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ON THE EMBEDDING OF A COMMUTATIVE RING INTO A 0-DIMENSIONAL COMMUTATIVE RING

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Abstract. Throughout this paper rings are understood to be commutative with unity, and subrings are understood to have same identity as their overrings. It is well known that the complete ring of quotients $Q(R)$ of a commutative semiprime ring R is a regular ring. Sometimes it is useful to embed a semiprime ring R into a regular ring, for example see [3]. This paper gives necessary and sufficient conditions that a commutative ring R can be embedded into a 0-dimensional overring. These conditions are applied to certain classes of commutative rings in order to conclude that for each ring in these classes there exists a 0-dimensional overring. It is constructed a commutative ring having no 0-dimensional overring.

A commutative ring with unity will be denoted by R. We will also use the symbol $T(R)$ when we want to emphasize the fact that $T(R)$ is a total quotient ring, i. e. a ring in which every regular element is invertible.

We next recall the following definitions.

Definition 1. A commutative ring R is called semiprime if R contains no nonzero nilpotent elements.

Definition 2. A commutative ring R is called regular if for $a \in R$ there exists $a' \in R$ such that $a^2 a' = a$.

Definition 3. A commutative ring R is called π-regular if for $a \in R$ there exist $a' \in R$ and a positive integer n such that $a^n = (a^n)^2 a'$.

Definition 4. A commutative ring R is called 0-dimensional if every prime ideal of R is a maximal ideal of R.

It is well known that R is a regular ring if and only if R is a semiprime 0-dimensional ring and that R is a π-regular ring if and only if R is a 0-dimensional ring.

We know that a domain D has a regular overring. Namely, every field containing D is a regular overring of D. The quotient field of D is a minimal regular overring of D. Let R be a semiprime ring. It is known that R has a regular overring. For example, it is known that the complete quotient ring $Q(R)$ of R is a regular overring of R. ([4], 2.4. Prop. 1.). It is easy to construct a regular overring of R. Namely,

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let $\{P_\lambda\}$ be the set of the prime ideals of R. Then $\bigcap_\lambda P_\lambda = (0)$ and we can realize R as a subdirect product of the domains $\{R/P_\lambda\}$. Let $K_\lambda = Q(R/P_\lambda)$ be the quotient field of R/P_λ, for every λ. Then $R \subseteq \prod_\lambda K_\lambda$ and $\prod_\lambda K_\lambda$ is, as a direct product of fields, a regular ring. Sometimes, it is useful to embed R into a regular overring (see [3]).

In the paper we ask the following question. When can we embed a commutative ring R into a 0-dimensional overring?

Let R be a commutative ring. Theorem 7 characterizes those rings that have a 0-dimensional overring. Theorem 1 and 2, Proposition 3 and 5 and Lemma 6 give the preliminary facts needed for the proof of Theorem 7.

THEOREM 1. *Let $\{R_\lambda\}$ be a collection of regular rings. Then $\prod_\lambda R_\lambda$ is also a regular ring. Let $\{R\}_{i=1}^n$ be a finite collection of 0-dimensional rings. Then $\prod_{i=1}^n R_i$ is also a 0-dimensional ring.*

Proof. This result is immediate.

THEOREM 2. *Let R be a commutative ring and $a \in R$, $a \neq 0$. Let $\{I_\lambda\}$ be the set of ideals of R with the property that $a \notin I_\lambda$, for every λ and let A denote a maximal element of $\{I_\lambda\}$ with respect to inclusion ($\subseteq$). Let $T(R/A)$ be the total quotient ring of R/A. The ring $T(R/A)$ is a quasi-local ring and $\bar{a}\bar{M} = 0$ for the maximal ideal $\bar{M}$ of $T(R/A)$ and $\bar{a} = a + A \in T(R/A)$. Furthermore $\bar{a}$ is contained in every nonzero ideal of $T(R/A)$. If R is a Noetherian ring, then the maximal ideal $\bar{M}$ of $T(R/A)$ is nilpotent.*

Proof. Let $\bar{a} = a + A \in T(R/A)$ and $\bar{b} \in T(R/A)$, $\bar{b} \neq 0$. It is easy to prove that $(\bar{a}) \subseteq (\bar{b})$. Let $\bar{x}$ be a zero divisor of $T(R/A)$ (if one exists) and let $\bar{y}$ be a nonzero element of $T(R/A)$ such that $\bar{x}\bar{y} = 0$. Since $\bar{a} \in (\bar{y})$, it follows that $\bar{x}\bar{a} = 0$. Hence $T(R/A)$ is a quasi-local ring and $\bar{a}\bar{M} = 0$ for the maximal ideal $\bar{M}$ of $T(R/A)$. Furthermore, if $\bar{M}$ contains a non-nilpotent element, then $a \in \bigcap_{n=1}^{\infty} \bar{M}^n$. If R is a Noetherian ring, $T(R/A)$ is also a Noetherian ring, hence $\bigcap_{n=1}^{\infty} \bar{M}^n = (0)$, and therefore $\bar{a} \notin \bigcap_{n=1}^{\infty} \bar{M}^n$ and $\bar{M}$ is a nilpotent ideal of $T(R/A)$.

PROPOSITION 3. *Let R be a ring, $a \in R$, $a \neq 0$ and let $\{I_\lambda\}$ be the set of ideals of R with the property that $a \notin I_\lambda$, for every λ. Let A be a maximal element of $\{I_\lambda\}$. If every zero divisor of $T(R/A)$ is nilpotent, then A is a primary ideal of R. If R is a Noetherian ring or a 0-dimensional ring, then A is a primary ideal of R.*

Proof. Clearly, if every zero divisor of $T(R/A)$ is nilpotent then A is a primary ideal of R. Hence, if R is a Noetherian ring then A is a primary ideal of R. If R is a 0-dimensional ring, then R/A is as a homomorphic image of a 0-dimensional ring also a 0-dimensional ring, and by the preceding theorem R/A is a quasi-local ring, hence every zero divisor of R/A is nilpotent. Therefore A is a primary ideal of R.

COROLLARY 4. *Let R be a 0-dimensional ring and let $\{Q_\lambda\}$ be the set of primary ideals of R. Then $\bigcap_\lambda Q_\lambda = (0)$.*

PROPOSITION 5. *Let R be a 0-dimensional ring and let $\{M_\lambda\}$ be the set of the maximal ideals of R. Then $S_{M_\lambda}(0) = \operatorname{Ker}(R \to R_{M_\lambda})$ is the minimal M_λ-primary ideal of R, for every λ. An ideal A of R is M_λ-primary if and only $S_{M_\lambda}(0) \subseteq A \subseteq M_\lambda$, furthermore $\bigcap_\lambda S_{M_\lambda}(0) = (0)$.*

Proof. The proof follows easily.

LEMMA 6. *Let $\bar{R}$ be a commutative ring and let R be a subring of $\bar{R}$. If Q is a P-primary ideal of $\bar{R}$ then $Q \cap R$ is a $P \cap R$-primary ideal of R.*

Proof. The proof is immediate.

THEOREM 7. *Let R be a commutative ring. Then there exists a 0-dimensional overring $\bar{R}$ of R if and only if there exists a family $\{Q_\lambda\}$ of primary ideals of R satisfying the properties: 1) $\bigcap_\lambda Q_\lambda = (0)$ and 2) if $a \in R$ then there exists a positive integer n_a such that $a^{n_a} \notin \bigcup_\lambda (P_\lambda \setminus Q_\lambda)$, where P_λ is the prime ideal of R associated with Q_λ, for every λ.*

Proof. Let R be a commutative ring having a 0-dimensional overring $\bar{R}$. Let $\{M_\lambda\}$ be the set of prime ideals of $\bar{R}$ and let $S_{M_\lambda}(0) = \operatorname{Ker}(\bar{R} \to \bar{R}_{M_\lambda})$, for every λ. Let $a \in R$. Since $\bar{R}$ is a 0-dimensional ring there exists an idempotent element e of $\bar{R}$ such that $a + (1 - e)$ is a regular and $a(1 - e)$ is a nilpotent element of $\bar{R}$. ([1], Theorem 6). Let $\{M_a^\lambda\}$ be the set of prime ideals of $\bar{R}$ containing a. Since $a \in M_\alpha$, it follows that $1 - e \notin M_\alpha$, for every α. Consider the ring $\bar{R}_S$, where $S = \bar{R} \setminus (\bigcup_\alpha M_\alpha)$ and let a' be the image of a in $\bar{R}_S$. $(1 - e')$ is the unity of $\bar{R}_S$ and since $(1 - e')a'$ is a nilpotent element of $\bar{R}_S$, it follows that a' is also a nilpotent element of $\bar{R}_S$, hence there exists a positive integer n_a such that $a^{n_a} \in \bigcap_\alpha S_{M_\alpha}(0)$. Clearly, $S_{M_\lambda}(0) \cap R$ is a $(M_\lambda \cap R)$-primary ideal of R for every λ and $\bigcap_\lambda (S_{M_\lambda}(0) \cap R) = (0)$, furthermore,

$$a^{n_a} \notin \bigcup_{\lambda} ((R \cap M_\lambda) \setminus (R \cap S_{M_\lambda}(0))).$$

Conversely, let $\{Q_\lambda\}$ be a set of primary ideals of R satisfying the conditions 1) $\bigcap Q_\lambda = (0)$ and 2) if $a \in R$ then there exists a positive integer a_a such that $a^{n_a} \notin \bigcup_{\lambda} (P_\lambda \setminus Q_\lambda)$, where P_λ is the prime ideal of R associated with Q_λ, for every λ. Consider the ring $\overline{R} = \prod_{\lambda} (R/Q_\lambda)_{P/Q_\lambda}$. Let $r \in R$ and let $\bar{r} = \prod_{\lambda} (r + Q) \in \prod_{\lambda} (R/Q_\lambda)_{P_\lambda/Q_\lambda}$. The mapping $\varphi(r) = \bar{r}, r \in R$, is a monomorphism on R into $\overline{R}$. If we identify R with $\varphi(R)$, $\overline{R}$ is an overring of R. Let $a \in R$ and let e_a be the idempotent element of $\overline{R}$ having as its component the unity in those places λ where $a \notin P_\lambda$ and zero in the remaining places. It is easy to verify that $a + (1 - e_a)$ is a regular and $a(1 - e_a)$ is a nilpotent element of $\overline{R}$. Let R_1 be the subring of $\overline{R}$ generated by R and $\{e_a \mid a \in R\}$. It is easy to see that the total quotient ring $T(R_1)$ of R_1 is a 0-dimensional overring of R. ([2], see the proof of Theorem 7).

Now we construct a commutative ring having no 0-dimensional overring.

THEOREM 8. *Let R be a total quotient ring having the following properties*: 1° *R contains a non-nilpotent zero-divisor*; 2° *there exists $a \in R$, $a \neq 0$ such that*: *$xa = 0$ and $(x) \supseteq (a)$ for every zero-divisor $x \in R$, $x \neq 0$. Then R has no 0-dimensional overring.*

Proof. Assume that there exists a 0-dimensional overring $\overline{R}$ of R. Let x be a non-nilpotent zero-divisor of R. Since $\overline{R}$ is a 0-dimensional ring there exists an idempotent element e of R such that $x + (1 - e)$ is a regular and $x(1 - e)$ is nilpotent element of $\overline{R}$. Therefore $x^n(1 - e) = 0$ for some positive integer n, i. e. $x^n = x^n e$. Since $x^n \neq 0$, 2° implies $(x^n) \supseteq (a)$, i. e. $a = x^n y = ex^n y$ for some $y \in R$. Since $a = ex^n y$, it follows that $a(1 - e) = 0$. Therefore $a[x + (1 - e)] = 0$ and $x + (1 - e)$ is a regular element of $\overline{R}$. This is a contradiction. Therefore there does not exist a 0-dimensional overring of R. This theorem can also be proved using Theorem 7. Namely, let $\{Q_\lambda\}$ be the set of primary ideals of R. It is easy to conclude that $\bigcap_{\lambda} Q_\lambda \neq (0)$. Therefore Theorem 7 shows that R has no 0-dimensional overring.

In the following example we shall construct a ring having the properties 1° and 2° of the preceding theorem.

Example. Let V be a valuation domain, dim $V > 1$. Let M be the maximal ideal of V and let P be a prime ideal of V, $P \neq M$. Let $a \in P$, $a \neq 0$, nad let A be the maximal ideal of V with the property that $a \notin A$. ($A = \{x \in V \mid v(x) > v(a)\}$). It is easy to prove that the ring $R = V/A$ satisfies the properties 1° and 2° of the preceding theorem. Here $\bar{a} = a + A \in V/A$ plays the role of the element a in 2° in the preceding theorem.

We now apply Theorem 7 to certain classes of commutative rings in order to conclude that for each ring in these classes there exists a 0-dimensional overring.

THEOREM 9. *Let R be a commutative ring such that the ideal* (0) *admits a primary decomposition. Then there exists a* 0-*dimensional overring of* R.

Proof. Since (0) admits a primary decomposition, $(0) = \bigcap_{i=1}^{n} Q_i$, where Q_i is a P_i-primary ideal, $i = 1, 2, \ldots, n$. $R \subseteq \bar{R} = \prod_{i=1}^{n} (R/Q_i)_{P_i/Q_i}$ and $\bar{R}$ is, as a finite product of 0-dimensional rings, a 0-dimensional overring of R.

COROLLARY 10. *Let R be a Noetherian ring. Since* (0) *admits a primary decomposition, R has a* 0-*dimensional overring.*

It is known that if R is a Noetherian ring then $T(R) = Q(R)$, where $T(R)$ is the total quotient ring and $Q(R)$ is the complete quotient ring of R. Therefore if $T(R)$ is a total quotient Noetherian ring that is not 0-dimensional (i. e. that it is not artinian), then its complete quotient ring is not 0-dimensional. It follows by Corollary 10 that there exists a commutative ring R having a 0-dimensional overring and the complete quotient ring $Q(R)$ of R is not a 0-dimensional ring.

THEOREM 11. *Let R be a commutative ring and let N be the nil-radical of R. If there exist primary ideals $\{Q_i\}_{i=1}^{n}$ such that $(\bigcap_{i=1}^{n} Q_i) \cap \cap N = (0)$, then R has a* 0-*dimensional overring.*

Proof. Let P_i denote the prime ideal of R associated with Q_i ($i = 1, 2, \ldots, n$) and let $\{P_\lambda\}$ be the set of minimal prime deals of R. Let $Q(R/P_\lambda)$ be the quotient field of the domain R/P_λ, for every λ. Then $R \subseteq \bar{R} = (\prod_{i=1}^{n} (R/Q_i)_{P_i/Q_i}) \oplus (\prod_{\lambda} Q(R/P_\lambda))$. $\prod_{i=1}^{n} (R/Q_i)_{P_i/Q_i}$ is, as a finite direct product of 0-dimensional rings, a 0-dimensional ring and $\prod_{\lambda} Q(R/P_\lambda)$ is, as a direct product of fields, a regular ring. Furthermore $\bar{R}$ is, as a direct product of two 0-dimensional rings, a 0-dimensional ring.

Example. Let R be a commutative semiprime ring and let M be a maximal ideal of R. If we define multiplication by the formula $(r, m)(r', m') = (rr', rm' + r'm)$ in the direct sum $R \oplus R/M$ of the additive abelian groups R and R/M, then $R \oplus R/M$ is a ring, R/M is the nilradical and $M \oplus R/M$ is a maximal ideal of $R \oplus R/M$. Since

$(M \oplus R/M)^2 = M^2$, $(M \oplus R/M)^2 \cap R/M = (0)$. Therefore the ring $R \oplus R/M$ satisfies the hypotheses of the preceding theorem.

THEOREM 12. *Let R be a ring having a 0-dimensional overring $\overline{R}$ and let $\{X_\lambda\}$ be a set of indeterminates over R. Then the polynomial ring $R[\{X_\lambda\}]$ also has a 0-dimensional overring.*

Proof. Clearly, $R[\{X_\lambda\}] \subseteq \overline{R}(\{X_\lambda\}] \subseteq T(\overline{R}[\{X_\lambda\}])$, where $T(\overline{R}[\{X_\lambda\}])$ denotes the total quotient ring of $\overline{R}[\{X_\lambda\}]$. Since $\overline{R}$ is a 0-dimensional ring, $T(\overline{R}[\{X_\lambda\}])$ is also a 0-dimensional ring by ([1], Proposition 8). Therefore $R[\{X_\lambda\}]$ also has a 0-dimensional overring.

THEOREM 13. *Let R be a commutative ring and let $\{Q_\lambda\}$ be the set of primary ideals of R. R can be embedded into a direct product of 0-dimensional rings if and only if $\bigcap\limits_\lambda Q_\lambda = (0)$.*

Proof. Let R be a commutative ring such that $\bigcap\limits_\lambda Q_\lambda = (0)$, where $\{Q_\lambda\}$ is the set of the primary ideals of R. Let P_λ be the prime ideal of R associated with Q_λ, for every λ. Then $R \subseteq \prod\limits_\lambda (R/Q_\lambda)_{P_\lambda/Q_\lambda}$ where $(R/Q_\lambda)_{P_\lambda/Q_\lambda}$ is a quasi-local 0-dimensional ring, for every λ. Therefore if $\bigcap\limits_\lambda Q_\lambda = (0)$, R can be embedded into a direct product of 0-dimensional rings. Conversely, suppose R can be embedded into a direct product of 0-dimensional rings and let $\{Q_\lambda\}$ be the set of primary ideals of R. It is easy to see that $\bigcap\limits_\lambda Q_\lambda = (0)$.

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O ULAGANJU KOMUTATIVNOG PRSTENA U KOMUTATIVAN PRSTEN DIMENZIJE NULA

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Sadržaj

U ovom radu se posmatraju komutativni prstenovi sa jedinicom. Dobro je poznato da je kompletni prsten razlomaka $Q(R)$ komutativnog poluprostog prstena R regularan prsten. Katkada je korisno da se komutativni poluprosti prsten R uloži u regularan prsten, na primjer vidjeti [3]. Ovaj rad daje potrebne i dovoljne uslove da se komutativni prsten R može uložiti u 0-dimenzionalan nadprsten. Ovi uslovi su primjenjeni na neke klase komutativnih prstenova da se zaključi da svaki prsten iz tih klasa posjeduje 0-dimenzionalni nadprsten. Konstruisan je komutativni prsten koji nema 0-dimenzionalni nadprsten.