ON H-SMOOTH AND H-CONVEX SETS IN LINEAR SPACES

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Abstract. In this paper the properties of H-smooth and H-convex sets are investigated. It is shown that any H-convex set is convex. The centric, balanced and convex hulls of an H-smooth set, as well as its radial frontier are studied. A necessary and sufficient condition is given for an H-convex set to be strictly convex.

1. Let \( X \) denote a linear space over the field of all real or complex numbers.

If \( M \subset X \) is an absorbent set, then the functional \( p_M : X \to \mathbb{R} \)
defined by
\[
p_M(x) := \inf \{ a > 0 : x \in aM \}, \quad x \in X
\]
is called the Minkowski functional of \( M \).

The notion of H-smooth and H-convex set in a linear space was introduced by T. Precupanu in [3]. Such sets are of interest because the Minkowski functional \( p_M \) corresponding to an absorbent and H-smooth or H-convex set \( M \subset X \) is a Hilbertian semi-norm, that is a semi-norm which satisfies the parallelogram law:
\[
p_M(x + y)^2 + p_M(x - y)^2 = 2 p_M(x)^2 + 2 p_M(y)^2, \quad x, y \in X
\]
(see [2] and [3]).

We modify slightly the definition of an H-smooth set in comparison with those occurring in [1] and [2].

Definition 1. A non-empty subset \( M \) of a linear space \( X \) is called H-smooth if and only if for any \( a, \beta \in \mathbb{R} \), \( a > 0, \beta > 0 \) and each \( x \in aM, \ y \in \beta M \) there exist \( a_0, \beta_0 \in \mathbb{R}, \ a_0 > 0, \beta_0 > 0 \) such that
\[
a_0^2 + \beta_0^2 < 2 (a^2 + \beta^2); \tag{1}
\]
\[
x + y \in a_0 M; \tag{2}
\]
\[
x - y \in \beta_0 M. \tag{3}
\]

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The conditions of the definition just proposed are easier to check in concrete cases. We shall prove that our definition is actually equivalent to the one in [1]. This fact is useful in the proofs of some theorems concerning properties of $H$-smooth sets.

**LEMMA 1.** Let $M \subset X$ be an $H$-smooth set. Then for any $x \in M$ there exists a $\lambda \in (0, 1]$ such that $-x \in \lambda M$.

**Proof.** Let us fix an $x \in M$. If $x = 0$, we can put $\lambda = 1$. Suppose that $x \neq 0$. We write $a = \inf \{\lambda > 0 : x \in \lambda M\}$. Since $x \in M$, we have $a < 1$. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ of positive numbers such that $a_n \to a$ and $x \in a_n M$ for each $n \in \mathbb{N}$. Put $\beta_n = 2a_n$. Then $2x \in \beta_n M$ for $n \in \mathbb{N}$. From the $H$-smoothness of the set $M$ it follows that for each $n \in \mathbb{N}$ there exist $\alpha_{0,n} > 0$ and $\beta_{0,n} > 0$ fulfilling the following conditions:

\[
\begin{align*}
\alpha_{0,n}^2 + \beta_{0,n}^2 &< 2(\alpha_n^2 + \beta_n^2) = 10 \alpha_n^2; \\
3x &= x + 2x \in \alpha_{0,n} M; \\
-x &= x - 2x \in \beta_{0,n} M.
\end{align*}
\]

If it were $\alpha_{0,n} = 0$ or $\beta_{0,n} = 0$, we would have $x = 0$, opposite to our hypothesis. So we have $\alpha_{0,n} > 0$ and $\beta_{0,n} > 0$. Hence and from (5) it follows that $\alpha_{0,n} > 3a$. In view of (4) we obtain

\[
\beta_{0,n} < \sqrt{10 \alpha_n^2 - 9a^2}, \text{ for each } n \in \mathbb{N}.
\]

Letting now $n$ tend to infinity we deduce that $\lim \inf \beta_{0,n} < a$. Hence and from (6):

\[
\inf \{\lambda > 0 : -x \in \lambda M\} < a < 1.
\]

If $a < 1$, we have $\inf \{\lambda > 0 : -x \in \lambda M\} < 1$. In such a case there exists an $\lambda \in (0, 1)$ for which $-x \in \lambda M$.

Suppose now that $a = 1$. Since $x \in M$, $2x \in 2M$, it follows from the $H$-smoothness of the set $M$ that there exist numbers $\alpha_0 > 0$ and $\beta_0 > 0$ fulfilling the conditions:

\[
\begin{align*}
3x &= x + 2x \in \alpha_0 M; \\
-x &= x - 2x \in \beta_0 M; \\
\alpha_0^2 + \beta_0^2 &< 2(1^2 + 2^2) = 10.
\end{align*}
\]

Since $x \neq 0$ and conditions (7) and (8) hold, we deduce that $\alpha_0 > 0$ and $\beta_0 > 0$. Hence, by (7) we get $\alpha_0 > 3a = 3$.

Consequently,

\[
\beta_0^2 < 10 - \alpha_0^2 < 10 - 9 = 1 \quad \text{whence } \beta_0 < 1 \quad \text{and} \quad -x \in \beta_0 M.
\]

This ends the proof of our lemma.
Example 1. An H-smooth set need not be symmetric. For, take $x_0 \in (-1, 1) \setminus \{0\}$, $M := (-1, 1) \setminus \{x_0\}$. If $\alpha > 0$, $\beta > 0$, $x \in \alpha M$, $y \in \beta M$, then $|x| < \alpha$, $|y| < \beta$. Hence

$$|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2) < 2(\alpha^2 + \beta^2).$$

One can find numbers $\alpha_0 > 0$, $\beta_0 > 0$ such that $|x + y| < \alpha_0$, $|x - y| < \beta_0$, $\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2)$ and $x + y \neq \alpha_0 x_0$, $x - y \neq \beta_0 x_0$. Thus $x + y \in \alpha_0 M$, $x - y \in \beta_0 M$, which shows that $M$ is an H-smooth set. If $x_0 \neq 0$, the set $M$ is not symmetric.

PROPOSITION 1. A non-empty set $M \subset X$ is H-smooth if and only if for each $\alpha > 0$, $\beta > 0$ and each $x \in \alpha M$, $y \in \beta M$ there exist numbers $\alpha_0 > 0$, $\beta_0 > 0$ fulfilling conditions (1), (2) and (3).

Proof. We have to prove necessity only. Suppose that $M$ is an H-smooth set and take $\alpha > 0$, $\beta > 0$, $x \in \alpha M$, $y \in \beta M$. Let us consider the following four cases:

1. $\alpha > 0$ and $\beta > 0$. Then the existence of numbers $\alpha_0 > 0$, $\beta_0 > 0$ with properties (1), (2) and (3) follows from the definition of H-smoothness.

2. $\alpha = 0$ and $\beta = 0$. In such a case $x = y = 0$ and we can put $\alpha_0 = \beta_0 = 0$.

3. $\alpha > 0$, $\beta = 0$. Then $y = 0$ and putting $\alpha_0 = \beta_0 := \alpha$ we obtain $x + y = x \in \alpha_0 M$, $x - y = x \in \beta_0 M$, $\alpha_0^2 + \beta_0^2 = 2\alpha^2 = 2(\alpha^2 + \beta^2)$.

4. $\alpha = 0$, $\beta > 0$. Then $x = 0$ and in view of Lemma 1 there exists $\beta_0 \in (0, \beta]$ such that $-y \in \beta_0 M$. Setting $\alpha_0 := \beta$ we have $x + y = y \in \alpha_0 M$, $x - y = -y \in \beta_0 M$, $\alpha_0^2 + \beta_0^2 < 2\beta^2 = 2(\alpha^2 + \beta^2)$. The above cases exhaust all the possibilities and the proof is completed.

Remark 1. Zero need not belong to an H-smooth set. The set $M := (-1, 1) \setminus \{0\}$ may be used as an example.

PROPOSITION 2. If $M \subset X$ is an H-smooth set, then the set $M_0 := \{0\} \cup M$ is H-smooth.

Proof. Let us take $\alpha > 0$, $\beta > 0$, $x \in \alpha M_0$, $y \in \beta M_0$. Then the following cases are possible:

1. $x \in \alpha M$, $y \in \beta M$;
2. $x = 0 \in 0 \cdot M$, $y \in \beta M$;
3. $x \in \alpha M$, $y = 0 \in 0 \cdot M$;
4. $x = 0 \in 0 \cdot M$, $y = 0 \in 0 \cdot M$. 
On account of Proposition 1 in each of the above cases there exist \(a_0 > 0, \beta_0 > 0\) such that
\[
\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2), \quad x + y \in \alpha_0 M \subset \alpha_0 M_0, \quad x - y \in \beta_0 M \subset \beta_0 M_0
\]
which proves that \(M_0\) is an H-smooth set.

2. **Definition 2.** An H-smooth and balanced subset of a linear space is said to be H-convex.

In [1] we find the definition of the so-called strictly H-convex set. We shall show that this definition does not distinguish any new class of sets. Every H-convex set satisfies the condition which appears in the definition. In the present paper the notion "strictly H-convex set" will be used in another sense.

**THEOREM 1.** If \(M \subset X\) is an H-smooth and absorbent set, then for any \(\alpha > 0, \beta > 0\) and any \(x \in \alpha M, \ y \in \beta M\) there exist \(a_0 > 0, \beta_0 > 0\) fulfilling conditions (1), (2), (3) and the following condition:
\[
\max(\alpha_0, \beta_0) < \alpha + \beta.
\] (9)

**Proof.** The Minkowski functional \(p_M\) of the set \(M\) is a Hilbertian semi-norm. If \(\alpha > 0, \beta > 0, \ x \in \alpha M, \ y \in \beta M\), then \(p_M(x) < \alpha\) and \(p_M(y) < \beta\). Suppose first that \(p_M(x) < \alpha\) or \(p_M(y) < \beta\). In such a case we have
\[
p_M(x + y)^2 + p_M(x - y)^2 = 2(p_M(x)^2 + p_M(y)^2) < 2(\alpha^2 + \beta^2)
\]
and
\[
p_M(x + y) < p_M(x) + p_M(y) < \alpha + \beta,
\]
\[
p_M(x - y) < p_M(x) + p_M(-y) < \alpha + \beta.
\]
Then we can find numbers \(\alpha_1 > 0\) and \(\beta_1 > 0\) such that
\[
p_M(x + y) < \alpha_1 < \alpha + \beta, \ p_M(x - y) < \beta_1 < \alpha + \beta
\]
and
\[
\alpha_1^2 + \beta_1^2 < 2(\alpha^2 + \beta^2).
\]
Hence it follows that there exist numbers \(\alpha_0 > 0, \beta_0 > 0, \alpha_0 < \alpha_1, \beta_0 < \beta_1\) for which \(x + y \in \alpha_0 M\) and \(x - y \in \beta_0 M\). Moreover,
\[
\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2), \quad \alpha_0 < \alpha + \beta \quad \text{and} \quad \beta_0 < \alpha + \beta.
\]
It remains to consider the case where \(p_M(x) = \alpha\) and \(p_M(y) = \beta\). From the H-smoothness of the set \(M\) it follows that there exist \(\alpha_1 > 0, \beta_1 > 0\), such that
\[
\alpha_1^2 + \beta_1^2 < 2(\alpha^2 + \beta^2),
\]
\[
x + y \in \alpha_1 M, \ x - y \in \beta_1 M.
\] (10)
Put $a_0 := p_M(x + y)$, $\beta_0 := p_M(x - y)$. From the definition of the Minkowski functional we get $a_0 < a_1$ and $\beta_0 < \beta_1$. If it were $a_0 < a_1$ or $\beta_0 < \beta_1$, we would have

$$a_1^2 + \beta_1^2 > a_0^2 + \beta_0^2 = p_M(x + y)^2 + p_M(x - y)^2 \Rightarrow$$

contrary to (10). Thus $a_0 = a_1$, $\beta_0 = \beta_1$, $x + y \in a_0 M$ and $x - y \in \beta_0 M$. Moreover,

$$a_0 = p_M(x + y) < p_M(x) + p_M(y) = a + \beta,$$

$$\beta_0 = p_M(x - y) < p_M(x) + p_M(-y) = a + \beta.$$  

This completes our proof.

The previous theorem remains true in the case where $M \subset X$ is an arbitrary H-smooth set (not necessarily absorbent). Namely, we have the following:

**Theorem 2.** If $M \subset X$ is an H-smooth set, then for any $a > 0$, $\beta > 0$ and any $x \in a M$, $y \in \beta M$ there exist numbers $a_0 > 0$, $\beta_0 > 0$ such that conditions (1), (2), (3) and (9) are fulfilled.

**Proof.** Put $M_0 := \{0\} \cup M$. On account of Proposition 2, $M_0$ is an H-smooth set. Let $Y$ be the set of all points $x \in X$ for which there exists $a > 0$ such that $x \in a M_0$. Since $M_0$ is an H-smooth set and $0 \in M_1$, in view of Lemma 1, one can easily check that $Y$ is a linear subspace of the space $X$ in which $M_0$ is an absorbent set. From Theorem 1 it follows that for any $a > 0$, $\beta > 0$, $x \in a M$, $y \in \beta M$ there exist $a_1 > 0$, $\beta_1 > 0$ fulfilling condition (10) and

$$x + y \in a_1 M_0, \quad x - y \in \beta_1 M_0, \quad \max(a_1, \beta_1) < a + \beta.$$  

Put

$$a_0 := \left\{ \begin{array}{ll} a_1, & \text{for } x + y \in a_1 M \\
0, & \text{for } x + y \notin a_1 M \quad (\text{i.e. } x + y = 0) \end{array} \right.$$  

$$\beta_0 := \left\{ \begin{array}{ll} \beta_1, & \text{for } x - y \in \beta_1 M \\
0, & \text{for } x - y \notin \beta_1 M \quad (\text{i.e. } x - y = 0) \end{array} \right.$$  

The numbers $a_0 > 0$, $\beta_0 > 0$ fulfil conditions (1), (2), (3) and (9).

The example of an H-convex but not convex set, which was given by E. Kramar in [1], and Muntean and Precupanu in [2], is improper. Namely, we have the following

**Theorem 3.** Every H-convex set $M \subset X$ is convex.
Proof. Take \(x, y \in M\) and \(t \in [0, 1]\). Then \(tx \in tM\) and \((1 - t)y \in (1 - t)M\). From Theorem 2 it follows, in particular, that there exists an \(a_0 > 0\) such that \(tx + (1 - t)y \in a_0 M\) and \(a_0 < t + (1 - t) = 1\). Since \(M\) is balanced, we have \(tx + (1 - t)y \in M\). This ends the proof.

In the proofs of Theorems 1 and 2 we have made use of the fact that the Minkowski functional of an \(H\)-smooth set is a Hilbertian seminorm. Now we shall give another quite elementary proof of the convexity of an \(H\)-convex set. Having such a proof one is able to obtain immediately the subadditivity of the Minkowski functional corresponding to an absorbent \(H\)-convex set. Now, we proceed with the

Proof. Let \(M \subset X\) be an \(H\)-convex set. If \(x, y \in M\), \(t \in (0, 1)\), then \(tx \in tM\), \((1 - t)y \in (1 - t)M\). From the \(H\)-convexity of the set \(M\) it follows that there exists an \(a_0 > 0\) such that

\[
 tx + (1 - t)y \in a_0 M \quad \text{and} \quad a_0^2 < 2(t^2 + (1 - t)^2).
\]

Since the set \(M\) is balanced we obtain

\[
 tx + (1 - t)y \in \sqrt{2}(t^2 + (1 - t)^2) M.
\]

Putting \(t = \frac{1}{2}\) we have \(\frac{x + y}{2} \in M\). By induction one can prove that

\[
 \frac{k}{2^n} x + \left(1 - \frac{k}{2^n}\right) y \in M \quad \text{for any} \quad k, n \in \mathbb{N}, \quad k < 2^n.
\]

Fix \(x, y \in M\), \(t \in (0, 1)\), put \(z = tx + (1 - t)y\) and take arbitrary numbers \(r, s \in (0, 1)\) such that \(r < t < s\) and \(t = \frac{r + s}{2}\).

Since the set \(A : = \left\{\frac{k}{2^n} \in (0, 1) : k, n \in \mathbb{N}, \ k < 2^n\right\}\) is dense in the interval \((0, 1)\) we can choose two sequences \((r_n)_{n \in \mathbb{N}}\) and \((s_n)_{n \in \mathbb{N}}\) such that \(r_n, s_n \in A\), \(r_n < r\), \(s < s_n\) for each \(n \in \mathbb{N}\) and \(r_n \to r\), \(s_n \to s\).

Defining \(t_n : = \frac{s_n - t}{s_n - r_n}\) we have \(t_n \in (0, 1)\), \(t = t_n r_n + (1 - t_n) s_n\) for each \(n \in \mathbb{N}\) and \(t_n \to \frac{s - t}{s - r} = \frac{1}{2}\). Hence:

\[
 z = tx + (1 - t)y = [t_n r_n + (1 - t_n) s_n] x + [1 - t_n] r_n - (1 - t_n) s_n] y = t_n [r_n x + (1 - r_n) y] + (1 - t_n) [s_n x + (1 - s_n) y].
\]

Since \(r_n x + (1 - r_n) y \in M\) and \(s_n x + (1 - s_n) y \in M\) the relation \(z \in \sqrt{2}(t_n^2 + (1 - t_n)^2) M\) holds for each \(n \in \mathbb{N}\). As a consequence of the fact that \(\sqrt{2}(t_n^2 + (1 - t_n)^2) \to 1\) we obtain \(\inf \{\alpha > 0 : z \in \alpha M\} < 1\), whence \(\lambda z \in M\) follows for each \(\lambda \in (0, 1)\) because \(M\).
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is a balanced set. We define $T(x, y) := \{u \in X : u = ax + by, a, b \in [0, 1), \alpha + \beta < 1\}$. Obviously $T(x, y) = T(y, x)$ and

$$T(x, y) = \{u \in X : u = \lambda (tx + (1 - t)y), \lambda \in [0, 1), t \in [0, 1]\}.$$  

Hence $T(x, y) \subseteq M$ for any $x, y \in M$.

Now we are going to prove that for any $x, y \in M$ we have

$$(x, y) := \{tx + (1 - t)y \in X : t \in (0, 1)\} \subseteq M.$$  

If $x$ and $y$ are linearly dependent over $\mathbb{R}$, then the fact that $M$ is a balanced set implies $(x, y) \subseteq M$. Suppose further on that $x$ and $y$ are linearly independent over $\mathbb{R}$. Put

$$P := \{u \in X : u = ax + by, a, b \in \mathbb{R}, a + b < 1\},$$  

$$S := \{u \in X : u = ax + by, a, b \in \mathbb{R}, a + b > 1\},$$  

$P \cap S = \emptyset, P \cup S = \text{Lin}_\mathbb{R}\{x, y\}$ and consider two cases.

Case 1. There exists a $v \in S \cap M$. We shall show that $(x, y) \subseteq T(x, y) \cup T(y, v)$. There exist $a, b \in \mathbb{R}$ such that $a + b > 1$ and $v = ax + by$. At least one of the numbers $a$ and $b$ has to be positive. Suppose e. g. that $a > 0$.

For $t \in \left(0, \frac{a}{a + b}\right)$ we define $\alpha := \frac{t}{a}, \beta := \frac{a - t(a + b)}{a}$.

Then $\alpha \in (0, 1), \beta \in [0, 1), \alpha + \beta < a(a + b) + \beta = 1,

$$t = \alpha a, 1 - t = ab + \beta$$  

and

$$tx + (1 - t)y = aax + (ab + \beta)y = a(\alpha x + by) + \beta y =$$

$$= av + \beta y \in T(v, y).$$  

Hence it follows that $(x, y) \subseteq T(v, y)$ provided $b < 0$.

If $b > 0$, then for $t \in \left[\frac{a}{a + b}, 1\right)$ we define

$$\alpha := \frac{t(a + b) - a}{a}, \beta := \frac{1 - t}{b}.$$  

Then $\alpha \in [0, 1), \beta \in (0, 1), \alpha + \beta < a + \beta(a + b) = 1, t = \alpha + \beta a, 1 - t = \beta b$ and

$$tx + (1 - t)y = (\alpha + \beta a)x + \beta by = ax + \beta(ax + by) =$$

$$= ax + \beta v \in T(x, v).$$
Consequently \((x, y) \subseteq T(v, y) \cup T(x, v)\) which ends the proof of the inclusion announced. Since \(T(x, v) \subseteq M\) and \(T(v, y) \subseteq M\) we obtain \((x, y) \subseteq M\).

Case 2. \(S \cap M = \emptyset\). Take \(\alpha > 0, \beta > 0, \alpha > \beta\). Then \(ax \in \alpha M, \beta y \in \beta M\) and from the \(H\)-convexity of \(M\) it follows that there exist \(\alpha_0 > 0, \beta_0 > 0\) such that \(\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2)\) and \(ax + \beta y \in \alpha_0 M, ax - \beta y \in \beta_0 M\). Since \(M \cap \text{Lin}_R \{x, y\} \subseteq P\) we have:

\[
ax + \beta y = \alpha_0 (ax + by) \text{ for some } a, b \in R, \ a + b < 1,
\]
\[
ax - \beta y = \beta_0 (cx + dy) \text{ for some } c, d \in R, \ c + d < 1.
\]

From the linear independence (over \(R\)) of the vectors \(x\) and \(y\) we obtain \(a = \alpha_0 a, \beta = \beta_0 b, a = \beta_0 c, -\beta = \beta_0 d\). Hence

\[
0 < a + \beta = \alpha_0 (a + b) \leq \alpha_0, \ 0 < a - \beta = \beta_0 (c + d) \leq \beta_0.
\]

If it were \(a + \beta < \alpha_0\) or \(a - \beta < \beta_0\), we would have

\[
\alpha_0^2 + \beta_0^2 > (a + \beta)^2 + (a - \beta)^2 = 2(\alpha^2 + \beta^2),
\]
which leads to a contradiction. So, we have \(\alpha_0 = a + \beta, \beta_0 = a - \beta\) and, in particular, \(ax + \beta y \in (a + \beta) M\), that is

\[
\frac{ax + \beta y}{a + \beta} \in M \quad \text{for } a > 0, \beta > 0, \alpha > \beta.
\]

Interchanging the roles of \(x\) and \(y\) we obtain the analogous relation for \(a > 0, \beta > 0, a < \beta\). Hence, for \(t \in (0, 1)\), we have \(tx + (1 - t)y \in M\). This ends the proof.

3. **Definition** 3. The set \(M \subseteq X\) is called centric if \(\lambda M \subseteq M\) for each \(\lambda \in [0, 1]\).

Any centric and symmetric set is balanced.

**Proposition 3.** If \(M \subseteq X\) is an \(H\)-smooth and centric set, then \(M\) is \(H\)-convex.

**Proof.** We shall prove that \(M\) is symmetric. Take an \(x \in M\). On account of Lemma 1, there exists a \(\lambda \in (0, 1]\) such that \(-x \in \lambda M \subseteq \subseteq M\), which ends the proof.

**Theorem 4.** The set \(M \subseteq X\) is \(H\)-convex if and only if it is \(H\)-smooth and convex.

**Proof.** In view of Theorem 3 one has only to prove that the condition is sufficient. For, suppose that \(M\) is an \(H\)-smooth and convex set and take an arbitrary \(x \in M\). From Lemma 1 it follows, in particular, that \(-x \in \lambda M\) for some \(\lambda > 0\). Hence \(-\frac{1}{\lambda} x \in M\) and from the convexity of the set \(M\) we obtain
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\[ \left[ -\frac{1}{\lambda}, x, x \right] := \left\{ t \left( -\frac{1}{\lambda} \right) x + (1 - t) x \in X : t \in [0, 1] \right\} \subset M. \]

Consequently, \( 0 \in M \). So we have \( \lambda x = \lambda x + (1 - \lambda) \cdot 0 \in M \) for each \( x \in M, \lambda \in [0, 1] \). This shows that the set \( M \) is centric and we can use the previous proposition to complete the proof.

Now we shall investigate some connections between an H-smooth set \( M \subset X \) and its centric, balanced and convex hulls i.e. the smallest sets containing \( M \) which are centric, balanced or convex, respectively. These results are complementary to those presented in [1] and [2]. A centric hull of the set \( M \) will be denoted by \( C_n M \), whereas the symbols \( B_n M \) and \( \text{Conv} M \) will stand for its balanced and convex hull, respectively.

**Lemma 2.** The centric hull of an H-smooth set is H-smooth.

*Proof.* If \( a > 0, \beta > 0, x \in a C_n M, y \in \beta C_n M \) then, according to definition of a centric hull, there exist \( \lambda, \mu \in [0, 1] \) such that \( x \in a\lambda M, y \in \beta\mu M \). We have \( a\lambda > 0, \beta\mu > 0 \) and from Proposition 1 it follows that there exist numbers \( a_0 > 0, \beta_0 > 0 \), fulfilling the conditions

\[ a_0^2 + \beta_0^2 < 2 (a\lambda)^2 + 2 (\beta\mu)^2 < 2a^2 + 2\beta^2, \]

\[ x + y \in a_0 M \subset a_0 C_n M, x - y \in \beta_0 M \subset \beta_0 C_n M. \]

Hence \( C_n M \) is an H-smooth set.

From Lemma 2 and Proposition 3 it follows:

**Theorem 5.** If \( M \subset X \) is an H-smooth set, then \( C_n M = B_n M \). In particular, the balanced hull of an H-smooth set is an H-convex set.

**Theorem 6.** If \( M \subset X \) is an H-smooth set, then \( \text{Conv} M = B_n M \). In particular, the convex hull of an H-smooth set is an H-convex set.

*Proof.* The set \( B_n M = C_n M \) is H-convex and so it is convex. Hence \( C_n M \subset B_n M = C_n M \). If \( x \in M \) then, in view of Lemma 1, we have \( \frac{-x}{\lambda} \in M \) for some \( \lambda > 0 \). Thus \( \left[ \frac{-x}{\lambda}, x \right] \subset \text{Conv} M \) and, in particular, \( 0 \in \text{Conv} M \).

If \( x \in C_n M \), then there exist \( \lambda \in [0, 1] \) and \( y \in M \) such that \( x = \lambda y = \lambda y + (1 - \lambda) \cdot 0 \in \text{Conv} M \). Consequently, \( C_n M \subset \text{Conv} M \) whence \( \text{Conv} M = C_n M = B_n M \).

Following T. Precupanu, by the radial frontier of a set \( M \subset X \) we mean the collection of all points \( x \in X \setminus \{0\} \) such that \( (x, \to) \cap \cap M = \emptyset \) and \( [x_1, x] \cap M \neq \emptyset \) for each \( x_1 \in (0, x) \), where \( (0, x) := \)
The radial frontier of $M$ will be denoted by $\text{Fr } M$.

**Lemma 3.** For any $M \subset X$ equality $\text{Fr } M = \text{Fr } \text{Cn } M$ holds.

**Proof.** Let us first fix an $x \in \text{Fr } M$. Then $x \neq 0$ and $(x, \rightarrow) \cap \text{Cn } M = \emptyset$. If it were $(x, \rightarrow) \cap \text{Cn } M \neq \emptyset$, it would exist a $t > 1$ such that $tx \in \text{Cn } M$. Then $tx \in \lambda M$ for some $\lambda \in (0, 1)$ and so we would have $\frac{t}{\lambda} > 1$ and $\frac{t}{\lambda} x \in M$, contrary to our hypothesis. Consequently, $(x, \rightarrow) \cap \text{Cn } M = \emptyset$. For any $x_1 \in (0, x)$, we have $[x_1, x] \cap \text{Cn } M = \emptyset$. Since $M \subset \text{Cn } M$, we get $[x_1, x] \cap \text{Cn } M \neq \emptyset$; hence $x \in \text{Fr } \text{Cn } M$. Now, suppose that $x \in \text{Fr } \text{Cn } M$. Then $x \neq 0$ and $(x, \rightarrow) \cap \text{Cn } M = \emptyset$, since $M \subset \text{Cn } M$. For any $x_1 \in (0, x)$, one has $[x_1, x] \cap \text{Cn } M \neq \emptyset$. If $x_1 = t_1 x$, $t_1 \in (0, 1)$ and $t x \in \text{Cn } M$ for some $t \in [t_1, 1]$, then there exists a $\lambda \in (0, 1)$ such that $tx \in \lambda M$. i.e. $\frac{t}{\lambda} x \in M$. Since $(x, \rightarrow) \cap \text{M} = \emptyset$, it must be $\frac{t}{\lambda} < 1$. On the other hand, $\frac{t}{\lambda} \geq t > t_1$; consequently $\frac{t}{\lambda} x \in [x_1, x] \cap M$ and $x \in \text{Fr } M$. This ends the proof.

From Lemma 3 and Theorems 5 and 6 it follows:

**Proposition 4.** If $M \subset X$ is an $H$-smooth set, then $\text{Fr } M = \text{Fr } \text{Bn } M = \text{Fr } \text{Conv } M$.

In [2] it has been proved the following lemma

**Lemma 4.** If $M \subset X$ is an absorbent set, then

$$\text{Fr } M = \{x \in X : p_M (x) = 1\}.$$ 

In view of the properties of the Minkowski functional one can therefore obtain

**Lemma 5.** If $M \subset X$ is an absorbent and centric set, then

$$M \cup \text{Fr } M = \{x \in X : p_M (x) < 1\}$$

and

$$M \setminus \text{Fr } M = \{x \in X : p_M (x) < 1\}.$$ 

The authors of [2] have introduced the concept of a radially bounded set, i.e. a set $M \subset X$ with the property that for each $x \in X \setminus \{0\}$ there exists a $\alpha_0 > 0$ such that $x \notin aM$ for $a > \alpha_0$. 
For an absorbent set $M \subset X$ the following conditions are equivalent (see [2] Lemma 2):

(i) $M$ is radially bounded;
(ii) $\rho_M(x) = 0$ if and only if $x = 0$;
(iii) $\{0\} \cup \text{Fr } M$ is an absorbent set.

Consequently, if a set $M \subset X$ is absorbent, H-convex and satisfies one of the two equivalent conditions (i) or (iii), then the Minkowski functional $\rho_M$ is a Hilbertian norm. It is easy to check that the following lemma is true.

**Lemma 6.** If $p : X \to \mathbb{R}$ is a Hilbertian norm, then the sets

$M_1 := \{x \in X : p(x) = 1\}$ and $M_2 := M_1 \cup \{0\}$ are H-smooth whereas $M_3 := \{x \in X : p(x) < 1\}$ is H-convex.

The next two theorems yield a completion of Theorem 2 from [2].

**Theorem 7.** Let $M \subset X$ be an absorbent, radially bounded and H-smooth set. Then

(a) $\text{Fr } M$ and $\{0\} \cup \text{Fr } M$ are H-smooth sets;
(b) if $M$ is an H-convex set, then so are $M \cup \text{Fr } M$ and $M \setminus \text{Fr } M$;
(c) $Bn M \cup \text{Fr } M$ and $Bn M \setminus \text{Fr } M$ are H-convex sets;
(d) $\text{Conv } M \cup \text{Fr } M$ and $\text{Conv } M \setminus \text{Fr } M$ are H-convex sets.

**Proof.** The Minkowski functional of the set $M$ is a Hilbertian norm. Assertions (a) and (b) follow immediately from Lemmas 4, 5 and 6. To prove (c) and (d) let us notice that $Bn M = \text{Conv } M$ is an absorbent, radially bounded and H-convex set as well as $\text{Fr } \text{Conv } M = \text{Fr } Bn M = \text{Fr } M$. It remains to use (b).

The result below has been obtained in [2] under the additional assumption that $M$ is a symmetric set. We will show that this assumption may be omitted.

**Theorem 8.** If $M \subset X$ is an absorbent, radially bounded set and $\text{Fr } M$ is an H-smooth set, then the Minkowski functional $\rho_M$ of the set $M$ is a Hilbertian norm.

**Proof.** The equivalence of conditions (i) and (ii) implies that $\{0\} \cup \text{Fr } M$ is an absorbent set and, on account of Proposition 2, this union is H-smooth. Thus, the Minkowski functional $\rho_{\{0\} \cup \text{Fr } M}$ is a Hilbertian semi-norm. However, $\rho_{\{0\} \cup \text{Fr } M} = \rho_M$ (see [2] Lemma 3) and since $M$ is radially bounded, we deal with a norm.

4. **Definition 4.** An absorbent subset $M$ of a space $X$ is said to be strictly convex if and only if it is convex and for any $x, y \in M$, $x \neq y$ and any $t \in (0, 1)$ there is $\rho_M(tx + (1 - t)y) < 1$, where $\rho_M$ denotes the Minkowski functional of $M$. 
Now we are going to give a necessary and sufficient condition for an $H$-convex set to be strictly convex. In [2] it has been pointed out that each absorbent, $H$-smooth and radially bounded set is strictly convex. There exist, however, $H$-convex and strictly convex sets which are not radially bounded. As an example one can take the set $M := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1\}.$

**Definition 5.** An absorbent set $M \subset X$ is called strictly $H$-convex if and only if it is balanced and for any $x, y \in M, x \neq y$ and each $\alpha > 0, \beta > 0$ there exist numbers $\alpha_0 > 0, \beta_0 > 0$ fulfilling the conditions:

$$\alpha_0^2 + \beta_0^2 < 2(\alpha^2 + \beta^2); \quad (1)$$
$$\alpha x + \beta y \in \alpha_0 M; \quad (11)$$
$$\alpha x - \beta y \in \beta_0 M; \quad (12)$$
$$\alpha_0 < \alpha + \beta. \quad (13)$$

**Remark 2.** Every strictly $H$-convex set is $H$-convex.

**Proof.** Let $M \subset X$ be a strictly $H$-convex set. It is sufficient to show that $M$ is $H$-smooth. Take $\alpha > 0, \beta > 0, x \in \alpha M, y \in \beta M.$ Then we have $x = au, y = \beta v$ for some $u, v \in M.$ If $u \neq v,$ there exist numbers $\alpha_0 > 0, \beta_0 > 0$ fulfilling condition (1) and such that

$$x + y = au + \beta v \in \alpha_0 M, x - y = au - \beta v \in \beta_0 M.$$ If $u = v,$ we put $\alpha_0 := \alpha + \beta, \beta_0 := |\alpha - \beta|$ getting

$$\alpha_0^2 + \beta_0^2 = 2(\alpha^2 + \beta^2), x + y = (\alpha + \beta) u \in \alpha_0 M,$$
$$x - y = (\alpha - \beta) u = \beta_0 \operatorname{sgn}(\alpha - \beta) u \in \beta_0 M.$$ Thus, our remark is proved.

**THEOREM 9.** Let $M \subset X$ be an absorbent set. The following three conditions are equivalent:

(a) $M$ is a strictly $H$-convex set;

(b) $M$ is a strictly convex and $H$-convex set;

(c) $M$ is a strictly convex and $H$-smooth set.

**Proof.** (a) $\Rightarrow$ (b). On account of Remark 2 it suffices to prove that $M$ is strictly convex. The set $M,$ being $H$-convex, is convex. Take $x, y \in M, x \neq y, t \in (0, 1).$ From the strict $H$-convexity of the set $M,$ setting $\alpha := t, \beta := 1 - t,$ we obtain, in particular, $tx + (1 - t)y = \alpha x + \beta y \in \alpha_0 M,$ for some $\alpha_0 > 0$ with the property $\alpha_0 < \alpha + \beta = 1.$ Hence:

$$p_M (tx + (1 - t)y) < 1.$$
(b) $\Rightarrow$ (a). The set $M$ is balanced. Let us fix $x, y \in M, x \neq y, a > 0, \beta > 0$. Then $ax \in aM, \beta y \in \beta M$ and, according to Theorem 1, there exist numbers $a_1 > 0, \beta_0 > 0$ such that $a_1^2 + \beta_0^2 < 2(a^2 + \beta^2)$, $ax + \beta y \in a_1 M, ax - \beta y \in \beta_0 M$ and $a_1 < a + \beta$.

Since $M$ is a strictly convex set the following inequality holds:

$$p_M \left( \frac{ax + \beta y}{a + \beta} \right) < 1,$$

whence $p_M (ax + \beta y) < a + \beta$. If $a_1 < a + \beta$, we put $a_0 := a_1$.

On the other hand, if $a_1 = a + \beta$, we can choose $a_0 > 0$ such that

$$p_M (ax + \beta y) < a_0 < a_1 = a + \beta.$$

In both cases, the numbers $a_0$ and $\beta_0$ fulfil conditions (1), (11), (12) and (13).

The equivalence of conditions (b) and (c) is a consequence of Theorem 4.

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