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A NOTE ON TOLERANCES OF IDEMPOTENT ALGEBRAS

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Abstract. Tolerances of algebras are defined as binary, reflexive, symmetric and compatible relations. It is advantageous to describe tolerances by certain covering systems, namely, the systems of all blocks. The present paper gives a characterization by covering systems associated with tolerances of idempotent algebras. This result makes possible to characterize tolerances of a wide class of lattices, containing all the ones of finite length, via covering systems.

Given an algebra (A, F), a binary, reflexive, symmetric and compatible relation $\varrho \subseteq A \times A$ is said to be a tolerance relation (or shortly tolerance) of the algebra A. (We denote algebras and their base sets by the same letter.) Tolerances of algebras were firstly investigated by B. Zelinka [5]. For a tolerance ϱ of the algebra A, a subset B of A is called a block of ϱ if $B^2 \subseteq \varrho$, but $C^2 \subseteq \varrho$ for no $B \subset C \subseteq A$. The set of all blocks of ϱ is denoted by $\mathscr{C}_\varrho$. There is a familiar correspondence between congruences (equivalence relations) and compatible partitions on a given algebra (set). I. Chajda in [1] (I. Chajda, J. Niederle and B. Zelinka in [2]) established a same type of connection between tolerances of algebras (of sets) and certain coverings. The aim of this note is to present a simpler and more handlable condition for coverings connected with tolerances of idempotent algebras. Since sets can be considered as idempotent algebras (having one projection only as fundamental operation) we can simplify the conditions given in [2] for sets.

The first author in [4] characterized the set of all blocks of a tolerance on a lattice. We can easily derive his result from our theorem for lattices satisfying the Chain Condition.

THEOREM. *For an idempotent algebra* (A, F) *and a family of subsets* $\mathscr{C} \subseteq P^+(A)$ *(the set of all non-void subsets of* A*) the following two conditions are equivalent*:

(i) $\mathscr{C}$ *is the set of all blocks of a suitable tolerance of* A.

(ii) $\mathscr{C}$ *has the following properties,*

(α) $\cup (B \mid B \in \mathscr{C}) = A$

(β) *For any* $B, D \in \mathscr{C}$, $B \subseteq D$ *implies* $B = D$.

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(γ) *For any n-ary operation* $f \in F$ *and* $B_1, \ldots, B_n \in \mathscr{C}$ *there exists a set* $D \in \mathscr{C}$ *such that* $f(B_1, \ldots, B_n) = \{f(b_1, \ldots, b_n) \mid b_i \in B_i, 1 \leqslant i \leqslant n\} \subseteq D$.

(δ) *For any subalgebra E of A if any two elements* $a, b \in E$ *are contained in a set* $D_{a,b} \in \mathscr{C}$, *then there exists a set* $B \in \mathscr{C}$ *containing E.*

Proof. The part $(i) \Rightarrow (ii)$ is obvious. To prove the implication $(ii) \Rightarrow (i)$, let $\mathscr{C}$ be a family of nonvoid subsets of A ($\mathscr{C} \subseteq P^+(A)$) satisfying the conditions (α), (β), (γ), (δ). Define a binary relation ϱ on A in the following way: for any $a, b \in A$, $(a, b) \in \varrho$ iff there exists a set $B \in \mathscr{C}$ such that $\{a, b\} \subseteq B$. Obviously ϱ is symmetric, moreover the conditions (α) and (γ) imply the reflexivity and compatibility of ϱ. Therefore ϱ is a tolerance of A, and let $\mathscr{C}_\varrho$ denote the set of its blocks. By the definition of the relation ϱ a set $B \in \mathscr{C}$ does not belong to $\mathscr{C}_\varrho$ iff there is an element $d \in A \setminus B$ such that $(b, d) \in \varrho$ for all $b \in B$. Using the Zorn-lemma we get a ϱ-block $D \in \mathscr{C}$ such that

$$B \subseteq \{d\} \cup B \subseteq D.$$

The idempotency of the algebra A implies that D is a subalgebra of A (this was observed by I. Chajda and B. Zelinka in [3]). Moreover D being a ϱ-block we have $(d, e) \in \varrho$ for all $d, e \in D$. By the definition of the relation ϱ each pair of elements of D is contained in some set in $\mathscr{C}$. Now applying the condition (δ) we get a set $E \in \mathscr{C}$ such that $D \subseteq E$. Thus $B \subseteq D \subseteq E$, and from this we can derive $B = D = E$ by the condition (β), which means $B \in \mathscr{C}_\varrho$. This proves $\mathscr{C} \subseteq \mathscr{C}_\varrho$.

On the other hand let D be an arbitrary ϱ-block. D is a subalgebra of A and for each pair of elements $a, b \in D$, $(a, b) \in \varrho$ holds. By the definition of the relation ϱ and the condition (δ) we have a set $E \in \mathscr{C}$ containing D. Using the Zorn-lemma again we get a ϱ-block $G \in \mathscr{C}_\varrho$ such that

$$D \subseteq E \subseteq G.$$

Since the set of all blocks of a tolerance satisfies the condition (β) we have $D = E = G$, i. e., $D \in \mathscr{C}$ and therefore $\mathscr{C}_\varrho \subseteq \mathscr{C}$. Thus we proved $\mathscr{C} = \mathscr{C}_\varrho$, which finishes our proof.

Now we are ready to derive the following

COROLLARY [4]. *Let L be a lattice satisfying the Chain Condition (i. e., any chain in L is finite) and let* $\mathscr{C} \subseteq P^+(L)$. *Then the following two conditions are equivalent:*

(*i*) $\mathscr{C}$ *is the set of all blocks of a suitable tolerance* ϱ *of L.*

(*ii*) $\mathscr{C}$ *is of the form* $\{[a_\gamma, b_\gamma] \mid \gamma \in \Gamma\}$ *where* $[a_\gamma, b_\gamma]$ *are intervals of L (thus* $a_\gamma \leqslant b_\gamma$*) and the following three axioms are satisfied,*

(C1) $\cup([a_\gamma, b_\gamma] \mid \gamma \in \Gamma) = L$;

(C2) *For any* $\gamma, \delta \in \Gamma$ $a_\delta = a_\gamma$ *is equivalent to* $b_\gamma = b_\delta$;

(*C*3) *For any* $\gamma, \delta \in \Gamma$ *there exists a* $\mu \in \Gamma$ *such that* $a_\mu = a_\gamma \vee a_\delta$, $b_\mu \geqslant b_\gamma \vee b_\delta$ *and, dually, there exists a* $\nu \in \Gamma$ *such that* $a_\nu \leqslant a_\gamma \wedge a_\delta$, $b_\nu = b_\gamma \wedge b_\delta$.

Proof. To prove the $(i) \Rightarrow (ii)$ implication first we show that the blocks of a tolerance ϱ are convex sublattices (cf. I. Chajda and B. Zelinka [3]). As we have mentioned, the blocks of a tolerance of an idempotent algebra are subalgebras. To prove the convexity assume $A \in \mathscr{C} = \mathscr{C}_\varrho$, $a \leqslant c \leqslant b$, and $a, b \in A$. Then by (α) $c \in C \in \mathscr{C}$ holds for some C. By making use of (γ) twice we obtain $D, E \in \mathscr{C}$ such that $D \supseteq \{x \vee y \mid x \in C,\ y \in A\}$ and $E \supseteq \{x \wedge y \mid x \in D,\ y \in A\}$. Since $c = c \vee a \in D$, $c = c \wedge b \in E$, and for any $x \in A$ $x = (c \vee x) \wedge x \in E$, we have $c \in E \supseteq A$. Thus (β) implies $c \in E = A$, showing the convexity of A. The Chain Condition yields that convex sublattices are intervals, whence $\mathscr{C}$ is of the form $\{[a_\gamma, b_\gamma] \mid \gamma \in \Gamma\}$. (*C*1) follows from (α). If $a = a_\gamma = a_\delta$ then by (γ) there exists an $A \in \mathscr{C}$ such that $A \supseteq \{x \vee y \mid x \in [a_\gamma, b_\gamma],\ y \in [a_\delta, b_\delta]\}$. By convexity, $A \supseteq [a_\gamma \vee a_\delta, b_\gamma \vee b_\delta] \supseteq [a, b_\iota]$ for $\iota \in \{\gamma, \delta\}$. Hence $b_\gamma = b_\delta$ follows from (β), showing (C2). Similarly, for any $\gamma, \delta \in \Gamma$ there exists a $\mu \in \Gamma$ such that

$$[a_\mu, b_\mu] \supseteq \{x \vee y \mid x \in [a_\gamma, b_\gamma],\ y \in [a_\delta, b_\delta]\}$$

and, by convexity, $[a_\mu, b_\mu] \supseteq [a_\gamma \vee a_\delta, b_\gamma \vee b_\delta]$. Hence $a_\mu \leqslant a_\gamma \vee a_\delta$ and $b_\mu \geqslant b_\gamma \vee b_\delta$. Now for $\iota \in \{\gamma, \delta\}$ by (γ) and convexity again, $[a_\iota, b_\iota] \subseteq [a_\iota \wedge a_\mu,\ b_\iota] = [a_\iota \wedge a_\mu,\ b_\iota \wedge b_\mu] \subseteq B_\iota \in \mathscr{C}$. From (β) we conclude $[a_\iota, b_\iota] = B_\iota$, so $a_\iota = a_\iota \wedge a_\mu$. Therefore $a_\iota \leqslant a_\mu$ $(\iota \in \{\gamma, \delta\})$, which implies $a_\gamma \vee a_\delta \leqslant a_\mu$. The duality principle settles the rest of (*C*3).

(*ii*) implies (*i*). Suppose (*ii*), then it suffices to show that (α), (β), (γ) and (δ) hold. Evidently (α) and (γ) follow from (*C*1) and (*C*3), respectively. If $\gamma, \delta \in \Gamma$ and $[a_\gamma, b_\gamma] \subseteq [a_\delta, b_\delta]$ then considering a suitable μ from (*C*3) and using (*C*2) twice we obtain

$$a_\gamma = a_\gamma \vee a_\delta = a_\mu,$$

$$b_\gamma \leqslant b_\delta = b_\gamma \vee b_\delta \leqslant b_\mu = b_\gamma, \quad a_\gamma = a_\delta.$$

Hence (β) holds. Since any sublattice of L is bounded by the Chain Condition, (δ) is obvious.

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BILJEŠKA O TOLERANCIJAMA IDEMPOTENTNIH ALGEBRI

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Sadržaj

Opis tolerancija pomoću sistema pokrivanja tj. sistema svih blokova, ima stanovite prednosti pred ostalima. U radu je dana karakterizacija sistema pokrivanja pridruženih tolerancijama idempotentnih algebri.