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GENERATING FUNCTIONS AND COMBINATORIAL IDENTITIES

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Abstract. Computation of generating functions for renewal sequences is performed by means of the multivariate Lagrange expansion formulae due to Good (1960), which yields the multifold analogue of Carlitz' mixed generating function. As applications, the natural transition is demonstrated from Euler's binomial theorem and the classical Vandermonde convolution formula to Abel identities and Hagen-Rothe formulas, as well as their multifold analogues due to Mohanty & Handa (1969) and Carlitz (1977), respectively.

For a complex parameter α and two complex functions $A(x)$ and $= B(x)$, consider the sequence $\{C_n(\alpha)\}$ defined by generating $=$ function

$$A(x)\,B^{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} C_n(\alpha).$$

In combinatorial computations and special functions, it is often necessary to determine the mixed generating function of the sequence $\{C_n(a+bn)\}$. For this purpose, Carlitz [2, 1977] found a very useful formula and pursued its application to special functions. Recently, the author noticed that the famous Abel identities and the Hagen-Rothe identities are equivalent, respectively, to Euler's binomial theorem and Vandermonde's classical convolution formula when the mixed generating function of Carlitz is assumed as precondition. This fact will be illustrated in the first section. As natural generalization, the second and the third sections will deal with two kinds of multifold analogues and their applications to combinatorial identities of multivariate convolutions.

Denote by $\boldsymbol{C}$ and $\boldsymbol{N}_0$, respectively, the sets of complex numbers and non-negative integers, with the n-fold tensor products $\boldsymbol{C}^n$ and $\boldsymbol{N}_0^n$ in which the linear ordering "$\leqslant$" is induced from the usual one. For two vectors $\bar{x} = (x_1, x_2, \cdots, x_n) \in \boldsymbol{C}^n$ and $\bar{m} = (m_1, m_2, \cdots, m_n) \in \boldsymbol{N}_0^n$, we formally define factorial product $\bar{m}! = \prod_{k=1}^{n} m_k!$, coordinate sum $|\bar{x}| = \sum_{k=1}^{n} x_k$, multivariate-monomial $\bar{x}^{\bar{m}} = \prod_{k=1}^{n} x_k^{m_k}$, scalar product $\langle \bar{x}, \bar{m} \rangle = \sum_{k=1}^{n} m_k x_k$, binomial product $\binom{\bar{x}}{\bar{m}} = \prod_{k=1}^{n} \binom{x_k}{m_k}$, and multinomial

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coefficient $\binom{x}{\bar{m}} = \frac{|\bar{m}!|}{\bar{m}!}\binom{x}{|\bar{m}|}$. These notation will be adopted, throughout the paper in order to avoid unnecessary complicated expressions.

1. Generating Functions: Abel and Hagen-Rothe

The classical Lagrange inversion formula (cf.[7, 13]) is very useful for formal power series calculus.

LEMMA 1.1. (The Lagrange inversion formula) *Let $\varphi(x)$ be a formal power series and $\varphi(0) \neq 0$. If two indeterminates x and y are related by*

$$y = x\,\varphi(y) \tag{1.1a}$$

then $f(y)$, a formal power series in y, can be expanded as a formal power series in x as follows:

$$f(y) = f(0) + \sum_{m=1}^{\infty} \frac{x^m}{m!} D_y^{m-1}\{f'(y)\varphi^m(y)\}_{y=0}. \tag{1.1b}$$

In particular, we have

$$\frac{f(y)}{1 - y\varphi'(y)/\varphi(y)} = \sum_{m=0}^{\infty} \frac{x^m}{m!} D_y^m\{f(y)\varphi^m(y)\}_{y=0} \tag{1.1c}$$

where $D_y^m = d^m/dy^m$ is the differential operator.

By means of this lemma, Carlitz (1977) estalished the following

THEOREM 1.2. (Carlitz [2]) *Let $A(x)$ and $B(x)$ be formal power series satisfying $A(0) = B(0) = 1$. For a complex parameter α, define the C-sequence by expansion*

$$A(x)\,B^{\alpha}(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}\, C_m(\alpha). \tag{1.2a}$$

Then for two complex numbers a and b, we have the generating function for renewal sequence $\{C_m(a+bm)\}_m$:

$$\frac{A(y)\,B^a(y)}{1 - byB'(y)/B(y)} = \sum_{m=0}^{\infty} \frac{x^m}{m!}\, C_m(a+bm), \tag{1.2b}$$

where x and y are two indeterminates related by

$$y = x\,B^b(y). \tag{1.2c}$$

This formula plays the fundamental role in calculating the mixed generating functions for renewal sequences. We take the Abel formulae and the Hagen-Rothe identities as two examples to illustrate this fact.

PROPOSITION 1.3. (Binomial expansions [11, 13]) *For two indeterminates x and y related by*

$$y = x\,(1+y)^b \tag{1.3a}$$

there hold expansions

$$\frac{(1+y)^{a+1}}{1+y-by}=\sum_{m=0}^{\infty}\binom{a+bm}{m}x^m \tag{1.3b}$$

$$(1+y)^a=\sum_{m=0}^{\infty}\frac{a}{a+bm}\binom{a+bm}{m}x^m. \tag{1.3c}$$

Proof. Put $A(x)=1$ and $B(x)=1+x$ in Theorem 1.2. Then the binomial expansion

$$(1+x)^\alpha \quad = \quad \sum_{m=0}^{\infty}\binom{\alpha}{m}x^m$$

confirms Eq.(1.2a) and Eq.(1.2b) reduces to Eq.(1.3b) directly. Noticing that

$$\begin{aligned}(1+y)^a &= (1+y)^a\frac{1+y-by}{1+y-by}\\ &= \frac{(1+y)^{a+1}}{1+y-by}-bx\frac{(1+y)^{a+b}}{1+y-by}\end{aligned}$$

in view of (1.3a), we may easily derive Eq.(1.3c) from Eq.(1.3b).

□

Applying the exponential law

$$\begin{aligned}(1+y)^{a+c} &= (1+y)^a\times(1+y)^c\\ \frac{(1+y)^{a+c}}{1+y-by} &= (1+y)^a\times\frac{(1+y)^c}{1+y-by}\end{aligned}$$

to the expansions in Proposition 1.3, we get the Hagen-Rothe convolution formulas.

COROLLARY 1.4. (Hagen-Rothe identities [10, 13]) *Let* $\{\mathscr{A}_a(k)\}$ *and* $\{\mathscr{A}'_a(k)\}$ *are two sequences defined by*

$$\mathscr{A}_a(k):=\binom{a+bk}{k}\frac{a}{a+bk} \tag{1.4a}$$

$$\mathscr{A}'_a(k):=\binom{a+bk}{k}. \tag{1.4b}$$

We have convolution identities

$$\mathscr{A}_{a+c}(m)=\sum_{k=0}^{m}\mathscr{A}_a(k)\ \mathscr{A}_c(m-k) \tag{1.4c}$$

$$\mathscr{A}'_{a+c}(m)=\sum_{k=0}^{m}\mathscr{A}_a(k)\ \mathscr{A}'_c(m-k). \tag{1.4d}$$

Instead, put $A(x) = 1$ and $B(x) = e^x$ in Theorem 1.2. Then the exponential expansion

$$e^{\alpha x} = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} x^m$$

corresponds to Eq.(1.2a). Similarly, we can estabilish, by means of Theorem 1.2, the mixed generating functions and convolution identities for the Abel coefficients.

PROPOSITION 1.5. (Exponential expansions [11, 13]) *For two indeterminates x and y related by*

$$y = x\,e^{by} \tag{1.5a}$$

there hold the expansions

$$\frac{e^{ay}}{1-by} = \sum_{m=0}^{\infty} \frac{(a+bm)^m}{m!} x^m \tag{1.5b}$$

$$e^{ay} = \sum_{m=0}^{\infty} \frac{a}{a+bm} \frac{(a+bm)^m}{m!} x^m. \tag{1.5c}$$

Recalling the exponential law

$$\begin{aligned} e^{(a+c)y} &= e^{ay} \times e^{cy} \\ \frac{e^{(a+c)y}}{1-by} &= e^{ay} \times \frac{e^{cy}}{1-by} \end{aligned}$$

we obtain the Abel identities via power series expansion.

COROLLARY 1.6. (Abel formulas [10, 13]) *Let $\{\mathscr{B}_a(k)\}$ and $\{\mathscr{B}'_a(k)\}$ are two sequences defined by*

$$\mathscr{B}_a(k) := \frac{(a+bk)^k}{k!} \frac{a}{a+bk} \tag{1.6a}$$

$$\mathscr{B}'_a(k) := \frac{(a+bk)^k}{k!}. \tag{1.6b}$$

We have convolution identities

$$\mathscr{B}_{a+c}(m) = \sum_{k=0}^{m} \mathscr{B}_a(k)\ \mathscr{B}_c(m-k) \tag{1.6c}$$

$$\mathscr{B}'_{a+c}(m) = \sum_{k=0}^{m} \mathscr{B}_a(k)\ \mathscr{B}'_c(m-k). \tag{1.6d}$$

2. Tensor Product Form: Carlitz

Now we use the multivariate Lagrange expansion formula due to Good (1960) to investigate the tensor product versions of the results displayed in last section.

LEMMA 2.1. (The multivariate Lagrange inversion formula [9]) *For* $k = 1, 2, \cdots, n$, *let* $= \varphi_k(\bar{x}) = \varphi_k(x_1, x_2, \cdots, x_n)$ *be formal power series with* $\varphi_k(\bar{0}) \neq 0$. *If two sets* $\{x_i\}_{i=1}^n$ *and* $\{y_j\}_{j=1}^n$ *of indeterminates are related by*

$$y_k = x_k\, \varphi_k(\bar{y}), k = 1, 2, \cdots, n \tag{2.1a}$$

then $F(\bar{y})$, *a formal power series in* $\{y_i\}$, *can be expanded as a formal power series in* $\{x_j\}$ *as follows:*

$$F(\bar{y}) = \sum_{\bar{m}\in \mathbf{N}_0^n} \frac{\bar{x}^{\bar{m}}}{\bar{m}!} D_{\bar{y}}^{\bar{m}} \left\{ F(\bar{y})\bar{\varphi}^{\bar{m}}(\bar{y}) \det_{n\times n}\left[\delta_{ij} - \frac{y_i}{\varphi_i(\bar{y})}\frac{\partial \varphi_i(\bar{y})}{\partial y_j}\right]\right\}_{\bar{y}=\bar{0}}. \tag{2.1b}$$

As the main theorem of this paper, the tensor-product version of Carlitz' formula may be stated as follows.

THEOREM 2.2. *Let* $A(\bar{x})$ *and* $B_k(\bar{x})$ *be multivariate formal power series satsfying* $A(\bar{0}) = B_k(\bar{0}) = 1$, $(k = 1, 2, \cdots, n)$. *For complex vector* $\bar{\alpha}$, *consider the expansion*

$$A(\bar{x})\, \bar{B}^{\bar{\alpha}}(\bar{x}) = \sum_{\bar{m}\in \mathbf{N}_0^k} \frac{\bar{x}^{\bar{m}}}{\bar{m}!} C_{\bar{m}}(\bar{\alpha}). \tag{2.2a}$$

Then for any complex vector $\bar{a} = (a_1, a_2, \cdots, a_n)$ *and complex matrix* $\mathbf{b} = [b_{ij}]_{(1\leqslant i,j\leqslant n)}$ *independent on* $\bar{x}$, *the generating function for renewal sequence* $\{C_{\bar{m}}(\bar{a} + \bar{m}\mathbf{b})\}_{\bar{m}}$ *is given by*

$$\sum_{\bar{m}\in \mathbf{N}_0^n} \frac{\bar{x}^{\bar{m}}}{\bar{m}!} C_{\bar{m}}(\bar{a} + \bar{m}\mathbf{b}) = \frac{A(\bar{y})\, \bar{B}^{\bar{a}}(\bar{y})}{\det\left[\delta_{ij} - \frac{y_i\partial}{\partial y_j}\log\prod_{k=1}^n B_k^{b_{ik}}(\bar{y})\right]} \tag{2.2b}$$

where $\{x_i\}$ *and* $\{y_j\}$ *are related by*

$$y_k = x_k \prod_{j=1}^n B_j^{b_{kj}}(\bar{y}), \quad k = 1, 2, \cdots, n. \tag{2.2c}$$

Proof. From Eq.(2.2a), it is trivial to have

$$\sum_{\bar{m}\in \mathbf{N}_0^n} \frac{\bar{x}^{\bar{m}}}{\bar{m}!} C_{\bar{m}}(\bar{a} + \bar{m}\mathbf{b}) \quad = \quad \sum_{\bar{m}\in \mathbf{N}_0^n} \frac{\bar{x}^{\bar{m}}}{\bar{m}!} D_{\bar{y}}^{\bar{m}} \left\{A(\bar{y})\bar{B}^{\bar{a}+\bar{m}\mathbf{b}}(\bar{y})\right\}_{\bar{y}=\bar{0}}$$

which can be reformulated as Eq.(2.2b) by means of Lemma 2.1 under replacement

$$\varphi_i(\bar{y}) \quad \rightarrow \quad \prod_{j=1}^n B_j^{b_{ij}}(\bar{y}), \quad i = 1, 2, \cdots, n$$

$$F(\bar{y}) \quad \rightarrow \quad \frac{A(\bar{y})\, \bar{B}^{\bar{a}}(\bar{y})}{\det\left[\delta_{ij} - \frac{y_i\partial}{\partial y_j}\log\prod_{k=1}^n B_k^{b_{ik}}(\bar{y})\right]}.$$

This completes the proof of the theorem.

□

Denote by

$$\epsilon_{\bar{\alpha}}(\bar{m}) = \det\left[\delta_{ij} - \frac{b_{ij}m_j}{\alpha_j + \sum_{k=1}^{n} m_k b_{kj}}\right]. \tag{2.3}$$

We can recover, through this theorem, the multivariate generating functions and the multiple convolution formulas due to Carlitz [1]

PROPOSITION 2.3. (Carlitz [1]) *For two sets $\{x_i\}$ and $\{y_j\}$ of indeterminates related by*

$$y_i = x_i \prod_{j=1}^{n}(1+y_j)^{b_{ij}}, \quad i = 1,2,\cdots,n \tag{2.4a}$$

there hold expansions

$$\frac{\prod_{k=1}^{n}(1+y_k)^{a_k}}{\det\left[\delta_{ij} - \frac{b_{ij}y_i}{1+y_j}\right]} = \sum_{\bar{m}\in \mathbf{N}_0^n} \binom{\bar{a}+\bar{m}\mathbf{b}}{\bar{m}} \bar{x}^{\bar{m}} \tag{2.4b}$$

$$\prod_{k=1}^{n}(1+y_k)^{a_k} = \sum_{\bar{m}\in \mathbf{N}_0^n} \epsilon_{\bar{a}}(\bar{m}) \binom{\bar{a}+\bar{m}\mathbf{b}}{\bar{m}} \bar{x}^{\bar{m}}. \tag{2.4c}$$

Proof. Put $A(\bar{x}) = 1$ and $B_i(\bar{x}) = 1 + x_i$ in Theorem 2.2. Then Eq.(2.2a) reads as the tensor product of the classical binomial expansions

$$\prod_{k=1}^{n}(1+x_k)^{\alpha_k} = \sum_{\bar{m}\in \mathbf{N}_0^n} \binom{\bar{\alpha}}{\bar{m}} \bar{x}^{\bar{m}}$$

and Eq.(2.2b) becomes Eq.(2.4b) directly.

For $\sigma \subseteq [n] = \{1,2,\cdots,n\}$, let $\theta(\sigma) = (-1)^p$, where p is the cardinality of σ. The matrix expansion

$$\det\left[\delta_{ij} - \frac{b_{ij}y_i}{1+y_j}\right] = \sum_{\sigma\subseteq[n]} \theta(\sigma) \det_{i,j\in\sigma}\left[\frac{b_{ij}y_i}{1+y_j}\right] \tag{2.5a}$$

$$= \sum_{\sigma\subseteq[n]} \theta(\sigma) \det_{i,j\in\sigma}[b_{ij}] \prod_{\kappa\in\sigma} \frac{y_\kappa}{1+y_\kappa} \tag{2.5b}$$

may be used to manipulate the formal power series

$$\begin{aligned}
\prod_{k=1}^{n}(1+y_k)^{a_k} &= \det\left[\delta_{ij}-\frac{b_{ij}y_i}{1+y_j}\right]\times\frac{\prod_{k=1}^{n}(1+y_k)^{a_k}}{\det\left[\delta_{ij}-\frac{b_{ij}y_i}{1+y_j}\right]}\\
&= \sum_{\sigma\subseteq[n]}\theta(\sigma)\det_{i,j\in\sigma}[b_{ij}]\prod_{\kappa\in\sigma}x_\kappa\frac{\prod_{k=1}^{n}(1+y_k)^{a_k+\sum_{\kappa\in\sigma}(b_{\kappa k}-\delta_{\kappa k})}}{\det\left[\delta_{ij}-\frac{b_{ij}y_i}{1+y_j}\right]}\\
&= \sum_{\sigma\subseteq[n]}\theta(\sigma)\det_{i,j\in\sigma}[b_{ij}]\sum_{\bar m\in\mathbf{N}_0^n}\binom{\bar a+\bar m\mathbf{b}}{\bar m}\prod_{\kappa\in\sigma}\frac{m_\kappa}{a_\kappa+\sum_{k=1}^{n}m_kb_{k\kappa}}\bar x^{\bar m}
\end{aligned}$$

in view of Eq.(2.4a) and Eq.(2.4b). Interchanging the summation order in the last equation and using matrix expansion

$$\epsilon_{\bar a}(\bar m) = \sum_{\sigma\subseteq[n]}\theta(\sigma)\det_{i,j\in\sigma}[b_{ij}]\prod_{\kappa\in\sigma}\frac{m_\kappa}{a_\kappa+\sum_{k=1}^{n}m_kb_{k\kappa}} \tag{2.6}$$

we arrive at Eq.(2.4c).

□

Applying the exponential law to the expansions in proposition 2.3 leads us to the multiple Hagen-Rothe convolutions due to Carlitz (1977).

COROLLARY 2.4. (Carlitz [1]) *Let* $\{\mathscr{C}_{\bar a}(\bar k)\}$ *and* $\{\mathscr{C}'_{\bar a}(\bar k)\}$ *are two sequences defined by*

$$\mathscr{C}_{\bar a}(\bar k) := \binom{\bar a+\bar k\mathbf{b}}{\bar k}\epsilon_{\bar a}(\bar k) \tag{2.7a}$$

$$\mathscr{C}'_{\bar a}(\bar k) := \binom{\bar a+\bar k\mathbf{b}}{\bar k}. \tag{2.7b}$$

We have convolution identities

$$\mathscr{C}_{\bar a+\bar c}(\bar m) = \sum_{\bar 0\leqslant\bar k\leqslant\bar m}\mathscr{C}_{\bar a}(\bar k)\,\mathscr{C}_{\bar c}(\bar m-\bar k) \tag{2.7c}$$

$$\mathscr{C}'_{\bar a+\bar c}(\bar m) = \sum_{\bar 0\leqslant\bar k\leqslant\bar m}\mathscr{C}_{\bar a}(\bar k)\,\mathscr{C}'_{\bar c}(\bar m-\bar k). \tag{2.7d}$$

The Abelian analogues may be fulfilled similarly.

PROPOSITION 2.5. (Carlitz [1]) *For two sets* $\{x_i\}$ *and* $\{y_j\}$ *of indeterminates related by*

$$y_i = x_i\,e^{\sum_{j=1}^{n}b_{ij}y_j},\quad i=1,2,\cdots,n \tag{2.8a}$$

there hold expansions

$$\frac{e^{\langle \bar{a},\bar{y}\rangle}}{\det[\delta_{ij}-b_{ij}y_i]} = \sum_{\bar{m}\in\boldsymbol{N}_0^n} \frac{(\bar{a}+\bar{m}\mathbf{b})^{\bar{m}}}{\bar{m}!}\,\bar{x}^{\bar{m}} \tag{2.8b}$$

$$e^{\langle \bar{a},\bar{y}\rangle} = \sum_{\bar{m}\in\boldsymbol{N}_0^n} \epsilon_{\bar{a}}(\bar{m})\,\frac{(\bar{a}+\bar{m}\mathbf{b})^{\bar{m}}}{\bar{m}!}\,\bar{x}^{\bar{m}}. \tag{2.8c}$$

Proof. Put $A(\bar{x})=1$ and $B_k(\bar{x})=e^{x_k}$ in Theorem 2.2. Then Eq.(2.2a) reads as the tensor product of the exponential expansions

$$e^{\langle \bar{\alpha},\bar{x}\rangle} = \sum_{\bar{m}\in\boldsymbol{N}_0^n} \frac{\bar{\alpha}^{\bar{m}}}{\bar{m}!}\,\bar{x}^{\bar{m}}$$

and Eq.(2.2b) reduces Eq.(2.8b) directly.

Similar to the matrix expansion (2.5a–2.5b) in the proof of Proposition 2.3, we have

$$\begin{aligned}\det[\delta_{ij}-b_{ij}y_i] &= \sum_{\sigma\subseteq[n]} \theta(\sigma)\det_{i,j\in\sigma}[b_{ij}y_i]\\ &= \sum_{\sigma\subseteq[n]} \theta(\sigma)\det_{i,j\in\sigma}[b_{ij}]\prod_{\kappa\in\sigma} y_\kappa\end{aligned}$$

which is used now to manipulate the formal power series

$$\begin{aligned}e^{\langle \bar{a},\bar{y}\rangle} &= \det[\delta_{ij}-b_{ij}y_i]\times\frac{e^{\langle \bar{a},\bar{y}\rangle}}{\det[\delta_{ij}-b_{ij}y_i]}\\ &= \sum_{\sigma\subseteq[n]} \theta(\sigma)\det_{i,j\in\sigma}[b_{ij}]\prod_{\kappa\in\sigma} x_\kappa\frac{\prod_{k=1}^n e^{y_k(a_k+\sum_{\kappa\in\sigma}b_{\kappa k})}}{\det[\delta_{ij}-b_{ij}y_i]}\\ &= \sum_{\sigma\subseteq[n]} \theta(\sigma)\det_{i,j\in\sigma}[b_{ij}]\sum_{\bar{m}\in\boldsymbol{N}_0^n}\frac{(\bar{a}+\bar{m}\mathbf{b})^{\bar{m}}}{\bar{m}!}\prod_{\kappa\in\sigma}\frac{m_\kappa}{a_\kappa+\sum_{k=1}^n m_k b_{k\kappa}}\,\bar{x}^{\bar{m}}\end{aligned}$$

in view of Eq.(2.8a) and Eq.(2.8b). Interchanging the summation order in the last equation and recalling matrix expansion (2.6), we arrive at Eq.(2.8c).

□

The application of the exponential law to both expansions in this proposition leads us to the multifold Abel identities due to Carlitz (1977).

COROLLARY 2.6. (Carlitz [1]) *Let* $\{\mathscr{D}_{\bar{a}}(\bar{k})\}$ *and* $\{\mathscr{D}'_{\bar{a}}(\bar{k})\}$ *are two sequences defined by*

$$\mathscr{D}_{\bar{a}}(\bar{k}) := \frac{(\bar{a}+\bar{k}\mathbf{b})^{\bar{k}}}{\bar{k}!}\,\epsilon_{\bar{a}}(\bar{k}) \tag{2.9a}$$

$$\mathscr{D}'_{\bar{a}}(\bar{k}) := \frac{(\bar{a}+\bar{k}\mathbf{b})^{\bar{k}}}{\bar{k}!}. \tag{2.9b}$$

We have convolution identities

$$\mathscr{D}_{\bar{a}+\bar{c}}(\bar{m}) = \sum_{\bar{0}\leqslant\bar{k}\leqslant\bar{m}} \mathscr{D}_{\bar{a}}(\bar{k})\ \mathscr{D}_{\bar{c}}(\bar{m}-\bar{k}) \tag{2.9c}$$

$$\mathscr{D}'_{\bar{a}+\bar{c}}(\bar{m}) = \sum_{\bar{0}\leqslant\bar{k}\leqslant\bar{m}} \mathscr{D}_{\bar{a}}(\bar{k})\ \mathscr{D}'_{\bar{c}}(\bar{m}-\bar{k}). \tag{2.9d}$$

3. Multinomial Coefficients: Mohanty-Handa

Putting

$$a = \sum_{i=1}^{n} a_i, \quad b_i = \sum_{j=1}^{n} b_{ij}, \qquad i = 1,2,\cdots,n$$

$$B_k(\bar{x}) \equiv B(\bar{x}), \qquad k = 1,2,\cdots,n$$

in Theorem 2.2, we get its multinomial version.

THEOREM 3.1. *Let $A(\bar{x})$ and $B(\bar{x})$ be multivariate formal power series satsfying $A(\bar{0}) = B(\bar{0}) = 1$. For complex number α, consider expansion*

$$A(\bar{x})\, B^{a}(\bar{x}) = \sum_{\bar{m}\in \boldsymbol{N}_0^n} \frac{\bar{x}^{\bar{m}}}{\bar{m}!}\ C_{\bar{m}}(\alpha). \tag{3.1a}$$

Then for any parameter vector $\bar{b} = (b_1, b_2, \cdots, b_n)$ independent on $\bar{x}$, we have the generating function for renewal sequence $\{C_{\bar{m}}(a + \langle\bar{b},\bar{m}\rangle)\}_{\bar{m}}$

$$\sum_{\bar{m}\in \boldsymbol{N}_0^n} \frac{\bar{x}^{\bar{m}}}{\bar{m}!} C_{\bar{m}}(a+\langle\bar{b},\bar{m}\rangle) = \frac{A(\bar{y})\, B^{a}(\bar{y})}{\det\left[\delta_{ij} - \frac{b_i\, y_i}{B(\bar{y})}\frac{\partial B(\bar{y})}{\partial y_j}\right]} \tag{3.1b}$$

where $\{x_i\}$ and $\{y_j\}$ are related by

$$y_k = x_k\, B^{b_k}(\bar{y}), \quad k = 1,2,\cdots,n. \tag{3.1c}$$

The formula displayed above may be used to revisit the following multivariate generating functions and convolution identities due to Mohanty-Handa (1969).

PROPOSITION 3.2. (Mohanty-Handa [12]) *For two sets $\{x_i\}$ and $\{y_j\}$ of indeterminates related by*

$$y_k = x_k\,(1+|\bar{y}|)^{b_k}, \quad k = 1,2,\cdots,n \tag{3.2a}$$

there hold expansions

$$\frac{(1+|\bar{y}|)^{a+1}}{1+|\bar{y}|-\langle\bar{b},\bar{y}\rangle} = \sum_{\bar{m}\in \boldsymbol{N}_0^n} \binom{a+\langle\bar{b},\bar{m}\rangle}{vtm}\ \bar{x}^{\bar{m}} \tag{3.2b}$$

$$(1+|\bar{y}|)^{a} = \sum_{\bar{m}\in \boldsymbol{N}_0^n} \frac{a}{a+\langle\bar{b},\bar{m}\rangle} \binom{a+\langle\bar{b},\bar{m}\rangle}{vtm}\ \bar{x}^{\bar{m}}. \tag{3.2c}$$

Proof. Put $A(\bar{x})=1$ and $B(\bar{x})=1+|\bar{x}|$ in Theorem 3.1. It is an easy exercise to check

$$\det\left[\delta_{ij}-\frac{b_i y_i}{B(\bar{y})}\frac{\partial B(\bar{y})}{\partial y_j}\right]=\det\left[\delta_{ij}-\frac{b_i y_i}{1+|\bar{y}|}\right]=1-\frac{\langle\bar{b},\bar{y}\rangle}{1+|\bar{y}|}.$$

Then Eq.(3.1a) reads as the multinomial theorem

$$(1+|\bar{x}|)^{\alpha}=\sum_{\bar{m}\in\boldsymbol{N}_0^n}\binom{\alpha}{\bar{m}}\bar{x}^{\bar{m}}$$

and Eq.(3.1b) reduces to Eq.(3.2b) directly. From Eq.(3.2a) and Eq.(3.2b), we can make expansion

$$\begin{aligned}(1+|\bar{y}|)^{a} &= (1+|\bar{y}|-\langle\bar{b},\bar{y}\rangle)\times\frac{(1+|\bar{y}|)^{a}}{1+|\bar{y}|-\langle\bar{b},\bar{y}\rangle}\\ &= \frac{(1+|\bar{y}|)^{a+1}-\sum_{k=1}^{n}b_k x_k(1+|\bar{y}|)^{a+b_k}}{1+|\bar{y}|-\langle\bar{b},\bar{y}\rangle}\\ &= \sum_{\bar{m}\in\boldsymbol{N}_0^n}\binom{a+\langle\bar{b},\bar{m}\rangle}{\bar{m}}\bar{x}^{\bar{m}}\times\left\{1-\sum_{k=1}^{n}\frac{b_k m_k}{a+\langle\bar{b},\bar{m}\rangle}\right\}.\end{aligned}$$

For the inner sum, we have a closed form

$$\frac{\langle\bar{b},\bar{m}\rangle}{a+\langle\bar{b},\bar{m}\rangle}=\sum_{k=1}^{n}\frac{b_k m_k}{a+\langle\bar{b},\bar{m}\rangle}\tag{3.3}$$

and Eq.(3.2c) is confirmed accordingly.

□

Applying the exponential law to the expansions in Proposition 3.2, we get the multinomial forms of the Hagen-Rothe convolution formulas due to Mohanty & Handa (1969).

COROLLARY 3.3. (Mohanty-Handa [12]) *Let $\{\mathscr{E}_a(\bar{k})\}$ and $\{\mathscr{E}'_a(\bar{k})\}$ are two sequences defined by*

$$\mathscr{E}_a(\bar{k}):=\binom{a+\langle\bar{b},\bar{k}\rangle}{\bar{k}}\frac{a}{a+\langle\bar{b},\bar{k}\rangle}\tag{3.4a}$$

$$\mathscr{E}'_a(\bar{k}):=\binom{a+\langle\bar{b},\bar{k}\rangle}{\bar{k}}.\tag{3.4b}$$

We have convolution identities

$$\mathscr{E}_{a+c}(\bar{m})=\sum_{\bar{0}\leqslant\bar{k}\leqslant\bar{m}}\mathscr{E}_a(\bar{k})\ \mathscr{E}_c(\bar{m}-\bar{k})\tag{3.4c}$$

$$\mathscr{E}'_{a+c}(\bar{m})=\sum_{\bar{0}\leqslant\bar{k}\leqslant\bar{m}}\mathscr{E}_a(\bar{k})\ \mathscr{E}'_c(\bar{m}-\bar{k}).\tag{3.4d}$$

The same approach may derive the Abelian analogues of the results on multinomial coefficients either.

PROPOSITION 3.4. (Mohanty-Handa [12]) *For two sets $\{x_i\}$ and $\{y_j\}$ of indeterminates related by*

$$y_k = x_k\, e^{b_k|\bar{y}|}, \quad k = 1, 2, \cdots, n \tag{3.5a}$$

there hold expansions

$$\frac{e^{a|\bar{y}|}}{1-\langle\bar{b},\bar{y}\rangle} = \sum_{\bar{m}\in\boldsymbol{N}_0^n} \frac{(a+\langle\bar{b},\bar{m}\rangle)^{|\bar{m}|}}{\bar{m}!}\,\bar{x}^{\bar{m}} \tag{3.5b}$$

$$e^{a|\bar{y}|} = \sum_{\bar{m}\in\boldsymbol{N}_0^n} \frac{a}{a+\langle\bar{b},\bar{m}\rangle}\,\frac{(a+\langle\bar{b},\bar{m}\rangle)^{|\bar{m}|}}{\bar{m}!}\,\bar{x}^{\bar{m}}. \tag{3.5c}$$

Proof. Put $A(\bar{x}) = 1$ and $B(\bar{x}) = e^{|\bar{x}|}$ in Theorem 3.1. It is not hard to see

$$\det\left[\delta_{ij} - \frac{b_i\,y_i}{B(\bar{y})}\frac{\partial B(\bar{y})}{\partial y_j}\right] = \det\left[\delta_{ij} - b_i\,y_i\right] = 1 - \langle\bar{b},\bar{y}\rangle.$$

Then Eq.(3.1a) reads as a multiple exponential expansion

$$e^{\alpha|\bar{x}|} = \sum_{\bar{m}\in\boldsymbol{N}_0^n} \frac{\alpha^{|\bar{m}|}}{\bar{m}!}\,\bar{x}^{\bar{m}}$$

and (3.1b) reduces to (3.5b) directly. From Eq.(3.5a) and Eq.(3.5b), we can perform formal power series calculus

$$\begin{aligned} e^{a|\bar{y}|} &= \left(1-\langle\bar{b},\bar{y}\rangle\right)\times\frac{e^{a|\bar{y}|}}{1-\langle\bar{b},\bar{y}\rangle} \\ &= \frac{e^{a|\bar{y}|}-\sum_{k=1}^{n} b_k x_k e^{(a+b_k)|\bar{y}|}}{1-\langle\bar{b},\bar{y}\rangle} \\ &= \sum_{\bar{m}\in\boldsymbol{N}_0^n} \frac{(a+\langle\bar{b},\bar{m}\rangle)^{|\bar{m}|}}{\bar{m}!}\,\bar{x}^{\bar{m}}\times\left\{1-\sum_{k=1}^{n}\frac{b_k m_k}{a+\langle\bar{b},\bar{m}\rangle}\right\} \end{aligned}$$

which leads us to Eq.(3.5c) in view of Eq.(3.3).

□

The application of the exponential law to the expansions in this proposition result in the multinomial Abel identities due to Mohanty & Handa (1969).

COROLLARY 3.5. (Mohanty-Handa [12]) *Let $\{\mathscr{F}_a(\bar{k})\}$ and $\{\mathscr{F}'_a(\bar{k})\}$ are two sequences defined by*

$$\mathscr{F}_a(\bar{k}) := \frac{(a+\langle\bar{b},\bar{k}\rangle)^{|\bar{k}|}}{\bar{k}!}\,\frac{a}{a+\langle\bar{b},\bar{k}\rangle} \tag{3.6a}$$

$$\mathscr{F}'_a(\bar{k}) := \frac{(a+\langle\bar{b},\bar{k}\rangle)^{|\bar{k}|}}{\bar{k}!}. \tag{3.6b}$$

We have convolution identities

$$\mathscr{F}_{a+c}(\bar{m}) = \sum_{\bar{0}\leqslant\bar{k}\leqslant\bar{m}} \mathscr{F}_a(\bar{k})\ \mathscr{F}_c(\bar{m}-\bar{k}) \tag{3.6c}$$

$$\mathscr{F}'_{a+c}(\bar{m}) = \sum_{\bar{0}\leqslant\bar{k}\leqslant\bar{m}} \mathscr{F}_a(\bar{k})\ \mathscr{F}'_c(\bar{m}-\bar{k}). \tag{3.6d}$$

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