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ONE–DIMENSIONAL FLOW OF A COMPRESSIBLE VISCOUS MICROPOLAR FLUID: A GLOBAL EXISTENCE THEOREM

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Abstract. An initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodinamicaly perfect and politropic. A global-in-time existence theorem is proved. The proof is based on a local existence theorem, obtained in the previous paper [4].

1. Statement of the problem and the main result

In this paper we consider an initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid, being in thermodinamical sense perfect and politropic (see [4] and references therein).

Let ρ, v, ω and θ denotes respectively the mass density, velocity, microrotation velocity and temperature in the Lagrangean description. Then the problem that we consider has the formulation as follows:

$$\frac{\partial\rho}{\partial t}+\rho^2\frac{\partial v}{\partial x}=0, \tag{1.1}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\Big(\rho\frac{\partial v}{\partial x}\Big)-K\frac{\partial}{\partial x}(\rho\theta), \tag{1.2}$$

$$\rho\frac{\partial\omega}{\partial t}=A\Big[\rho\frac{\partial}{\partial x}\Big(\rho\frac{\partial\omega}{\partial x}\Big)-\omega\Big], \tag{1.3}$$

$$\rho\frac{\partial\theta}{\partial t}=-K\rho^2\theta\frac{\partial v}{\partial x}+\rho^2\Big(\frac{\partial v}{\partial x}\Big)^2+\rho^2\Big(\frac{\partial\omega}{\partial x}\Big)^2+\omega^2+D\rho\frac{\partial}{\partial x}\Big(\rho\frac{\partial\theta}{\partial x}\Big) \tag{1.4}$$

in $]0,1[\times\mathbf{R}_+$,

$$v(0,t)=v(1,t)=0, \tag{1.5}$$

$$\omega(0,t)=\omega(1,t)=0, \tag{1.6}$$

$$\frac{\partial\theta}{\partial x}(0,t)=\frac{\partial\theta}{\partial x}(1,t)=0, \tag{1.7}$$

for $t\in\mathbf{R}_+$,

$$\rho(x,0)=\rho_0(x), \tag{1.8}$$

$$v(x,0)=v_0(x), \tag{1.9}$$

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$$\omega(x,0) = \omega_0(x), \tag{1.10}$$
$$\theta(x,0) = \theta_0(x) \tag{1.11}$$

for $x \in]0,1[$. Here K, A and D are given positive constants; ρ_0, v_0, ω_0 and θ_0 are given functions, satisfying the conditions

$$\rho_0, \theta_0 > 0 \quad \text{in} \quad [0,1]. \tag{1.12}$$

Let $T \in \mathbf{R}_+$; a generalised solution of the problem (1.1)–(1.11) in the domain $Q_T =]0,1[\times]0,T[$ is a function

$$(x,t) \to (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{1.13}$$

where

$$\rho \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T), \tag{1.14}$$
$$v, \omega, \theta \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T) \cap L^2(0,T;H^2(]0,1[)), \tag{1.15}$$

that satisfies the equations (1.1)–(1.4) a.e. in Q_T, the conditions (1.5)-(1.11) in the sense of traces and the conditions

$$\inf_{Q_T} \rho > 0. \tag{1.16}$$

From embedding and interpolation theorems ([3]) one can conclude that from (1.14) and (1.15) it follows:

$$\rho \in C([0,T], L^2(]0,1[)) \cap L^\infty(0,T;C([0,1])), \tag{1.17}$$
$$v, \omega, \theta \in L^2(0,T;C^{(1)}([0,1])) \cap C([0,T], H^1(]0,1[)), \tag{1.18}$$
$$v, \omega, \theta \in C(\overline{Q}_T). \tag{1.19}$$

Specially, the condition (1.16) has a sense.

Assuming the conditions

$$\rho_0, \theta_0 \in H^1(]0,1[),\ v_0, \omega_0 \in H_0^1(]0,1[) \tag{1.20}$$

and the inequalities (1.12), in the previous paper [4] we proved a uniqueness of a generalised solution and the following local existence theorem: there exists $T_0 \in \mathbf{R}_+$, such that in the domain $Q_{T_0} =]0,1[\times]0,T_0[$ there exists a generalised solution, having the property

$$\theta > 0 \text{ in } \overline{Q}_{T_0}. \tag{1.21}$$

With the use of that theorem, in this paper we shall prove the following result.

THEOREM 1.1. *Let the conditions (1.20) and (1.12) be fulfiled. Then for each $T \in \mathbf{R}_+$, in the domain Q_T there exists a generalised solution (1.13) of the problem (1.1)–(1.11), having the property*

$$\theta > 0 \text{ in } \overline{Q}_T.$$

In our proof we apply the method of the book [1], where the Theorem 1.1 was proved for the classical fluid ($\omega = 0$); for this case see also [2].

2. The proof of Theorem 1.1

Because of the local existence result, Theorem 1.1 is an immediate consequence of the following statement.

PROPOSITION 2.1. *Let $T \in \mathbf{R}_+$ and let a function*

$$(x, t) \to (\rho, v, \omega, \theta)(x, t), \quad (x, t) \in Q_T \tag{2.1}$$

satisfies the condition:

for each $T' \in]0, T[$, (2.1) is a generalised solution of the problem (1.1)–(1.11) in the domain $Q_{T'} =]0, 1[\times]0, T'[$ and the inequality $\theta > 0$ in $\overline{Q}_{T'}$ holds true.

Then (2.1) is a generalised solution of the same problem in the domain Q_T and inequality $\theta > 0$ in $\overline{Q}_T$ holds true.

The above statement is a consequence of results below. In that what follows we assume that the function (2.1) satisfies the condition of the Proposition 2.1. By $C \in \mathbf{R}_+$ we denote a generic constant, having possibly different values at different places; we also use the notation $\|f\| = \|f\|_{L^2(]0,1[)}$. Because of the fact that equations (1.2) and (1.3) don't contain the function ω, some of our considerations are identical to that of classical fluid. In these cases we omit proofs or details of proofs, making reference to correspondent pages of the book [1].

LEMMA 2.1. *It holds*

$$v, \omega \in L^\infty(0, T; L^2(]0, 1[)), \tag{2.2}$$

$$\theta \in L^\infty(0, T; L^1(]0, 1[)), \tag{2.3}$$

Proof. Multiplying the equations (1.2), (1.3) and (1.4) respectively by $v, A^{-1}\rho^{-1}\omega$ and ρ^{-1}, integrating over $]0, 1[$ and making use of (1.5)–(1.7), after addition of the obtained equalities we find that

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta\right)dx = 0 \text{ on }]0, T[. \tag{2.4}$$

Integrating over $]0,t[$, $t \in]0, T[$ and making use of (1.9)–(1.11), we obtain

$$\int_0^1 \left(\frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta\right)dx = \frac{1}{2}\|v_0\|^2 + \frac{1}{2A}\|\omega_0\|^2 + \|\theta_0\|_{L^1(]0,1[)} \text{ on }]0, T[, \tag{2.5}$$

or

$$\|v\|^2 + \|\omega\|^2 + \|\theta\|_{L^1(]0,1[)} \leqslant C \text{ on }]0, T[. \tag{2.6}$$

From (2.6) there follow the statements (2.2) and (2.3). □

LEMA 2.2. ([1], pp. 47–48, 50–52). *Let* $t \in]0,T[$ *and*

$$M_\theta(t) = \max_{[0,1]} \theta(\cdot, t), \tag{2.7}$$

$$m_\rho(t) = \min_{[0,1]} \rho(\cdot, t), \tag{2.8}$$

$$I_1(t) = \int_0^1 \rho(x,t)\Big(\frac{\partial\theta}{\partial x}(x,t)\Big)^2 dx, \tag{2.9}$$

$$I_2(t) = \int_0^t I_1(\tau)d\tau. \tag{2.10}$$

Then there exist $C \in \mathbf{R}_+$ *and (for each* $\varepsilon > 0$*)* $C_\varepsilon \in \mathbf{R}^+$, *such that for each* $t \in]0,T[$ *the inequalities*

$$M_\theta^2(t) \leqslant \varepsilon I_1(t) + C_\varepsilon(1 + I_2(t)), \tag{2.11}$$

$$m_\rho(t) \geqslant C\Big(I + \int_0^t M_\theta(\tau)d\tau\Big)^{-1} \tag{2.12}$$

hold true.

LEMA 2.3. *It holds*

$$\inf_{Q_T} \theta > 0, \tag{2.13}$$

$$\rho \in L^\infty(Q_T). \tag{2.14}$$

Proof. Let $W = \theta^{-1}$ and $p > 1$. Multiplying the equation (1.4) by $2p\rho^{-1}W^{2p+1}$ and integrating over $]0,1[$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 W^{2p}dx = &\int_0^1 \Big[2DpW^{2p-1}\frac{\partial}{\partial x}\Big(\rho\frac{\partial W}{\partial x}\Big) - 2p\Big(2D\rho\theta\Big(\frac{\partial W}{\partial x}\Big)^2 + \rho W^2\Big(\frac{\partial v}{\partial x} - \frac{K\theta}{2}\Big)^2 \\ &+ \frac{\omega^2}{\rho}W^2 + \rho W^2\Big(\frac{\partial\omega}{\partial x}\Big)^2\Big)W^{2p-1} + \frac{K^2p}{2}\rho W^{2p-1}\Big]dx \\ \leqslant &\int_0^1 \Big[2DpW^{2p-1}\frac{\partial}{\partial x}\Big(\rho\frac{\partial W}{\partial x}\Big) + \frac{K^2p}{2}\rho W^{2p-1}\Big]dx \text{ on }]0,T[.\end{aligned} \tag{2.15}$$

Integrating the first term on right-hand side by parts and making use of (1.7), we find that

$$\frac{d}{dt}\int_0^1 W^{2p}dx \leqslant \int_0^1 \Big[-2Dp(2p-1)W^{2p-2}\Big(\frac{\partial W}{\partial x}\Big)^2 + \frac{K^2p}{2}\rho W^{2p-1}\Big]dx, \tag{2.16}$$

or

$$\frac{d}{dt}\int_0^1 W^{2p}dx \leqslant \frac{pK^2}{2}\int_0^1 \rho W^{2p-1}dx \text{ on }]0,T[. \tag{2.17}$$

The conclusions (2.13) and (2.14) follow now from (2.17) as in the case of classical fluid ([1], pp. 48–50). □

LEMMA 2.4. *It holds*

$$M_\theta \in L^2(]0,T[), \tag{2.18}$$

$$\inf_{Q_T} \theta > 0, \tag{2.19}$$

$$\theta \in L^\infty(0,T;L^2(]0,1[)) \cap L^2(0,T;H^1(]0,1[)). \tag{2.20}$$

Proof. Let

$$\Phi = \frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta. \tag{2.21}$$

Multiplying the equations (1.2), (1.3) and (1.4) respectively by $v\Phi$, $A^{-1}\rho^{-1}\omega\Phi$ and $\rho^{-1}\Phi$, integrating over $]0,1[$ and making use of (1.5)–(1.7), after addition of the obtained equations, we find that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2 dx + \int_0^1 \rho\left(\frac{\partial\Phi}{\partial x}\right)^2 dx + (D-1)\int_0^1 \rho\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}dx$$
$$+\left(1-\frac{1}{A}\right)\int_0^1 \rho\omega\frac{\partial\omega}{\partial x}\frac{\partial\Phi}{\partial x}dx - K\int_0^1 \rho\theta v\frac{\partial\Phi}{\partial x}dx = 0 \text{ on }]0,T[, \tag{2.22}$$

or

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2 dx + \int_0^1 \rho\left(\frac{\partial\Phi}{\partial x}\right)^2 dx + (D-1)\int_0^1 \rho\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}dx$$
$$\leqslant L\int_0^1 \rho\left|\omega\frac{\partial\omega}{\partial x}\frac{\partial\Phi}{\partial x}\right|dx + K\int_0^1 \rho\theta\left|v\frac{\partial\Phi}{\partial x}\right|dx \text{ on }]0,T[, \tag{2.23}$$

where $L = |1 - A^{-1}|$. Applying on the right-hand side the Young inequality with a parameter $\delta > 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2 dx + \int_0^1 \rho\left[(1-2\delta)\left(\frac{\partial\Phi}{\partial x}\right)^2 + (D-1)\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}\right]dx$$
$$\leqslant C\delta^{-1}\int_0^1 \rho\left[\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2 + \theta^2 v^2\right]dx \text{ on }]0,T[. \tag{2.24}$$

One can easily see that the following inequality holds true

$$(1-2\delta)\Big(\frac{\partial\Phi}{\partial x}\Big)^2+(D-1)\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x} \geqslant (D-6\delta)\Big(\frac{\partial\theta}{\partial x}\Big)^2-\Big(4\delta+\frac{(1-4\delta+D)^2}{8\delta}\Big)v^2\Big(\frac{\partial v}{\partial x}\Big)^2$$
$$-\frac{1}{4\delta}\Big((1-2\delta)^2+\frac{1}{2}(1-4\delta+D)^2\Big)\frac{\omega^2}{A^2}\Big(\frac{\partial\omega}{\partial x}\Big)^2. \quad (2.25)$$

Let $\delta = 24^{-1}\min\{1, D\}$. From (2.24) and (2.25) it follows the inequality

$$\frac{d}{dt}\int_0^1 \Phi^2 dx + \frac{3D}{2}\int_0^1 \rho\Big(\frac{\partial\theta}{\partial x}\Big)^2 dx$$
$$\leqslant C_1\int_0^1 \rho\Big[v^2\Big(\frac{\partial v}{\partial x}\Big)^2+\omega^2\Big(\frac{\partial\omega}{\partial x}\Big)^2+\theta^2v^2\Big]dx \text{ on }]0,T[, \quad (2.26)$$

where

$$C_1 = 2\max\{4\delta+\frac{(1-4\delta+D)^2}{8\delta}, \frac{C}{\delta}+\frac{2(1-2\delta)^2+(1-4\delta+D)^2}{8\delta}, \frac{C}{\delta}\}.$$

Multiplying (1.2) and (1.3) respectively by v^3 and $\rho^{-1}\omega^3$, integrating over $]0,1[$ and making use of (1.5) and (1.6), after applying the Young inequality we obtain the inequalities

$$\frac{d}{dt}\int_0^1 v^4 dx + \int_0^1 \rho v^2\Big(\frac{\partial v}{\partial x}\Big)^2 dx \leqslant 6K^2\int_0^1 \rho\theta^2v^2 dx \text{ on }]0,T[, \quad (2.27)$$

$$\frac{d}{dt}\int_0^1 \omega^4 dx + A\int_0^1 \rho\omega^2\Big(\frac{\partial\omega}{\partial x}\Big)^2 dx \leqslant 0 \text{ on }]0,T[. \quad (2.28)$$

Multiplying (2.27) by C_1 and (2.28) by $C_2 = A^{-1}C_1$, after addition of the obtained inequalities with (2.26), we find that

$$\frac{d}{dt}\int_0^1\Big(\Phi^2+C_1v^4+C_2\omega^4\Big)dx+D\int_0^1\rho\Big(\frac{\partial\theta}{\partial x}\Big)^2 dx \leqslant C\int_0^1 \rho\theta^2v^2\,dx \text{ on }]0,T[\quad (2.29)$$

or, taking into account (2.2), (2.14) and (2.11),

$$\frac{d}{dt}\Big(\int_0^1(\Phi^2+C_1v^4+C_2\omega^4)dx+DI_2\Big) \leqslant C(1+DI_2)$$
$$\leqslant C\Big(1+\int_0^1(\Phi^2+C_1v^4+C_2\omega^4)dx+DI_2\Big) \text{ on }]0,T[. \quad (2.30)$$

From (2.30) it follows the inequality

$$\int_0^1(\Phi^2+C_1v^4+C_2\omega^4)dx+DI_2 \leqslant C \text{ on }]0,T[\quad (2.31)$$

and therefore it holds

$$I_2 \in L^\infty(]0,T[), \tag{2.32}$$

$$\Phi \in L^\infty\big(0,T;L^2(]0,1[)\big). \tag{2.33}$$

From (2.32) and (2.11) we conclude that (2.18) holds true. The inequality (2.19) follows now from (2.18) and (2.12); the inclusion (2.20) follows from (2.33), (2.19) and (2.32). □

LEMMA 2.5. ([1], pp. 53–54) *It holds*

$$\rho \in L^\infty\big(0,T;H^1(]0,1[)\big) \cap H^1(Q_T). \tag{2.34}$$

LEMMA 2.6. ([1], pp. 53–54) *It holds*

$$v \in L^\infty\big(0,T;H^1(]0,1[)\big) \cap H^1(Q_T) \cap L^2\big(0,T;H^2(]0,1[)\big). \tag{2.35}$$

LEMMA 2.7. *It holds*

$$\omega \in L^\infty\big(0,T;H^1(]0,1[)\big) \cap H^1(Q_T) \cap L^2\big(0,T;H^2(]0,1[)\big). \tag{2.36}$$

Proof. Multiplying the equation (1.3) by $\rho^{-1}\omega$, integrating over $]0,1[$ and making use of (1.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + A\int_0^1 \left[\rho\left(\frac{\partial\omega}{\partial x}\right)^2 + \frac{\omega^2}{\rho}\right]dx = 0 \quad \text{on} \quad]0,T[, \tag{2.37}$$

or

$$\frac{1}{2}\|\omega(.,t)\|^2 + A\int_0^t d\tau \int_0^1 \left[\rho\left(\frac{\partial\omega}{\partial x}\right)^2 + \frac{\omega^2}{\rho}\right](x,\tau)dx$$

$$= \frac{1}{2}\int_0^1 \omega_0^2(x)dx \leqslant C, \quad t \in]0,T[. \tag{2.38}$$

Using (2.19), we conclude that

$$\omega \in L^2\big(0,T;H^1(]0,1[)\big). \tag{2.39}$$

Multiplying (1.3) by $A^{-1}\rho^{-1}\dfrac{\partial^2\omega}{\partial x^2}$ and integrating over $]0,1[$, after integration by parts on the left-hand side and making use of (1.6), we find that

$$\frac{1}{2A}\frac{d}{dt}\left\|\frac{\partial\omega}{\partial x}\right\|^2 + \int_0^1 \rho\left(\frac{\partial^2\omega}{\partial x^2}\right)^2 dx = \int_0^1 \left(\frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2} - \frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}\right)dx \quad \text{on} \quad]0,T[. \tag{2.40}$$

In that what follows we use the inequalities

$$|f|^2 \leqslant 2\|f\| \left\|\frac{\partial f}{\partial x}\right\|, \quad \left|\frac{\partial f}{\partial x}\right| \leqslant 2\left\|\frac{\partial f}{\partial x}\right\| \left\|\frac{\partial^f}{\partial x^2}\right\|, \tag{2.41}$$

valid for a function f vanishing at $x = 0$ and $x = 1$ or having derivatives that vanish at the same points.

With the help of (2.19) and (2.41) and using the Young inequality with a parameter $\delta > 0$, for the terms on the right-hand side of (2.40) we find estimates on $]0, T[$ as follows:

$$\left|\int_0^1 \frac{\omega}{\rho} \frac{\partial^2 \omega}{\partial x^2} dx\right| \leqslant \delta \left\|\frac{\partial^2 \omega}{\partial x^2}\right\|^2 + C\|\omega\|^2, \tag{2.42}$$

$$\left|\int_0^1 \frac{\partial \rho}{\partial x} \frac{\partial \omega}{\partial x} \frac{\partial^2 \omega}{\partial x^2} dx\right| \leqslant 2\left\|\frac{\partial \omega}{\partial x}\right\|^{\frac{1}{2}} \left\|\frac{\partial^2 \omega}{\partial x^2}\right\|^{\frac{1}{2}} \int_0^1 \left|\frac{\partial^2 \omega}{\partial x^2} \frac{\partial \rho}{\partial x}\right| dx$$

$$\leqslant 2\left\|\frac{\partial \omega}{\partial x}\right\|^{\frac{1}{2}} \left\|\frac{\partial^2 \omega}{\partial x^2}\right\|^{\frac{3}{2}} \left\|\frac{\partial \rho}{\partial x}\right\| \leqslant \delta \left\|\frac{\partial^2 \omega}{\partial x^2}\right\| + C\left\|\frac{\partial^2 \omega}{\partial x^2}\right\|^2 \left\|\frac{\partial \rho}{\partial x}\right\|^4. \tag{2.43}$$

Using again (2.19), from (2.40), (2.42) and (2.43) we obtain (making use of (1.10))

$$\left\|\frac{\partial \omega}{\partial x}(.,t)\right\|^2 + \int_0^t \left\|\frac{\partial^2 \omega}{\partial x^2}(.,\tau)\right\|^2 d\tau \leqslant \|\omega_0'\|^2 + C\int_0^t \left(\|\omega\|^2 + \left\|\frac{\partial \omega}{\partial x}\right\|^2 \left\|\frac{\partial \rho}{\partial x}\right\|^4\right) d\tau$$

$$\leqslant C\left(1 + \int_0^t \left(\|\omega\|^2 + \left\|\frac{\partial \omega}{\partial x}\right\|^2 \left\|\frac{\partial \rho}{\partial x}\right\|^4\right) d\tau\right), \quad t \in]0, T[. \tag{2.44}$$

With the help of (2.34) and (2.39), from (2.44) we find that

$$\left\|\frac{\partial \omega}{\partial x}(.,t)\right\|^2 + \int_0^t \left\|\frac{\partial^2 \omega}{\partial x^2}(.,\tau)\right\|^2 d\tau \leqslant C, \quad t \in]0, T[. \tag{2.45}$$

Using (2.14) and (2.19), from (1.3) we obtain

$$\left\|\frac{\partial \omega}{\partial t}\right\|^2 \leqslant C\left(\|\omega\|^2 + \left\|\frac{\partial \omega}{\partial x}\right\|^2 + \left\|\frac{\partial^2 \omega}{\partial x^2}\right\|^2\right) \quad \text{on }]0, T[. \tag{2.46}$$

and, because of (2.39) and (2.45),

$$\int_0^t \left\|\frac{\partial \omega}{\partial t}(.,\tau)\right\|^2 d\tau \leqslant C, \quad t \in]0, T[. \tag{2.47}$$

The conclusion (2.36) follows from (2.45) and (2.47). □

LEMMA 2.8. *It holds*

$$\theta \in L^\infty\big(0, T; H^1(]0,1[)\big) \cap H^1(Q_T) \cap L^2(0,T;H^2(]0,1[)). \tag{2.48}$$

Proof. Multiplying (1.4) by $\rho^{-1}\dfrac{\partial^2\theta}{\partial x^2}$ and integrating over $]0,1[$, after integration by parts on the left-hand side and making use of (1.7), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial\theta}{\partial x}\Big\|^2 + D\int_0^1 \rho\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dx = K\int_0^1 \rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx - \int_0^1 \rho\Big(\frac{\partial v}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx$$

$$-\int_0^1 \rho\Big(\frac{\partial \omega}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx - \int_0^1 \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx - D\int_0^1 \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx \quad \text{on }]0,T[. \tag{2.49}$$

With the help of (2.14), (2.35), (2.41) and (2.36) and using the Young inequality with a parametar $\delta > 0$, for the terms on the right-hand side of (2.49) we find estimates on $]0,T[$ as follows:

$$\Big|\int_0^1 \rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant CM_\theta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + CM_\theta^2, \tag{2.50}$$

$$\Big|\int_0^1 \rho\Big(\frac{\partial v}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant C\Big\|\frac{\partial v}{\partial x}\Big\|^{\frac{3}{2}}\Big\|\frac{\partial^2 v}{\partial x^2}\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big\|\frac{\partial^2 v}{\partial x^2}\Big\|$$

$$\leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big(1+\Big\|\frac{\partial^2 v}{\partial x^2}\Big\|^2\Big), \tag{2.51}$$

$$\Big|\int_0^1 \rho\Big(\frac{\partial \omega}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant C\Big\|\frac{\partial \omega}{\partial x}\Big\|^{\frac{3}{2}}\Big\|\frac{\partial^2 \omega}{\partial x^2}\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big\|\frac{\partial^2 \omega}{\partial x^2}\Big\|$$

$$\leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big(1+\Big\|\frac{\partial^2 \omega}{\partial x^2}\Big\|^2\Big), \tag{2.52}$$

$$\Big|\int_0^1 \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant C\|\omega\|\Big\|\frac{\partial\omega}{\partial x}\Big\|\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C, \tag{2.53}$$

$$\Big|\int_0^1 \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant 2\Big\|\frac{\partial\theta}{\partial x}\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^{\frac{3}{2}}\Big\|\frac{\partial\rho}{\partial x}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big\|\frac{\partial\theta}{\partial x}\Big\|^2. \tag{2.54}$$

Using again (2.19), from (2.49)–(2.54) (making use of (1.11)) we obtain

$$\Big\|\frac{\partial\theta}{\partial x}(.,t)\Big\|^2 + \int_0^t\Big\|\frac{\partial^2\theta}{\partial x^2}(.,\tau)\Big\|^2 d\tau \leqslant \|\theta_0'\|^2 + C\Big(1+\int_0^t\Big(M_\theta^2(\tau)$$

$$+\Big\|\frac{\partial^2 v}{\partial x^2}(.,\tau)\Big\|^2 + \Big\|\frac{\partial^2\omega}{\partial x^2}(.,\tau)\Big\|^2 + \Big\|\frac{\partial\theta}{\partial x}(.,\tau)\Big\|^2\Big)d\tau\Big). \tag{2.55}$$

With the help of (2.18), (2.35), (2.36) and (2.20), from (2.55) we find that

$$\left\|\frac{\partial\theta}{\partial x}(.,t)\right\|^2+\int_0^t\left\|\frac{\partial^2\theta}{\partial x^2}(.,\tau)\right\|^2 d\tau\leqslant C,\quad t\in]0,T[. \tag{2.56}$$

Using (2.14), (2.19), (2.34), (2.35), (2.36), (2.41) and (2.53), from (1.4) we obtain

$$\left\|\frac{\partial\theta}{\partial t}\right\|^2\leqslant C\Big(1+M_\theta^2+\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2+\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2+\left\|\frac{\partial^2\theta}{\partial x^2}\right\|^2\Big)\quad\text{on }]0,T[\tag{2.57}$$

and, because of (2.18), (2.35), (2.36) and (2.56),

$$\int_0^t\left\|\frac{\partial\theta}{\partial t}(.,\tau)\right\|^2 d\tau\leqslant C,\quad t\in]0,T[. \tag{2.58}$$

The conclusion (2.48) follows from (2.56) and (2.58). □

The Proposition 2.1 follows immediately from (2.13), (2.19), (2.34), (2.35), (2.36) and (2.48).

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