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SOME FUNCTIONAL EQUATIONS RELATED TO QUADRATIC FUNCTIONS

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Abstract. It is introduced the functional equation $F(x+u)+F(x-u) = F(x+v)+F(x-v)$ for any $x, u, v \in L$, $\|u\| = \|v\| = 1$, where L is a real inner product space of dim $L \geqslant 2$. Our main result states that its only »regular« solution is essentially the norm square function $\|\cdot\|^2$.

Introduction

In this paper $(L, \|.\|)$ denotes a real linear normed space with dim $L \geqslant 2$ and $(\mathscr{A}, +)$ an abelian group such that for any $a \in \mathscr{A}$ there exists a unique $b \in \mathscr{A}$ (denoted by $\frac{1}{2} a$) with $b + b = a$.

Consider the functional equation

$$F(x+u)+F(x-u) = F(x+v)+F(x-v), \quad x, u, v \in L, \quad \|u\| = \|v\| = 1 \tag{1}$$

where $F : L \to \mathscr{A}$ is the unknown function. This equation has appeared in the following statement (see [7]): If a continuous function $F : R^2 \to R$ has the same integral on every semidisk of the unit radius, then it satisfies (1).

We also deal with the slightly modified equation

$$F(x+u)+F(x-u) = F(x+v)+F(x-v), \quad x, u, v \in L, \quad \|u\| = \|v\|;\ F(0) = 0, \tag{2}$$

and show that if the norm on L is derived from an inner product $\langle .,. \rangle : L \times L \to R$, then (2) is equivalent to the more familiar equation

$$F(x+y) = F(x) + F(y), \quad x, y \in L, \quad \langle x, y \rangle = 0. \tag{3}$$

Nonnegative or continuous solutions of (3) have been known before (see e. g. [8], [3]), but we are informed that J. Rätz has recently found

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the general solution, discussing the equation (3) on a so called »orthogonality space« (see [6]). The solutions of (3) are called orthogonally additive functions.

In Section 1 we investigate the connection between the equations (1), (2) and (3) on inner product spaces obtaining the one point continuous solutions of (1).

The case of spaces without an inner product is studied in Section 2. We also obtain the general solution of (2) without using results of [6], but with the help of the well known norm square equation (see [2])

$$F(x+y)+F(x-y)=2F(x)+2F(y), \quad x, y \in L. \tag{4}$$

We conclude that there are generally no other than additive solutions of the above equations (1)—(3) on such a space.

1. The case of inner product spaces

In this Section we assume that $(L, \langle .\,, .\rangle)$ is an inner product space with $\dim L \geqslant 2$ and the norm $\|\cdot\| = \sqrt{\langle .\,, .\rangle}$. The general solution of (3) is given by

1.1. THEOREM ([6]). *The function $F: L \to \mathscr{A}$ is orthogonally additive if and only if there exist additive mappings $a: R \to \mathscr{A}$, $A: L \to \mathscr{A}$ such that*

$$F(x) = a(\|x\|^2) + A(x), \; x \in L. \tag{1.1}$$

1.2. COROLLARY. *Equations* (2) *and* (3) *are mutually equivalent.*

Proof. The implication (3) $\Rightarrow$ (2) can be verified by a simple computation according to Theorem 1.1. The converse implication can be proved as follows. Let $x, y \in L$, $\langle x, y\rangle = 0$. Then $\left\|\frac{1}{2}(x+y)\right\| = \left\|\frac{1}{2}(x-y)\right\|$ from which it follows

$$F(x+y) = F\left(\frac{1}{2}[x+y]+\frac{1}{2}[x+y]\right) + F\left(\frac{1}{2}[x+y]-\frac{1}{2}[x+y]\right) = F\left(\frac{1}{2}[x+y]+\frac{1}{2}[x-y]\right) + F\left(\frac{1}{2}[x+y]-\frac{1}{2}[x-y]\right) = F(x)+F(y).$$

1.3. COROLLARY. *The equation* (3) *implies the equation* (1).

Now, we restrict ourselves to the solutions of (1).

1.4. LEMMA. *For any $\lambda \in R$ there is a dense additive subgroup $D_\lambda \subset R$ such that if F is a solution of* (1) *with $F(0) = 0$, then for all $w, z \in L$, $\|w\| = \|z\| = 1$, $\langle w, z\rangle = 0$,*

$$F(\lambda w + \mu z) = F(\lambda w) + F(\mu z), \quad \mu \in D_\lambda,$$

holds true.

Proof. Consider the additive subgroup D_λ generated by the set

$$P_\lambda = \{2p_\lambda^{(k)} = 2\sqrt{1 - \lambda^2/4k^2} \mid k \in N,\ k > |\lambda|/2\}.$$

Since $\lim\limits_{k \to \infty} (p_\lambda^{(k+1)} - p_\lambda^{(k)}) = 0$, the subgroup D_λ is dense in R.

Now let F be a solution of (1) with $F(0) = 0$ and $w, z \in L$, $\|w\| = \|z\| = 1$, $\langle w, z\rangle = 0$. Define the functions ${}_\lambda\Delta$ and ${}_\lambda\Delta_j^{(k)}$ ($k \in N$, $k > |\lambda|/2$, $j = 1, 2, \ldots, k$) on R by

$$\begin{gathered}{}_\lambda\Delta(\mu) = F(\lambda w + \mu z) - F(\mu z), \\ {}_\lambda\Delta_j^{(k)}(\mu) = F\left(\frac{j}{k}\lambda w + \mu z\right) - F\left(\frac{j-1}{k}\lambda w + \mu z\right), \quad \mu \in R.\end{gathered} \tag{1.2}$$

Obviously ${}_\lambda\Delta = \sum\limits_{j=1}^{k} {}_\lambda\Delta_j^{(k)}$ for all k. Now we show that the functions ${}_\lambda\Delta_j^{(k)}$ are periodic with the period $2p_\lambda^{(k)} = 2\sqrt{1 - \lambda^2/4k^2}$. Substituting into (1) the vectors

$$\left(\frac{2j-1}{2k}\lambda w + [\mu + p_\lambda^{(k)}]\, z\right), \quad \left(\frac{1}{2k}\lambda w + p_\lambda^{(k)} z\right), \quad \left(\frac{1}{2k}\lambda w - p_\lambda^{(k)} z\right)$$

for x, u, v respectively, we get the desired result

$$\begin{aligned}{}_\lambda\Delta_j^{(k)}(\mu + 2p_\lambda^{(k)}) &= F\left(\frac{j}{k}\lambda w + [\mu + 2p_\lambda^{(k)}]\, z\right) - F\left(\frac{j-1}{k}\lambda w + \right. \\ &\left. + [\mu + 2p_\lambda^{(k)}]\, z\right) = F\left(\frac{j}{k}\lambda w + \mu z\right) - F\left(\frac{j-1}{k}\lambda w + \mu z\right) = {}_\lambda\Delta_j^{(k)}(\mu).\end{aligned}$$

Thus ${}_\lambda\Delta$ is periodic with the period $2p_\lambda^{(k)}$ for all k, and hence any $\mu \in D_\lambda$ is a period of ${}_\lambda\Delta$. This means that

$${}_\lambda\Delta(\mu) = {}_\lambda\Delta(0) = F(\lambda w) - F(0) = F(\lambda w), \quad \mu \in D_\lambda.$$

Equating this with (1.2) completes the proof.

1.5. LEMMA. *If F is a solution of* (1), *then for any $y, w, z \in L$, $\|w\| = \|z\| = 1$, $\langle w, z\rangle = 0$, it holds*

$$F(\lambda w + \mu z + y) = F(\lambda w + y) + F(\mu z + y) - F(y), \ \mu \in D_\lambda. \quad (1.3)$$

Proof. Define the function $G_y : L \to \mathscr{A}$ by

$$G_y(x) = F(x + y) - F(y), \ x \in L.$$

A simple computation shows that G_y is a solution of (1) with $G_y(0) = 0$. Thus by Lemma 1.4, for every $w, z \in L$, $\|w\| = \|z\| = 1$, $\langle w, z\rangle = 0$ it follows that

$$G_y(\lambda w + \mu z) = G_y(\lambda w) + G_y(\mu z), \ \mu \in D_\lambda.$$

Hence, according to the definition of G_y the lemma is proved.

1.6. THEOREM. *Let $\mathscr{A}$ be equipped with a Hausdorff topology for which the operations $a + b$, $-a$, $\frac{1}{2}a$ are continuous. If a solution F of* (1) *is continuous at* 0 *with $F(0) = 0$, then F is orthogonally additive.*

Proof. Let a solution F of (1) be continuous at 0 with $F(0) = 0$ and let $x, y \in L$, $\langle x, y\rangle = 0$. Set $\|x\| = \lambda$, $\|y\| = \mu$, $\lambda^{-1}x = w$, $\mu^{-1}y = z$. Choose a sequence $\lambda_n \in D_\mu$, $\lambda_n \to \lambda$ and then another sequence $\mu_n \in D_{\lambda - \lambda_n}$, $\mu_n \to \mu$. Set $y'_n = [\lambda - \lambda_n]w$ and $y''_n = [\mu - \mu_n]z$. Then by the previous lemma

$$F(\lambda w + \mu z) = F(\lambda_n w + \mu z + y'_n) =$$

$$= F(\lambda_n w + y'_n) + F(\mu z + y'_n) - F(y'_n) =$$

$$= F(\lambda w) + F([\lambda - \lambda_n]w + \mu_n z + y''_n) - F(y'_n) =$$

$$= F(\lambda w) + F([\lambda - \lambda_n]w + y''_n) + F(\mu_n z + y''_n) - F(y''_n) - F(y'_n) =$$

$$= F(\lambda w) + F(\mu z) + F(y'_n + y''_n) - F(y'_n) - F(y''_n).$$

Letting $n \to \infty$, the continuity at 0 completes the proof.

1.7. COROLLARY. *If the solution F of* (1) *is continuous at* 0, *then it has the form*

$$F(x) = c(\|x\|^2) + C(x) + b, \quad x \in L \quad (1.4)$$

where $c : R \to \mathscr{A}$, $C : L \to \mathscr{A}$ are continuous additive mappings and $b \in \mathscr{A}$.

Proof. By Theorems 1.6 and 1.1, the function $F - F(0)$ can be written in the form (1.1). It is clear that a and A are continuous functions at 0. Because of their additivity, they are continuous everywhere, so if we denote them by c and C respectively, the statement is obtained with $b = F(0)$.

1.8. THEOREM (Main result). *Let $\mathscr{A} = R$, then the function $F = \|\cdot\|^2$ is the only solution of* (1) *which is continuous at* 0 *with* $F(0) = 0$ *and* $F(u) = 1$ *for all* $u \in L$, $\|u\| = 1$.

Proof. By Corollary 1.7, for every $u \in L$, $\|u\| = 1$ we have

$$C(u) = \frac{1}{2}(F(u) - F(-u)) = 0.$$

Thus C is equal to zero by its linearity. Furthermore $b = F(0) = 0$ and c is a continuous solution of the Cauchy equation, and so it has the form $c(\lambda) = c(1) \cdot \lambda$. Here $c(1) = c(\|u\|^2) = F(u) = 1$ (see [6]).

1.9. *Remark.* In each of the statements proved above, the continuity at 0 can be replaced by the continuity at any point of L, and it implies the continuity on the whole L. Indeed, if the solution F of (1) is continuous at $y \in L$, then the function G_y defined in the proof of Lemma 1.5 is of the form (1.4). Thus G_y is continuous on the whole L, and so is F.

2. The case of normed linear spaces

In what follows we consider the equations (1)—(3) on a real normed linear space $(L, \|\cdot\|)$, $\dim L \geqslant 2$. For (3) to make a sense on such a space, it is necessary to define an »orthogonality relation« in L which turns it into an orthogonality space. Because of the lack of such a natural concept, several ones have been introduced (see e. g. [4], [9]). In [8] the following concept of orthogonality is considered: The elements x and y of L are said to be orthogonal (shortly $x \perp y$), if $\|x\| \leqslant \|x + \lambda y\|$ for all $\lambda \in R$.

Anyway, the additive functions will solve (3) as well as (1) and (2). However, nontrivial continuous even solutions do not exist in general. Namely, we can quote here the result obtained by K. Sundaresan.

2.1. THEOREM ([8]). *A nontrivial even continuous orthogonally* ($\perp$) *additive real valued function on* L *can exist only if* L *is an inner product space.*

2.2. *Remark.* In 1980 J. Rätz found the following unpublished result, which generalizes Theorem 2.1 in a certain sense: If $L = R^2$*) and the norm on L is such that the unit ball is a polygon, then any even orthogonally ($\perp$) additive mapping from L into an abelian group $\mathscr{A}$ is identically zero.

Concerning the equation (1), we can only state that the continuous even solutions found in Corollary 1.7 do not satisfy it on spaces without inner product. Also we can show an example of a normed linear space for which the assertion of Theorem 2.1 holds. We shall make use

*) Since then it has appeared: J. Rätz, On orthogonally additive mappings II, Proc. of 19th International Symp. on Functional Equations, Nantes, (1981), 39.

of the following characterization of inner product spaces and the general solution of a version of the norm square equation (4).

2.3. LEMMA ([5]). *The normed linear space L is an inner product space with the same norm if and only if there is a fixed constant $\gamma \neq 0, \pm 1$ such that if $x, y \in L$ and $\|x\| = \|y\|$ then $\|\gamma x + y\| = \|x + \gamma y\|$.*

2.4. LEMMA. *A function $F : L \to \mathscr{A}$ satisfies the equation*

$$F(x+y) + F(x-y) = 2F(x) + F(y) + F(-y), \quad x, y \in L \quad (2.1)$$

if and only if there exist a biadditive (additive in both variables separately) $B : L \times L \to \mathscr{A}$ and an additive mapping $A : L \to \mathscr{A}$ such that

$$F(x) = B(x, x) + A(x), \quad x \in L. \quad (2.2)$$

Proof. Denoting by F_e and F_o the even and odd parts of F respectively, we have from (2.1)

$$F_e(x+y) + F_o(x+y) + F_e(x-y) + F_o(x-y) =$$
$$= 2F_e(x) + 2F_o(x) + 2F_e(y)$$
$$F_e(x+y) - F_o(x+y) + F_e(x-y) - F_o(x-y) = 2F_e(x) - 2F_o(x) +$$
$$+ 2F_e(y).$$

Summing up these equations, we obtain the equation (4) in F_e, therefore it has the required form $B(.,.)$ (see [2]). Now subtracting them we have

$$F_o(x+y) + F_o(x-y) = 2F_o(x).$$

It gives that $2F_o(x) = F_o(2x)$ by taking $x = y$. Then choosing $w, z \in L$ arbitrarily and $x = \frac{1}{2}(w+z)$, $y = \frac{1}{2}(w-z)$, the additivity of F_o is proved.

2.5. THEOREM. *If the function $c(\|\cdot\|^2)$ is a solution of (1) on L, where $c : R \to \mathscr{A}$ is a non-zero continuous additive mapping, then L is an inner product space.*

Proof. First we show that c is a one-to-one mapping. If it would not be so, then there were real numbers $\nu_1 \neq \nu_2$ such that $c(\nu_1) = c(\nu_2)$. Thus with $\nu = \nu_2 - \nu_1$ we have $c(\nu) = 0$, and by the additivity, c is periodic with the period $\nu \neq 0$. Since $c(\nu/2) + c(\nu/2) = c(\nu) = 0$ and the decomposition $0 + 0 = 0$ in $\mathscr{A}$ is unique, we obtain $c(\nu/2) = 0$ and $\nu/2$ is also a period of c. By induction we see that the numbers $\nu/2^k$, $k \in N$, are all periods of c. Since $\lim_{k\to\infty} \nu/2^k = 0$, c is constant on a dense subset of R, and by the continuity, c is identically zero on R, which is a contradiction.

Now let $x, y \in L$, $\|x\| = \|y\| = \lambda \neq 0$ and $u = \lambda^{-1} x$, $v = \lambda^{-1} y$. Then the equation (1) for $c(\|\cdot\|^2)$ at $(u + v)$, u, v gives the equality

$$c(\|2u + v\|^2) + c(\|v\|^2) = c(\|u + 2v\|^2) + c(\|u\|^2).$$

Since c is one-to-one, it follows $\|2u + v\| = \|u + 2v\|$. Consequently the sufficient condition of Lemma 2.3 holds with $\gamma = 2$.

2.6. THEOREM. *Let* $L = R^2$ *with the norm* $\|(\lambda, \mu)\| = |\lambda| + |\mu|$. *Then the equation* (1) *has only trivial continuous even solutions with values in* R.

Proof. First, in a similar way as in the proof of Lemma 1.4, we show that for every continuous solution F of (1) with $F(0, 0) = 0$ (this may be supposed) it holds

$$F(\lambda, \mu) = F(\lambda, 0) + F(0, \mu), \quad (\lambda, \mu) \in R^2 \tag{2.3}$$

(Obviously this does not mean the orthogonal additivity in any sense). Define the functions ${}_\lambda\Delta$ and ${}_\lambda\Delta_j^{(k)}$ on R by

$$\begin{aligned} {}_\lambda\Delta(\mu) &= F(\lambda, \mu) - F(0, \mu), \quad \mu \in R \\ {}_\lambda\Delta_j^{(k)}(\mu) &= F\left(\frac{j}{k}\lambda, \mu\right) - F\left(\frac{j-1}{k}\lambda, \mu\right), \end{aligned} \tag{2.4}$$

$(k \in N,\ k > |\lambda|/2,\ j = 1, 2, \ldots, k)$. Evidently for all k ${}_\lambda\Delta = \sum_{j=1}^{k} {}_\lambda\Delta_j^{(k)}$. Now we show that the functions ${}_\lambda\Delta_j^{(k)}$, $j = 1, 2, \ldots, k$, are periodic with the period $2p_\lambda^{(k)} = 2(1 - |\lambda|/2k)$. Setting

$$x = \left(\frac{2j-1}{2k}\lambda, \mu + p_\lambda^{(k)}\right), \quad u = \left(\frac{1}{2k}\lambda, p_\lambda^{(k)}\right), \quad v = \left(\frac{1}{2k}\lambda, -p_\lambda^{(k)}\right)$$

into (1), we get

$$\begin{aligned} {}_\lambda\Delta_j^{(k)}(\mu + 2p_\lambda^{(k)}) &= F\left(\frac{j}{k}\lambda, \mu + 2p_\lambda^{(k)}\right) - F\left(\frac{j-1}{k}\lambda, \mu + 2p_\lambda^{(k)}\right) = \\ &= F\left(\frac{j}{k}\lambda, \mu\right) - F\left(\frac{j-1}{k}\lambda, \mu\right) = {}_\lambda\Delta_j^{(k)}(\mu). \end{aligned}$$

Hence ${}_\lambda\Delta$ is periodic with the period $2p_\lambda^{(k)}$ and also with $2(p_\lambda^{(k+1)} - p_\lambda^{(k)})$ for all k. Since $\lim_{k\to\infty}(p_\lambda^{(k+1)} - p_\lambda^{(k)}) = 0$, ${}_\lambda\Delta$ is constant on a dense subgroup of R, and because if its continuity, it is constant on R:

$${}_\lambda\Delta(\mu) = {}_\lambda\Delta(0) = F(\lambda, 0) - F(0, 0) = F(\lambda, 0), \quad \mu \in R.$$

This and (2.4) together imply (2.3).

Let the functions f and g be defined by

$$f(\lambda) = F(\lambda, 0), \ g(\mu) = F(0, \mu), \quad \lambda, \mu \in R.$$

If F is even, then so are f and g. These functions satisfy the well-known quadratic equation (4), i. e. for f

$$f(\lambda + \xi) + f(\lambda - \xi) = 2f(\lambda) + 2f(\xi), \quad \lambda, \xi \in R. \tag{2.5}$$

If $|\xi| < 1$, then (2.5) is a direct consequence of (1) and (2.3). Indeed, for $\eta = 1 - |\xi|$ we have

$$2[f(\xi) + g(\eta)] = F(\xi, \eta) + F(-\xi, -\eta) = F(0, 1) + F(0, -1) =$$
$$= 2g(1) = 2[f(\lambda) + g(1)] - 2f(\lambda) = F(\lambda, 1) + F(\lambda, -1) - 2f(\lambda) =$$
$$= F(\lambda + \xi, \eta) + F(\lambda - \xi, -\eta) - 2f(\lambda) =$$
$$= [f(\lambda + \xi) + g(\eta)] + [f(\lambda - \xi) + g(\eta)] - 2f(\lambda).$$

Now suppose that (2.5) holds for $|\xi| < n \in N$. Let $n \leqslant \xi < n + 1$. Then by (1), (2.3) and our assumption

$$f(\lambda + \xi) + f(\lambda - \xi) - 2f(\lambda) = [2f(\lambda + \xi - 1) + 2g(1) - f(\lambda +$$
$$+ \xi - 2)] + [2f(\lambda - \xi + 1) + 2g(1) - f(\lambda - \xi + 2)] - 2f(\lambda) =$$
$$= 4[f(\lambda) + f(\xi - 1)] - 2[f(\lambda) + f(\xi - 2)] + 4g(1) - 2f(\lambda) =$$
$$= 4f(\xi - 1) - 2f(\xi - 2) + 4g(1).$$

It is seen that $f(\lambda + \xi) + f(\lambda - \xi) - 2f(\lambda)$ does not depend on λ, thus choosing $\lambda = 0$, we obtain (2.5) for $|\xi| < n + 1$. This proves (2.5) by induction.

By Lemma 2.4, it is clear that the continuous solutions f and g of (2.5) are of the form

$$f(\lambda) = c' \lambda^2, \ g(\mu) = c'' \mu^2, \quad \lambda, \mu \in R.$$

Since F satisfies (1), we have

$$c' = f(1) = F(1, 0) = F(0, 1) = g(1) = c''.$$

Finally from (1) with $c = c' = c''$

$$c = f(1) + g(0) = f\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right) = \frac{1}{4} c + \frac{1}{4} c = \frac{1}{2} c$$

which gives $c = 0$. This completes the proof.

2.7. *Remark.* It is not difficult to see that the assertion of Theorem 2.6 holds for functions continuous only at a single point.

In the rest of the paper we reduce the problem of equation (2) to the norm square equation.

2.8. THEOREM. *The equation* (2) *implies the modified norm square equation* (2.1).

Proof. Let $F : L \to \mathscr{A}$ be a solution of (2), and $x, y \in L$, $\|y\| \leqslant \|x\|$. Consider the connected sphere

$$Z_y = \{z \in L \mid \|z\| = \|y\|\}$$

and define the function $D : Z_y \to R$ by

$$D(z) = \|x + z\| - \|x - z\|, \quad z \in Z_y.$$

For $z_1 = \frac{\|y\|}{\|x\|} x$ and $z_2 = -z_1$ we have $z_1, z_2 \in Z_y$, and

$$D(z_1) = \left|1 + \frac{\|y\|}{\|x\|}\right| \|x\| - \left|1 - \frac{\|y\|}{\|x\|}\right| \|x\| = 2\|y\| \geqslant 0$$

$$D(z_2) = D(-z_1) = -D(z_1) \leqslant 0.$$

Since D is continuous, there is an element $z_0 \in Z_y$ such that $D(z_0) = 0$. Let $u = \frac{1}{2}(x + z_0)$, $v = \frac{1}{2}(x - z_0)$. It is clear that $\|u\| = \|v\|$, so the equation (2) at u, u, v and v, v, u is of the form

$$F(x + z_0) + F(0) = F(x) + F(z_0)$$

and

$$F(x - z_0) + F(0) = F(x) + F(-z_0),$$

respectively. Finally from these equations we get the desired result by using (2) with $\|y\| = \|z_0\|$:

$$F(x + y) + F(x - y) = F(x + z_0) + F(x - z_0) =$$
$$= F(x) + F(z_0) + F(x) + F(-z_0) = 2F(x) + F(y) + F(-y).$$

2.9. COROLLARY. *If L is an inner product space, then a function F is a solution of the equation* (2) *if and only if there exist additive mappings $a : R \to \mathscr{A}$ and $A : L \to \mathscr{A}$ such that*

$$F(x) = a(\|x\|^2) + A(x), \quad x \in L. \tag{2.6}$$

Proof. By Theorem 2.8 and Lemma 2.4, F can be written in the form (2.2). Obviously

$$B(x, x) = \frac{1}{2}(F(x) + F(-x)), \quad x \in L,$$

thus $B(x, x) = B(y, y)$ if $x, y \in L$, $\|x\| = \|y\|$. Now let $a : R \to \mathscr{A}$ be defined by

$$a(\lambda) = \operatorname{sgn}(\lambda)\, B(\sqrt{|\lambda|}\, u, \sqrt{|\lambda|}\, u), \quad \lambda \in R$$

where $u \in L$ is arbitrary with $\|u\| = 1$. It follows that for any $x \in L$ $B(x, x) = a(\|x\|^2)$. To show the additivity of a let $\lambda, \mu \in R_+$ and $x, y \in L$ with $\|x\|^2 = \lambda$, $\|y\|^2 = \mu$, $\langle x, y\rangle = 0$. Then by the equation (2) for $a(\|\cdot\|^2)$ with $u = \frac{1}{2}(x + y)$, $v = \frac{1}{2}(x - y)$ at u, u, v we have

$$a(\lambda + \mu) = a(\|x + y\|^2) = a(\|x\|^2) + a(\|y\|^2) = a(\lambda) + a(\mu).$$

2.10. THEOREM. *If L is not an inner product space, then the solutions of the equation* (2) *are exactly the additive functions $A : L \to \mathscr{A}$.*

Proof. By Lemma 2.4 and Theorem 2.8, any solution F of (2) has the form (2.2). We show that B is identically zero. Notice that by (2) for the even part F_e of F

$$B(u, u) = F_e(u) = \frac{1}{2}(F_e(0 + u) + F_e(0 - u)) =$$

$$= \frac{1}{2}(F_e(0 + v) + F_e(0 - v)) = F_e(v) = B(v, v) \qquad (2.7)$$

for all $u, v \in L$, $\|u\| = \|v\|$.

Further, by Lemma 2.3, there are elements u', $v' \in L$ such that $\|u'\| = \|v'\|$ and $\|2u' + v'\| < \|u' + 2v'\|$. Consider the connected sphere

$$U = \{u \in L \mid \|u\| = \|u'\|\}$$

and define the continuous function $q : U \to R$ by

$$q(u) = \frac{\|2u + v'\|}{\|u + 2v'\|}, \quad u \in U.$$

Clearly $q(v') = 1 > q(u')$, thus there is an element $u_0 \in U$ such that $k/l = q(u_0) < 1$ is rational. Setting $v_0 = v'$, $x = 2u_0 + v_0$, $y = u_0 + 2v_0$ for any $w \in L$, $\lambda = \|w\|/\|x\|$ we have $\|w\| = \|\lambda x\|$ and so by (2.7) $F_e(w) = F_e(\lambda x)$. Further the equation (2) at the points $\lambda(u_0 + v_0)$, λu_0, λv_0 gives

$$F_e(\lambda x) + F_e(\lambda v_0) = F_e(\lambda[u_0 + v_0] + \lambda u_0) + F_e(\lambda v_0) =$$

$$= F_e(\lambda[u_0 + v_0] + \lambda v_0) + F_e(\lambda u_0) = F_e(\lambda y) + F_e(\lambda u_0).$$

Hence we have $F_e(w) = F_e(\lambda x) = F_e(\lambda y)$ regarding the equality $\|\lambda u_0\| = \|\lambda v_0\|$ and (2.7). Then according to $\|l\lambda x\| = \|k\lambda y\|$, (2.7) and the biadditivity of B imply that

$$l^2 F_e(w) = l^2 F_e(\lambda x) = l^2 B(\lambda x, \lambda x) = B(l\lambda x, l\lambda x) =$$
$$= B(k\lambda y, k\lambda y) = k^2 B(\lambda y, \lambda y) = k^2 F_e(\lambda y) = k^2 F_e(w).$$

Thus for any $w \in L$ $(l^2 - k^2) F_e(w) = 0$ holds, and regarding the inequalities $q(u_0) < k/l \leqslant (l-1)/l < l/(l+1) < 1 = q(v_0)$ we obtain the following particular equalities

$$(2l - 1) F_e(w) = 0, \qquad (2l + 1) F_e(w) = 0.$$

Finally substracting them we have $2 F_e(w) = 0$ and so $F_e(w) = 0$ for any $w \in L$ because of the uniqueness of the 2-division in $\mathscr{A}$.

2.11. COROLLARY. *A real normed linear space L is an inner product space with the same norm if and only if the equation* (2) *allows a nonadditive solution on L.*

2.12. *Remark.* The problem whether the equation (1) has a solution of the form (2.2) different from (1.1) on an inner product space L, is open. However, we notice that for a function F derived from a biadditive mapping $B: L \times L \to \mathscr{A}$ to be a solution of (1), it is necessary and sufficient that $B(u, u) =$ constant holds for $u \in L$, $\|u\| = 1$.

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NEKE FUNKCIONALNE JEDNADŽBE KOJE SE ODNOSE NA KVADRATNE FUNKCIJE

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Sadržaj

U radu je uvedena funkcionalna jednadžba $F(x+u)+F(x-u)=$ $=F(x+v)+F(x-v)$, $x, u, v \in L$, $\|u\|=\|v\|=1$, gdje je L realan prostor sa skalarnim produktom i $\dim L \geq 2$. Glavni rezultat kaže da je njeno jedino »regularno« rješenje u biti kvadrat norme tj. $\|\cdot\|^2$.