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# A FUNCTORIAL DESCRIPTION OF COHERENT MAPPINGS BETWEEN INVERSE SYSTEMS

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In this note an answer to a question posed by S. Mardešić is given. It concers the abstract nature of coherent mappings in the form in which he needed them in the strong shape category he developed [4].

Let $\Lambda$ be a *directed set,* ie, a partially ordered set such that every finite subset has an upper bound.

An *inverse system* $\underline{X} = (X_\lambda, p_{\lambda\lambda'})$ (over $\Lambda$) consists of (topological) spaces $X_\lambda$ and (continuous) maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \leq \lambda' \in \Lambda$, $p_{\lambda\lambda'} = \mathrm{id}$ if $\lambda = \lambda'$ [4].

To the index set $\Lambda$ there is associated a simplicial set $\mathrm{N}\Lambda$, the nerve of $\Lambda$ [1]. The $n$-simplices of $\mathrm{N}\Lambda$ are the weakly increasing sequences

$$\underline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n), \quad \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$$

of elements of $\Lambda$ of length $n+1$. Such an $n$-simplex is turned by an operator $\alpha \in \Delta[n]$ (weakly increasing function $\alpha : [k] \to [n]$) into the $k$-simplex

$$\underline{\lambda}\alpha = (\lambda_{\alpha(0)}, \lambda_{\alpha(1)}, \ldots, \lambda_{\alpha(k)}) .$$

Now, let $\underline{X} = (X_\lambda, p_{\lambda\lambda'})$ be an inverse system over $\Lambda$ and let $\underline{Y} = (Y_\mu, q_{\mu\mu'})$ be an inverse system over the directed set $M$. A *coherent mapping* $\underline{f} = (f, f_{\underline{\mu}}) : \underline{X} \to \underline{Y}$ consists of an increasing function $f : M \to \Lambda$ and of mappings $f_{\underline{\mu}} : X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$, $\underline{\mu} = (\mu_0, \ldots, \mu_n) \in \mathrm{N}M$, such that for every operator $\alpha \in \Delta[n]$, one has

$$f_{\underline{\mu}}(x, \Delta^\alpha(t)) = q_{\mu_0\mu_{\alpha(0)}} f_{\underline{\mu}\alpha}(p_{f(\mu_{\alpha(k)})f(\mu_n)}(x), t). \tag{1}$$

for all $x \in X_{f((\mu_n)}$ and $t \in \Delta^k$ [4].

S. Mardešić asked for a more functorial description of these coherent mappings. The starting point for such a task is clear. A partially ordered set $\Lambda$ can be seen as a small category: The objects are elements of $\Lambda$, the morphisms the ordered pairs $(\lambda, \lambda')$ with $\lambda \leq \lambda'$ and the operations given by

$$\mathrm{dom}(\lambda, \lambda') = \lambda\,, \;\; \mathrm{cod}(\lambda, \lambda') = \lambda'\,, \;\; \mathrm{id}(\lambda) = (\lambda, \lambda)\,, \;\; (\lambda', \lambda'') \circ (\lambda, \lambda') = (\lambda, \lambda'') .$$

The same holds for any directed index set $M$ and a (weakly) increasing function $f : M \to \Lambda$ can be considered as a functor.

Then inverse systems are nothing but contravariant functors from the *index category* to the category of topological spaces, formally

$$\begin{aligned} \underline{X} &: \Lambda^{op} \to \mathrm{T}op\,, \\ \underline{Y} &: M^{op} \to \mathrm{T}op\,. \end{aligned}$$

Now, the representation of a coherent map by a family of maps indicates that a coherent map should be considered as a natural transformation. This becomes still more clear if one writes the condition (1) as a commutative diagram:

$$(2)\qquad \begin{array}{ccc} X_{f(\mu_n)} \times \Delta^n & \xrightarrow{f_{\underline{\mu}}} & Y_{\mu_0} \\ {\scriptstyle \mathrm{id} \times \Delta^\alpha}\uparrow & & \\ X_{f(\mu_n)} \times \Delta^m & & \uparrow {\scriptstyle q_{\mu_0 \mu_{\alpha(0)}}} \\ {\scriptstyle p_{f(\mu_{\alpha(k)})f(\mu_n)} \times \mathrm{id}}\downarrow & & \\ X_{f(\mu_{\alpha(k)})} \times \Delta^m & \xrightarrow[f_{\underline{\mu}\alpha}]{} & Y_{\mu_{\alpha(0)}} \end{array}$$

From the point of view of a natural transformation the index set should be the set of objects of a category. Thus one has to consider the category of simplices $\Gamma \mathrm{N}M$ of the simplicial set $\mathrm{N}M$ [2, p.141]. The objects of this category are the simplices of the simplicial set $\mathrm{N}M$; as morphisms one can take the pairs $(\underline{\mu}, \alpha)$ with $\underline{\mu} \in (\mathrm{N}M)_n$ and $\alpha : [k] \to [n]$ and the operations given by

$$\mathrm{dom}(\underline{\mu}, \alpha) = \underline{\mu}\alpha\,,\ \ \mathrm{cod}(\underline{\mu}, \alpha) = \underline{\mu}\,,\ \ \mathrm{id}(\underline{\mu}) = (\underline{\mu}, \mathrm{id})\,,$$
$$(\underline{\mu}, \alpha) \circ (\underline{\mu}\alpha, \beta) = (\underline{\mu}, \alpha\beta)\,.$$

The categories $M$ and $\Gamma\mathrm{N}M$ are connected by two interesting functors:

$$\begin{aligned} \Phi : \Gamma\mathrm{N}M \to M^{op} \quad &, \quad \underline{\mu} \mapsto \mu_0\,,\ \ (\underline{\mu}, \alpha) \mapsto (\mu_{\alpha(0)}, \mu_0)\,, \\ \Psi : \Gamma\mathrm{N}M \to M \quad &, \quad \underline{\mu} \mapsto \mu_n\,,\ \ (\underline{\mu}, \alpha) \mapsto (\mu_{\alpha(k)}, \mu_n)\,. \end{aligned}$$

Now looking at diagram (2) one could think of the composed functor

$$\underline{Y} \circ \Phi$$

as the target of the natural transformation in question. But unfortunately its source is not so easy to describe. For that an enlargement of the category Top of spaces is necessary. The new category Topd should have the same objects as Top, ie, the spaces, but as morphisms from a space $X'$ to $X$ one would like to take equivalence classes of diagrams $X' \leftarrow X'' \to X$ where the diagrams $X' \leftarrow X'' \to X$ and $X' \leftarrow X''' \to X$ are said to be equivalent if there is a homeomorphism $h : X'' \to X'''$ rendering the generated triangles commutative.

A definition of this sort would cause troubles with respect to the foundations of set theory. The class of all equivalence classes of such diagrams with fixed ends $X'$ and $X$ is a proper class and not a set as required in the definition of a category. There are two ways out of this difficulty; one may look for a suitable universe or – more down to earth – one may work in a small category of spaces, a full subcategory Tops of Top, that contains all spaces $X_\lambda$, $Y_\mu$, $\Delta^n$ and is – as a subcategory of Top – closed under products and subspaces, and therefore also under pullbacks. The latter approach will be adopted in the following. In order to distinguish honest maps $X' \to X$ in Top from the new morphisms in Topd one adopts, for the latter, the notation $X' \overset{\leftarrow}{\longrightarrow} X$.

It is still necessary to describe how morphisms $X' \overset{\leftarrow}{\longrightarrow} X$ and $Z' \overset{\leftarrow}{\longrightarrow} Z$ with $Z' = X$ are composed. To this end one forms the diagram where the right upper corner is constructed as a pullback

$$\begin{array}{ccccc} X' & \leftarrow & X'' & \leftarrow & \bullet \\ & & \downarrow & & \downarrow \\ & & X = Z' & \leftarrow & Z'' \\ & & & & \downarrow \\ & & & & Z \end{array}$$

and takes the equivalence class of the induced diagram $X' \leftarrow \bullet \to Z$. This definition is dual to the composition of correspondences in abelian categories described by P. Hilton in [3].

The category Tops is considered as a subcategory of Topd. An actual map $X' \to X$ is identified with the equivalence class $[X' \overset{\text{id}}{\leftarrow} X' \to X]$ of the diagram $X' \overset{\text{id}}{\leftarrow} X' \to X$. In this sense the composed functor $\tilde{\Phi} = \underline{Y} \circ \Phi$ is considered as a functor $\Gamma \text{N} \Lambda \to \text{T}opd$.

The desired second functor $\tilde{\Psi} : \Gamma \text{N} M \to \text{T}opd$ can now be described as follows:

$$\begin{aligned} \tilde{\Psi}(\underline{\mu}) &= \underline{X} \circ f \circ \Psi(\underline{\mu}) \times \Delta^n = X_{f(\mu_n)} \times \Delta^n\,, \\ \tilde{\Psi}(\underline{\mu}, \alpha) &= [\tilde{\Psi}(\underline{\mu}\alpha) \xleftarrow{\underline{X} \circ f \circ \Psi(\underline{\mu}, \alpha) \times \text{id}} \underline{X} \circ f \circ \Psi(\underline{\mu}) \times \Delta^k \xrightarrow{\text{id} \times \Delta^\alpha} \tilde{\Psi}(\underline{\mu})] = \\ &= [X_{f(\mu_{\alpha(k)})} \times \\ &\qquad \Delta^k \xleftarrow{p_{f(\mu_{\alpha(k)}) f(\mu_n)} \times \text{id}} X_{f(\mu_n)} \times \Delta^k \xrightarrow{\text{id} \times \Delta^\alpha} X_{f(\mu_n)} \times \Delta^n]\,. \end{aligned}$$

Now the work is done. A coherent map is a natural transformation

$$\underline{f} : \tilde{\Psi} \overset{\bullet}{\to} \tilde{\Phi}\,!$$

**Remark.** This all reminds us of the story how the notions of category, functor and natural transformation were invented. The aim was to understand the canonical embedding of a vector space in its double dual, nowadays considered as a *natural transformation.* But transformation from what to what? From one *functor* to another, from the identity functor to the double dual. And finally to understand functors one needed *categories.*

## References

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