum_{r=1}^{t} \gamma_{ir} = k,$$

$$(2) \qquad \sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}\cdot(k-\lambda).$$

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DEFINITION 1. *The $(t \times t)$-matrix (γ_{ir}) with entries satisfying properties* (1) *and* (2) *is called the orbit structure for parameters (v, k, λ) and orbit distribution $(\omega_1, \dots, \omega_t)$, $(\Omega_1, \dots, \Omega_t)$.*

The first step of the construction of designs is to find all orbit structures (γ_{ir}) for some parameters and orbit distribution. The next step, called indexing, is to determine for each number γ_{ir} exactly which points from the point orbit $\mathcal{P}_r$ are incident with representative of the the block orbit $\mathcal{B}_i$. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Symmetric $(36,15,6)$ designs are designs of Menon series (see [8]). According to [2], there are 25634 known symmetric $(36,15,6)$ designs. Lot of those designs have trivial automorphism groups (see [1]).

Let ρ be an automorphism of prime order of a symmetric $(36,15,6)$ design. It is easy to prove that then must be $\rho \leq 7$.

2. AUTOMORPHISM GROUPS OF ORDER 5 AND 7

LEMMA 1. *Let ρ be an automorphism of a symmetric* $(36,15,6)$ *design. If $|\rho| = 5$, then $F(\rho) = 1$.*

PROOF. It is known that $F(\rho) < k + \sqrt{n}$ and $F(\rho) \equiv v(mod\ |\rho|)$. Therefore, $F(\rho) \in \{1, 6, 11, 16\}$. For $F(\rho) \in \{6, 11, 16\}$ one can not build orbit structures. □

THEOREM 1. *Up to isomorphism there are* 38 *symmetric* $(36,15,6)$ *designs with automorphisms of order* 5. *Let us denote them by $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{38}$. Among them there are* 20 *self-dual, and* 9 *pairs of dual designs. Full automorphism groups of those designs are: $Aut\mathcal{D}_1 \cong Aut\mathcal{D}_2 \cong Aut\mathcal{D}_7 \cong Aut\mathcal{D}_8 \cong Aut\mathcal{D}_9 \cong Aut\mathcal{D}_{10} \cong Aut\mathcal{D}_{15} \cong Aut\mathcal{D}_{16} \cong Aut\mathcal{D}_{20} \cong Aut\mathcal{D}_{21} \cong Aut\mathcal{D}_{22} \cong Aut\mathcal{D}_{23} \cong Aut\mathcal{D}_{24} \cong Aut\mathcal{D}_{26} \cong Aut\mathcal{D}_{27} \cong Aut\mathcal{D}_{28} \cong Aut\mathcal{D}_{29} \cong Z_5$,*

$Aut\mathcal{D}_3 \cong Aut\mathcal{D}_4 \cong Aut\mathcal{D}_{11} \cong Aut\mathcal{D}_{12} \cong Aut\mathcal{D}_{17} \cong Z_{10}$, $Aut\mathcal{D}_{14} \cong Aut\mathcal{D}_{31} \cong Aut\mathcal{D}_{34} \cong Aut\mathcal{D}_{35} \cong Aut\mathcal{D}_{36} \cong Aut\mathcal{D}_{38} \cong Frob_{10}$, $Aut\mathcal{D}_{32} \cong Frob_{20}$, $Aut\mathcal{D}_{33} \cong Aut\mathcal{D}_{37} \cong Frob_{10} \times Z_3$, $Aut\mathcal{D}_{19} \cong Aut\mathcal{D}_{25} \cong E_{16} : Z_5$, $Aut\mathcal{D}_6 \cong Z_2 \times S_5$, $Aut\mathcal{D}_{30} \cong E_{16} : Frob_{20}$, $Aut\mathcal{D}_5 \cong Aut\mathcal{D}_{13} \cong S_3 \times A_5$, $Aut\mathcal{D}_{18} \cong PSp(4,3) : Z_2$.

PROOF. Solving equations (1) and (2) we got ten orbit structures. Let us denote them by OS1, OS2, ...,OS10. Indexing of those structures led to 38 mutually nonisomorphic designs: orbit structure OS1 led to 5 designs, $\mathcal{D}_1$ to $\mathcal{D}_5$; OS2 led to design $\mathcal{D}_6$; OS3 led to designs $\mathcal{D}_7$ and $\mathcal{D}_8$; OS4 led to $\mathcal{D}_9$ to $\mathcal{D}_{13}$; OS5 led to $\mathcal{D}_{14}$; OS6 to $\mathcal{D}_{15}$ to $\mathcal{D}_{30}$, OS7 to $\mathcal{D}_{31}$; OS8 to $\mathcal{D}_{32}$ and $\mathcal{D}_{33}$; OS9 to $\mathcal{D}_{34}$; finally, OS10 led to designs $\mathcal{D}_{35}$ to $\mathcal{D}_{38}$. Generators and orders of their full automorphism groups are found with the help of the computer program by V. Tonchev. Full automorphism groups are determined with the help of the GAP [5]. The most interesting automorphism group is

$PSp(4,3) : Z_2$ of order 51840. Using computer programs by V. Ćepulić we have found out that those 38 designs are mutually nonisomorphic, and that pairs of mutually dual designs are: $(\mathcal{D}_1, \mathcal{D}_2)$, $(\mathcal{D}_3, \mathcal{D}_4)$, $(\mathcal{D}_7, \mathcal{D}_8)$, $(\mathcal{D}_{14}, \mathcal{D}_{34})$, $(\mathcal{D}_{15}, \mathcal{D}_{22})$, $(\mathcal{D}_{16}, \mathcal{D}_{33})$, $(\mathcal{D}_{24}, \mathcal{D}_{28})$, $(\mathcal{D}_{27}, \mathcal{D}_{29})$, $(\mathcal{D}_{35}, \mathcal{D}_{36})$. □

In the similar way as the Theorem 1, the following theorem has been proved (see [3]):

THEOREM 2. *Up to isomorphism there are four symmetric* $(36,15,6)$ *designs with automorphisms of order* 7. *Let us denote them by* $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ *and* $\mathcal{D}_4$. *Full automorphism groups of those designs are:* $Aut\mathcal{D}_1 \cong Aut\mathcal{D}_2 \cong Frob_{21}$, $Aut\mathcal{D}_3 \cong G(2,2)$, $Aut\mathcal{D}_4 \cong Frob_{21} \times Z_2$. *Those four designs are self-dual.*

3. Automorphism groups of order 3

We have constructed many thousands of symmetric $(36,15,6)$ designs with automorphisms of order 3. Therefore we did not determine the actual number of nonisomorphic designs and their full automorphism groups. We have determine just orders of those groups.

One can not construct orbit structures for symmetric $(36,15,6)$ designs and automorphism groups of order 3 acting with 15 and 12 fixed points.

With the help of the computer program by V. Ćepulić we got two orbit structures for symmetric $(36,15,6)$ designs and automorphism groups of order 3 acting with 9 fixed points, and fourteen orbit structures for automorphism groups of order 3 acting with 6 fixed points. Indexing of those structures led to following results:

THEOREM 3. *Let* $\mathcal{D}$ *be a symmetric* $(36,15,6)$ *design admitting an automorphism of order* 3 *acting with* 9 *fixed points. Then* $|Aut\mathcal{D}| \in \{3,6,9,12,18,27,36,54,72,108,162,243,324,486,648,1944,3888\}$.

THEOREM 4. *Let* $\mathcal{D}$ *be a symmetric* $(36,15,6)$ *design admitting an automorphism of order* 3 *having* 6 *fixed points. Then* $|Aut\mathcal{D}| \in \{3,6,9,12,18,24,30,36,42,48,54,72,81,96,108,144,162,432,648,51840\}$.

We didn't cover the case with 3 fixed points and fixed point free action, because there are too many orbit structures for such actions of automorphisms of order 3.

4. Cyclic automorphism groups of order 4 acting standardly

THEOREM 5. *Up to isomorphism there are* 226 *symmetric* $(36,15,6)$ *designs admitting cyclic automorphism groups of order* 4 *acting standardly. Orders of full automorphism groups of those designs are* 4, 8 *and* 24.

PROOF. Up to isomorphism there are 24 orbit structures for parameters $(36, 15, 6)$ and orbit distribution $(4, 4, 4, 4, 4, 4, 4, 4, 4)$. None of them give rise to designs. Four orbit structures for orbit distribution

$$(1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 4, 4)$$

produce 226 nonisomorphic designs. For other orbit distributions one can not construct orbit structures. □

REMARK 4.1. *Groups Z_6, Z_9 and S_3 can not act standardly as automorphism groups of symmetric* $(36, 15, 6)$ *designs.*

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