Compatible Discrete Operator (CDO) schemes for elliptic problems on polyhedral meshes

Jérôme Bonelle[1,2] and Alexandre Ern[2]

[1]EDF R&D
[2]Université Paris Est - CERMICS (ENPC)

NM2PorousMedia, 29 Sept. - 03 Oct. 2014, Dubrovnik

Context

→ EDF has been developing several in-house CFD codes for 30 years
  - *Code_Saturne* (*open-source*) single-phase flow solver based on co-located Finite Volume schemes (since 1998)
  - Approach close to commercial codes like Star-CD or FLUENT

→ Re-open numerical work to improve numerical methods
  - Axes: physical fidelity, robustness on complex geometry, efficiency
  - New developments based on structure-preserving schemes

→ The Compatible Discrete Operator (CDO) framework
  - Inspired by seminal ideas of Tonti (1974-) and Bossavit (1988-)
  - Elliptic problems (Bonelle & AE, M2AN, 2014)
  - Stokes equations (Bonelle & AE, HAL preprint, 2014)

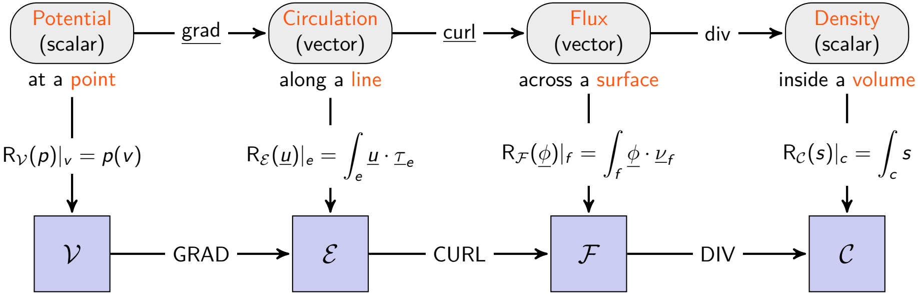
Outline

Outline

Compatible Discrete Operator (CDO) framework

De Rham complex

- → Degrees of freedom (DoFs) defined by de Rham maps
- → Definition of DoFs in agreement with physical nature of fields



Discrete differential operators

1. Metric-free: algebraically defined by incidence matrices
2. Commuting with de Rham maps

$$\mathrm{GRAD}\cdot\mathrm{R}_{\mathcal{V}} = \mathrm{R}_{\mathcal{E}}\cdot\underline{\mathrm{grad}},\quad \mathrm{CURL}\cdot\mathrm{R}_{\mathcal{E}} = \mathrm{R}_{\mathcal{F}}\cdot\underline{\mathrm{curl}},\quad \mathrm{DIV}\cdot\mathrm{R}_{\mathcal{F}} = \mathrm{R}_{\mathcal{C}}\cdot\mathrm{div}$$

3. Cochain complex: $\mathrm{CURL}\cdot\mathrm{GRAD}\equiv 0_{\mathcal{F}}$ and $\mathrm{DIV}\cdot\mathrm{CURL}\equiv 0_{\mathcal{C}}$

Two meshes: a primal and a dual mesh

Primal mesh : $\mathrm{M} = \{\mathrm{V}, \mathrm{E}, \mathrm{F}, \mathrm{C}\}$

→ Carry the information on geometry, material properties and BCs

→ Only mesh seen by the end-user

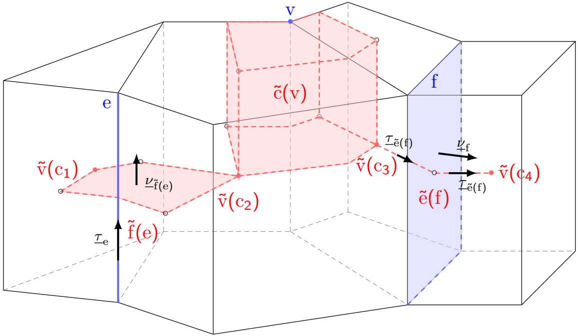
Dual mesh : $\widetilde{\mathrm{M}} = \{\widetilde{\mathrm{V}}, \widetilde{\mathrm{E}}, \widetilde{\mathrm{F}}, \widetilde{\mathrm{C}}\}$

→ Only for defining the scheme

→ Several definitions: barycentric, Voronoï...

→ One-to-one pairing: $v \leftrightarrow \tilde{c}(v)$, $e \leftrightarrow \tilde{f}(e)$, $f \leftrightarrow \tilde{e}(f)$ and $c \leftrightarrow \tilde{v}(c)$

→ Transfer of orientation: $\underline{\tau}_e \rightarrow \underline{\nu}_{\tilde{f}(e)}$ and $\underline{\nu}_f \rightarrow \underline{\tau}_{\tilde{e}(f)}$



Discrete setting

DoFs on the dual mesh $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{C}}$ are also defined by de Rham maps

$$\mathcal{V} \xrightarrow{\ \mathsf{GRAD}\ } \mathcal{E} \xrightarrow{\ \mathsf{CURL}\ } \mathcal{F} \xrightarrow{\ \mathsf{DIV}\ } \mathcal{C}$$

$$\widetilde{\mathcal{C}} \xleftarrow{\ \widetilde{\mathsf{DIV}}\ } \widetilde{\mathcal{F}} \xleftarrow{\ \widetilde{\mathsf{CURL}}\ } \widetilde{\mathcal{E}} \xleftarrow{\ \widetilde{\mathsf{GRAD}}\ } \widetilde{\mathcal{V}}$$

Duality Products

$$x\tilde{y} \in \{v\tilde{c}, e\tilde{f}, f\tilde{e}, c\tilde{v}\}$$

$$[\![\mathbf{a}, \mathbf{b}]\!]_{\mathcal{X}\tilde{\mathcal{Y}}} := \sum_{\mathrm{x}\in\mathrm{X}} \mathbf{a}_{\mathrm{x}} \mathbf{b}_{\tilde{\mathrm{y}}(\mathrm{x})} = \mathbf{a}^{\mathrm{T}}\, \mathbf{b}$$

where $\mathbf{a} \in \mathcal{X}$, $\mathbf{b} \in \widetilde{\mathcal{Y}}$, $\mathrm{X} \in \{\mathrm{V}, \mathrm{E}, \mathrm{F}, \mathrm{C}\}$

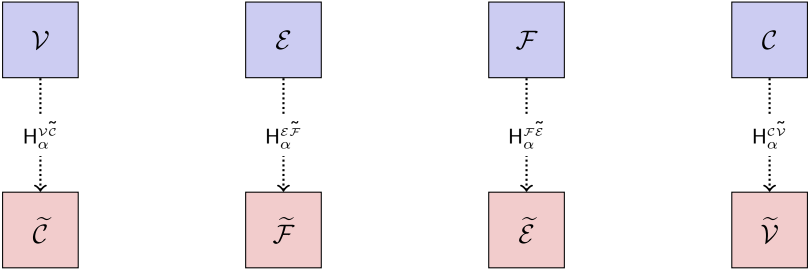
Adjunction properties

$$-[\![\mathsf{GRAD}(\mathbf{p}), \phi]\!]_{\mathcal{E}\tilde{\mathcal{F}}} = [\![\mathbf{p}, \widetilde{\mathsf{DIV}}(\phi)]\!]_{\mathcal{V}\tilde{\mathcal{C}}}$$

$$[\![\mathsf{CURL}(\mathbf{u}), \mathbf{v}]\!]_{\mathcal{F}\tilde{\mathcal{E}}} = [\![\mathbf{u}, \widetilde{\mathsf{CURL}}(\mathbf{v})]\!]_{\mathcal{E}\tilde{\mathcal{F}}}$$

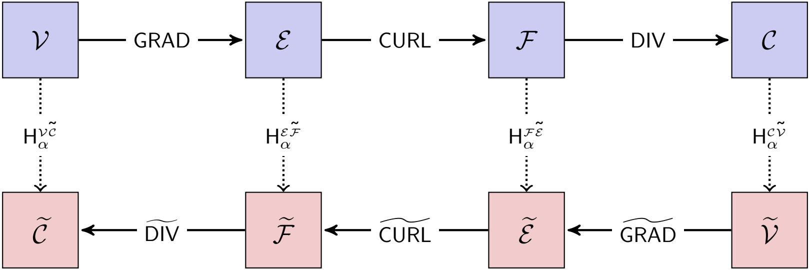
$$-[\![\mathsf{DIV}(\phi), \mathbf{p}]\!]_{\mathcal{C}\tilde{\mathcal{V}}} = [\![\phi, \widetilde{\mathsf{GRAD}}(\mathbf{p})]\!]_{\mathcal{F}\tilde{\mathcal{E}}}$$

Discrete Hodge operators $\mathsf{H}_{\alpha}^{\mathcal{X}\tilde{\mathcal{Y}}}$



- Links DoF spaces in duality $\mathcal{X}\tilde{\mathcal{Y}} \in \{\mathcal{V}\tilde{\mathcal{C}}, \mathcal{E}\tilde{\mathcal{F}}, \mathcal{F}\tilde{\mathcal{E}}, \mathcal{C}\tilde{\mathcal{V}}\}$
- Depends on a metric induced by a material property α
- $\mathsf{H}_{\alpha}^{\mathcal{X}\tilde{\mathcal{Y}}}$ is built by a cellwise assembly process
- Definition hinges on two local design properties
  1. Stability Upper/lower eigenvalues are uniformly bounded
  2. $\mathbb{P}_0$-consistency Exactly represents piecewise constant field on each $c \in \mathrm{C}$
- Multiple definitions $\rightarrow$ multiple schemes

Synthesis of discrete setting



Discrete Differential Operators	Discrete Hodge Operators

Topological laws	Constitutive relations
Error-free	Approximation
Unique definition	Multiple definitions

Literature overview

	CDO	HFV MFV	DDFV	MFD VEM	HHO	FEEC MSE
HO/LO	LO	LO	LO	LO/HO	HO	HO
Setting	NC	NC	NC	C	NC	C
Element	Poly.	Poly.	Poly.	Poly.	Poly.	Spe.
Meshes	P+D	P	P+D+$\diamond$	P	P	P
Key Op.	disc. Hodge	grad reco.	grad/div reco.	inner prod.	grad reco.	cochain proj.

HO/LO = Higher-order/Lower-order
NC/C = nonconforming/conforming reconstruction operator
Poly. / Spe. = Polyhedral / Specific (i.e. tetrahedral, hexahedral meshes)
P/D/$\diamond$ = primal / dual / diamond meshes are explicitly considered

- HFV/MFV *Hybrid/Mixed Finite Volume*: Droniou, Eymard, Herbin, Gallouët
- DDFV *Discrete Duality Finite Volume*: Andreianov, Hubert, Krell et al.
- MFD *Mimetic Finite Differences* & VEM *Virtual Element Method*: Beirão da Veiga, Brezzi, Lipnikov, Shashkov, Manzini et al.
- HHO *Hybrid High-Order*: Di Pietro & Ern
- FEEC *Finite Element Exterior Calculus*: Arnold, Falk, Whinteret al.
- MSE *Mimetic Spectral Element*: Kreeft, Gerritsma, Pahla et al.

Outline

CDO for elliptic problems

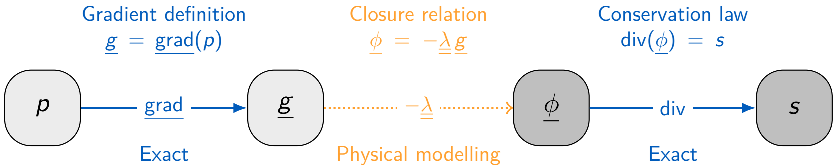
- Discretization process
- Analysis: algebraic viewpoint
- Analysis using reconstruction functions
- Links with other schemes
- Numerical results

Discretization process

$$\boxed{-\operatorname{div}(\underline{\underline{\lambda}}\,\underline{\operatorname{grad}}(p)) = s}$$

→ Topological laws
$$\begin{cases} \underline{g} = \underline{\operatorname{grad}}(p) \\ \operatorname{div}(\underline{\phi}) = s \end{cases}$$

→ Constitutive relations $\underline{\phi} = -\underline{\underline{\lambda}}\,\underline{g}$



Potential evaluated at a point

Gradient evaluated along a line

Flux evaluated across a surface

Density evaluated in a volume

Two CDO schemes depending on positioning of potential DoFs

- vertex-based schemes for potential DoFs located at primal vertices
- cell-based schemes for potential DoFs located at dual vertices (in one-to-one pairing with primal cells)

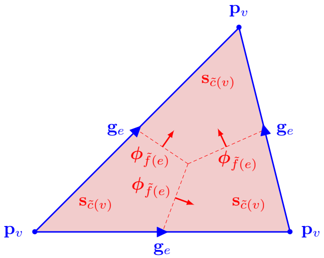
Vertex-based schemes

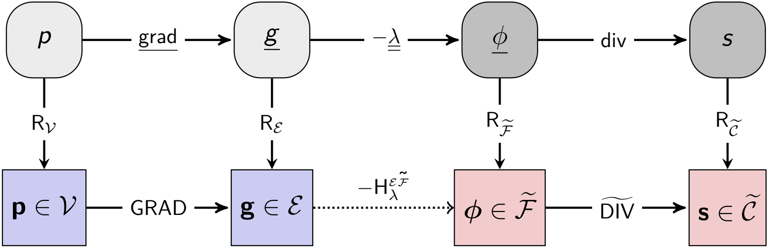
→ Potential DoFs located at primal vertices

→ Dirichlet BCs strongly enforced by removing boundary DoFs

→ DoF spaces still denoted $\{\mathcal{V}, \mathcal{E}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{C}}\}$

- $\mathcal{V} \equiv \widetilde{\mathcal{C}}$ and $\mathcal{E} \equiv \widetilde{\mathcal{F}}$





$$-\widetilde{\mathsf{DIV}} \cdot \mathsf{H}_{\lambda}^{\mathcal{E}\tilde{\mathcal{F}}} \cdot \mathsf{GRAD}(\mathbf{p}) = \mathsf{R}_{\widetilde{\mathcal{C}}}(s)$$

Symmetric Positive Definite (SPD) formulation

FV scheme on dual mesh

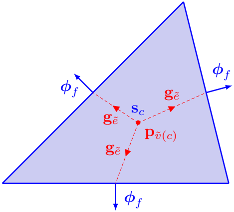
Cell-based schemes

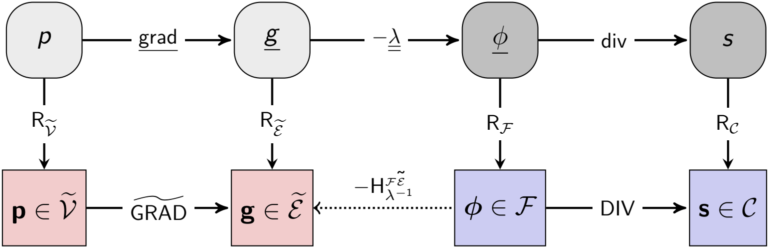
→ Potential DoFs located at dual vertices

→ DoF spaces: $\{\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}}, \mathcal{F}, \mathcal{C}\}$

- $\mathcal{C} \equiv \widetilde{\mathcal{V}}$ and $\mathcal{F} \equiv \widetilde{\mathcal{E}}$

→ Dirichlet BCs weakly enforced by modifying $\widetilde{\mathsf{GRAD}}$ near boundary





$$\begin{cases} \mathsf{H}^{\mathcal{F}\tilde{\mathcal{E}}}_{\lambda^{-1}}\boldsymbol{\phi} + \widetilde{\mathsf{GRAD}}(\mathbf{p}) = 0 \\ \mathsf{DIV}(\boldsymbol{\phi}) = \mathsf{R}_{\mathcal{C}}(s) \end{cases}$$

Saddle-point formulation
Mixed FV scheme on primal mesh

Analysis: algebraic viewpoint

Focus on vertex-based schemes (analysis of cell-based schemes also performed)

Stability & well-posedness hinge on Poincaré inequality + Hodge stability

→ Mesh regularity: Assume there exists a shape-regular simplicial submesh

→ • $\|\mathbf{a}\|^2_{2,\mathcal{X}_c} = \sum_{\mathrm{x}\in\mathrm{X}_c} h_c^3 (\frac{\mathbf{a}_\mathrm{x}}{|\mathrm{x}|})^2$ where $\mathbf{a} \in \mathcal{X} \in \{\mathcal{V}, \mathcal{E}, \mathcal{F}\}$ and $\mathrm{X} \in \{\mathrm{V}, \mathrm{E}, \mathrm{F}\}$

• $\|\mathbf{a}\|^2_{2,\mathcal{X}}$ collects local contributions

1. Discrete Poincaré inequality

There exists $C_P^{(0)} > 0$ s.t. $\forall \mathbf{p} \in \mathcal{V}$

$$\|\mathbf{p}\|_{2,\mathcal{V}} \leq C_P^{(0)} \|\mathsf{GRAD}(\mathbf{p})\|_{2,\mathcal{E}}$$

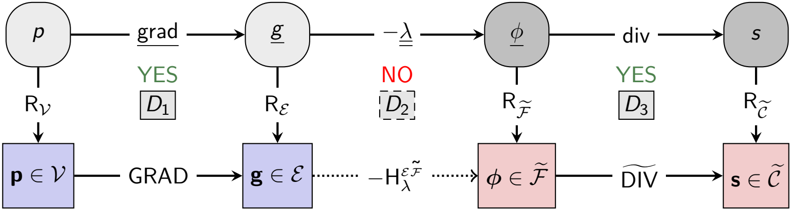
2. Extension to discrete Sobolev inequality possible

Sketch

→ Bound on BV-norm as in [Eymard et al., 2010]

→ Commuting with de Rham maps and adjunction between GRAD and $\widetilde{\mathsf{DIV}}$

Consistency error



Commuting operator: $\lfloor \lambda,\, \mathsf{R} \rceil := \mathsf{H}_\lambda^{\mathcal{E}\tilde{\mathcal{F}}} \mathsf{R}_{\mathcal{E}} - \mathsf{R}_{\widetilde{\mathcal{F}}} \underline{\underline{\lambda}}$

Consistency error: Bossavit (1999), Hiptmair (2001), Codecasa & Trevisan (2010)

$$\max\left(\|\mathsf{R}_{\mathcal{E}}(\underline{g}) - \mathbf{g}\|_{\mathsf{H}_\lambda}, \|\mathsf{R}_{\widetilde{\mathcal{F}}}(\underline{\phi}) - \boldsymbol{\phi}\|_{\mathsf{H}_\lambda^{-1}} \right) \le \| \lfloor \lambda,\, \mathsf{R} \rceil (\underline{g}) \|_{\mathsf{H}_\lambda^{-1}}$$

→ Discrete energy norm: $\|\mathbf{g}\|^2_{\mathsf{H}_\lambda} := \left[\!\left[\mathbf{g}, \mathsf{H}_\lambda^{\mathcal{E}\tilde{\mathcal{F}}} \mathbf{g} \right]\!\right]_{\mathcal{E}\tilde{\mathcal{F}}}$

→ Discrete complementary energy norm: $\|\boldsymbol{\phi}\|^2_{\mathsf{H}_\lambda^{-1}} := \left[\!\left[(\mathsf{H}_\lambda^{\mathcal{E}\tilde{\mathcal{F}}})^{-1} \boldsymbol{\phi}, \boldsymbol{\phi} \right]\!\right]_{\mathcal{E}\tilde{\mathcal{F}}}$

Error estimate for smooth solutions

Variational formulation of VB scheme: Find $\mathbf{p} \in \mathcal{V}$ s.t.

$$\llbracket \mathrm{GRAD}(\mathbf{p}), \mathrm{H}_\lambda^{\mathcal{E}\tilde{\mathcal{F}}} \mathrm{GRAD}(\mathbf{q}) \rrbracket_{\mathcal{E}\tilde{\mathcal{F}}} = \llbracket \mathbf{q}, \mathrm{R}_{\tilde{\mathcal{C}}}(s) \rrbracket_{\mathcal{V}\tilde{\mathcal{C}}} \qquad \forall \mathbf{q} \in \mathcal{V}$$

Assume mesh regularity. Assume stability and $\mathbb{P}_0$-consistency for Hodge operator. Assume $\underline{g}, \underline{\phi} \in [H^1(\mathrm{C})]^d$. Then,

$$\max(\|\mathrm{R}_{\mathcal{E}}(\underline{g}) - \mathbf{g}\|_{\mathrm{H}_\lambda}, \|\mathrm{R}_{\tilde{\mathcal{F}}}(\underline{\phi}) - \boldsymbol{\phi}\|_{\mathrm{H}_\lambda^{-1}}) \lesssim h_{\mathrm{M}}$$

→ Sketch of proof:

1. Choose $(\underline{G}, \underline{\Phi})$ as the piecewise constant approx. of $(\underline{g}, \underline{\phi})$
2. $\mathbb{P}_0$-consistency $\Rightarrow \|\lfloor \lambda, \mathrm{R} \rceil(\underline{g})\|_{\mathrm{H}_\lambda^{-1}} \le \|\mathrm{R}_{\mathcal{E}}(\underline{g} - \underline{G})\|_{\mathrm{H}_\lambda} + \|\mathrm{R}_{\tilde{\mathcal{F}}}(\underline{\phi} - \underline{\Phi})\|_{\mathrm{H}_\lambda^{-1}}$
3. Stability + mesh reg. $\Rightarrow \int_e (\underline{g} - \underline{G}) \cdot \underline{\tau}_e \lesssim h_c^{1/2} \|\underline{g}\|_{H^1(c)}$
4. Stability + mesh reg. $\Rightarrow \int_{\tilde{f}_c(e)} (\underline{\phi} - \underline{\Phi}) \cdot \underline{\nu}_{\tilde{f}_c(e)} \lesssim h_c^{3/2} \|\underline{\phi}\|_{H^1(c)}$

Extends result for piecewise Lipschitz solutions (Codecasa & Trevisan, 2010)

Analysis using reconstruction functions

→ Purpose: reconstruction (lifting) of gradients from edge DoFs

- local reconstruction on each primal cell
- vector-valued reconstruction functions $\{\underline{\ell}_{e,c}\}_{e\in \mathrm{E}_c}$

→ In general, reconstruction is nonconforming in $H(curl, c)$

Local reconstruction operator: $\forall \mathbf{g} \in \mathcal{E}_c$, $$\underline{\mathrm{L}}_{\mathcal{E}_c}(\mathbf{g}) := \sum_{e\in \mathrm{E}_c} \mathbf{g}_e \, \underline{\ell}_{e,c}(\underline{x})$$

New definition of local Hodge operator: $$(\mathrm{H}_{\lambda}^{\mathcal{E}_c\tilde{\mathcal{F}}_c})_{e,e'} := \int_c \underline{\ell}_{e,c}(\underline{x}) \cdot \underline{\underline{\lambda}}\, \underline{\ell}_{e',c}(\underline{x})$$

→ Global reconstruction operator: $\underline{\mathrm{L}}_{\mathcal{E}} : \mathcal{E} \longmapsto L^2(\Omega)^d$

→ Simply collects local reconstructions

$$\forall \mathbf{g} \in \mathcal{E}, \forall c \in \mathrm{C}, \qquad \underline{\mathrm{L}}_{\mathcal{E}}(\mathbf{g})|_c := \underline{\mathrm{L}}_{\mathcal{E}_c}(\mathrm{P}_{\mathcal{E},c}(\mathbf{g}))$$

Local design properties

(I1) Stability: $\exists \eta > 0$ s.t. $\forall \mathbf{g} \in \mathcal{E}_c$,

$$\eta \|\mathbf{g}\|^2_{2,\mathcal{E}_c} \leq \|\underline{\mathsf{L}}_{\mathcal{E}_c}(\mathbf{g})\|^2_{L^2(c)^d} \leq \eta^{-1} \|\mathbf{g}\|^2_{2,\mathcal{E}_c}$$

(I2) Partition of Unity: for all $\underline{x} \in c$,

$$\sum_{e \in \mathrm{E}_c} \underline{\ell}_{e,c}(\underline{x}) \otimes \underline{e} = \underline{\underline{\mathsf{Id}}}$$

(I3) Dual consistency: for all $e \in \mathrm{E}_c$,

$$\int_c \underline{\ell}_{e,c}(\underline{x}) = \int_{\tilde{f}_c(e)} \underline{\nu}_{\tilde{f}(e)}$$

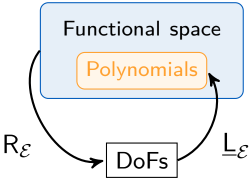
(I4) Unisolvence: for all $e, e' \in \mathrm{E}_c$,

$$\int_{e'} \underline{\ell}_{e,c}(\underline{x}) \cdot \underline{\tau}_{e'} = \delta_{e,e'}$$

$\rightarrow$ Based on ideas of Codecasa & Trevisan (2010) + concept of local stability

(I2) $\iff$ $\underline{\mathsf{L}}_{\mathcal{E}_c} \mathsf{R}_{\mathcal{E}_c}(\underline{G}) = \underline{G} \quad \forall \underline{G} \in (\mathbb{P}_0(c))^3$

(I4) $\iff$ $\mathsf{R}_{\mathcal{E}_c} \underline{\mathsf{L}}_{\mathcal{E}_c} = \mathsf{Id}$



Barycentric dual mesh + **(I1)-(I3)** $\Longrightarrow$ Stability and $\mathbb{P}_0$-consistency for Hodge op.

Error estimate for smooth solutions

Variational formulation: Find $\mathbf{p} \in \mathcal{V}$ s.t.

$$\int_\Omega \underline{\mathsf{L}}_{\mathcal{E}}(\mathsf{GRAD}(\mathbf{p})) \cdot \underline{\underline{\lambda}}\, \underline{\mathsf{L}}_{\mathcal{E}}(\mathsf{GRAD}(\mathbf{q})) = \int_\Omega s\, \mathsf{L}^0_{\mathcal{V}}(\mathbf{q}) \qquad \forall \mathbf{q} \in \mathcal{V}$$

where $\mathsf{L}^0_{\mathcal{V}}$ is a piecewise constant function on the dual mesh

Assume mesh regularity, **(I1)-(I3)** for rec. fn's, and $\underline{g}, \underline{\phi} \in [H^1(\mathrm{C})]^d$. Then,

$$\|\underline{g} - \underline{\mathsf{L}}_{\mathcal{E}}\mathbf{g}\|_\lambda = \|\underline{\phi} - \underline{\mathsf{L}}_{\widetilde{\mathcal{F}}}\boldsymbol{\phi}\|_{\lambda^{-1}} \lesssim h_{\mathrm{M}}$$

where $\underline{\mathsf{L}}_{\widetilde{\mathcal{F}}}\boldsymbol{\phi} := -\underline{\underline{\lambda}}\,\underline{\mathsf{L}}_{\mathcal{E}}\mathbf{g}$, $\|\underline{g}\|_\lambda := \int_\Omega \underline{g} \cdot \underline{\underline{\lambda}}\, \underline{g}$, and $\|\underline{\phi}\|_{\lambda^{-1}} := \int_\Omega \underline{\phi} \cdot \underline{\underline{\lambda}}^{-1}\, \underline{\phi}$

Additionally assume elliptic regularity and **(I4)** for re. fn's. Then,

$$\|p - \mathsf{L}^{\mathrm{conf}}_{\mathcal{V}}\mathbf{p}\|_{L^2(\Omega)} \lesssim h^2_{\mathrm{M}}$$

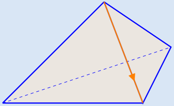
where $\mathsf{L}^{\mathrm{conf}}_{\mathcal{V}}$ is a conforming potential reconstruction operator

Links with other schemes

Simple examples on specific meshes

Tetrahedral meshes

→ $\underline{\ell}_e^{\mathrm{W}}$:= Nédélec FE

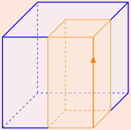


→ Conforming reconstruction

→ Lagrange P_1 finite elements

Cartesian meshes

→ $\underline{\ell}_{e,c}^{\mathrm{TPF}}(\underline{x})|_{\mathfrak{s}_{e,c}^{\mathrm{TPF}}} := \frac{\underline{\tilde{f}}_c(e)}{\underline{e}\cdot\underline{\tilde{f}}_c(e)}$



→ Nonconforming reconstruction

→ Two-Point Flux Approximation

- MAC (Harlow & Welch, 1965)
- FIT (Weiland, 70s)

These functions verify **(I1)-(I4)**

→ Abundant literature for conforming reconstruction of potentials

- Generalized barycentric coordinates (Gillette et al., 2010)
- FEEC (Arnold et al., 2006) or Mimetic Spectral Element (Kreeft et al., 2011)
- Extension of Whitney forms to prisms, pyramids and cubes
  - Bossavit (2003, 2010), Gradinaru & Hiptmair (1999)

Discrete Geometric Approach

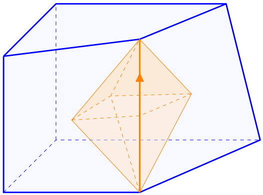
Codecasa, Specogna, and Trevisan (2009)

→ Nonconforming reconstruction

→ Handles polyhedral meshes with barycentric subdivision

$$\sum_{e \in \mathrm{E}_c} \tilde{f}_c(e) \otimes \underline{e} = |c| \underline{\underline{\mathsf{Id}}}$$

→ Piecewise constant reconstruction on submesh



Partitioning of c into sub-volumes $\{\mathfrak{s}_e\}_{e \in \mathrm{E}_c}$

Local function related to e' is constant in each sub-volume $\{\mathfrak{s}_e\}_{e \in \mathrm{E}_c}$

$$\underline{\ell}^{\mathrm{DGA}}_{e',c}|_{\mathfrak{s}_e} := \frac{\tilde{f}_c(e)}{\tilde{f}_c(e) \cdot \underline{e}} \delta_{e,e'} + \left(\underline{\underline{\mathsf{Id}}} - \frac{\tilde{f}_c(e) \otimes \underline{e}}{\tilde{f}_c(e) \cdot \underline{e}} \right) \frac{\tilde{f}_c(e')}{|c|}$$

$\{\underline{\ell}^{\mathrm{DGA}}_{e,c}\}_{e \in \mathrm{E}_c}$ verify **(I1)-(I4)**

Overview

Vertex-based schemes

$\rightarrow$ recover Lagrange P_1 finite elements on tetrahedral meshes

$\rightarrow$ fit the framework of Nodal MFD (Brezzi et al., 2009)

$\rightarrow$ fit the framework of Approximate Gradient Schemes (Eymard et al., 2012)

Cell-based schemes

$\rightarrow$ recover RTN_0 mixed finite elements on tetrahedral meshes

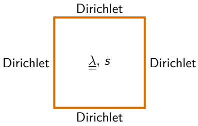
$\rightarrow$ can be hybridized by introducing face unknowns related to flux continuity

$\rightarrow$ fit the framework of MFD, HFV, MFV (Droniou et al., 2010)

DDFV schemes use simultaneously DoFs and operators from VB and CB schemes

Numerical results

Vertex-based scheme with Discrete Geometric Approach



$$p(x,y,z) := 1 + \sin(\pi x)\sin\left(\pi\left(y+\frac{1}{2}\right)\right)\sin\left(\pi\left(z+\frac{1}{3}\right)\right)$$

$$\underline{\underline{\lambda}} := \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix}$$

→ Error measures

- $erl2 := \frac{\|p - \mathsf{L}_{\mathcal{V}}(\mathbf{p})\|_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)}}$, $ergl2 := \frac{\|\underline{g} - \underline{\mathsf{L}}_{\mathcal{E}}(\mathbf{g})\|_{L^2(\Omega)^d}}{\|\underline{g}\|_{L^2(\Omega)^d}}$, $enerd := \|\mathsf{R}_{\mathcal{E}}(\underline{g}) - \mathbf{g}\|_{\lambda}$
- $\mathsf{L}_{\mathcal{V}}(\mathbf{p})(\underline{x})$ is a piecewise affine reconstruction on a submesh

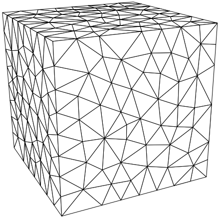
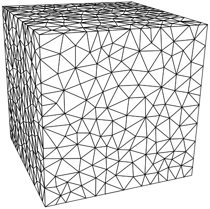
→ Mesh regularity criteria

- $\Gamma_1 := \min_{c\in\mathrm{C}}(\gamma_1)_c$ where $(\gamma_1)_c := \min_{e\in\mathrm{E}_c} \frac{\underline{e}\cdot\underline{\tilde{f}}_c(e)}{|e||\tilde{f}_c(e)|}$
- $\Gamma_2 := \min_{c\in\mathrm{C}}(\gamma_2)_c$ where $(\gamma_2)_c := \min_{e\in\mathrm{E}_c}\left(\left(\frac{|\tilde{f}_c(e)|}{|e|\,h_c}\right), \left(\frac{|e|\,h_c}{|\tilde{f}_c(e)|}\right)\right)$
- $\Gamma_3 := \min_{v\in\mathrm{V}}(\gamma_3)_v$ where $(\gamma_3)_v := |\tilde{c}(v)|\left(\sum_{e\in\mathrm{E}_v}|e||\tilde{f}(e)|\right)^{-1}$

→ Γ_1, Γ_2 and Γ_3 are the geometrical constants entering in the error estimates

FVCA6 test case

Tetrahedral meshes

$\#C$	Γ_1	Γ_2	Γ_3
215	1.7e-01	6.1e-03	1.3e-01
2 003	1.7e-01	4.7e-03	1.3e-01
3 898	1.5e-01	3.9e-03	1.3e-01
7 711	1.6e-01	4.0e-03	1.2e-01
15 266	1.5e-01	4.0e-03	1.4e-01
30 480	1.4e-01	4.1e-03	1.3e-01
61 052	1.5e-01	3.8e-03	1.3e-01

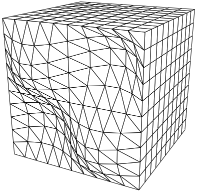
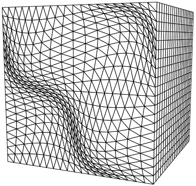
Solver: CG with precond. SSOR, $n_{iter}^{max} = 84$ for a residual 2.10^{-15}

Min/Max principle

Nsys ($\#V$)	Stencil	erl2	Rate	ergl2	Rate	enerd	Rate
8	6	8.2e-02		5.9e-01		2.6e-01	
184	11	2.1e-02	1.3	2.7e-01	0.8	1.7e-01	0.4
403	12	1.3e-02	2.0	2.1e-01	0.9	1.2e-01	1.3
873	13	8.2e-03	1.7	1.7e-01	0.9	9.9e-02	0.7
1 836	13	5.3e-03	1.8	1.3e-01	0.9	7.9e-02	0.9
3 863	14	3.3e-03	1.9	1.1e-01	0.9	6.3e-02	0.9
8 108	14	2.1e-03	1.9	8.3e-02	1.0	5.1e-02	0.8

FVCA6 test case

Prism meshes

#C	Γ_1	Γ_2	Γ_3
2 000	2.5e-01	1.0e-02	1.0e-01
16 000	3.1e-01	1.3e-02	9.1e-02
54 000	3.1e-01	1.3e-02	9.0e-02
128 000	3.1e-01	1.3e-02	8.9e-02

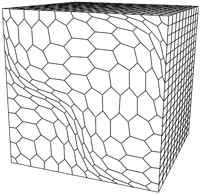
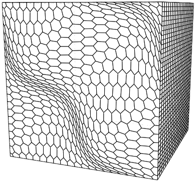
Solver: CG with precond. + SSOR,
$n_{iter}^{max} = 112$ for a residual 8.10^{-15}

Min/Max principle

Nsys (#V)	Stencil	erl2	Rate	ergl2	Rate	enerd	Rate
729	17	1.3e-02		1.9e-01		5.9e-02	
6 859	19	3.4e-03	1.8	9.4e-02	0.9	1.6e-02	1.7
24 389	19	1.5e-03	1.9	6.3e-02	1.0	7.5e-03	1.8
59 319	20	8.7e-04	1.9	4.7e-02	1.0	4.3e-03	1.9

FVCA6 test case

Prism-G meshes

#C	Γ_1	Γ_2	Γ_3
1 210	2.1e-01	1.7e-02	7.4e-02
8 820	1.9e-01	1.5e-02	6.7e-02
28 830	1.8e-01	1.4e-02	6.5e-02
67 240	1.8e-01	1.4e-02	6.5e-02

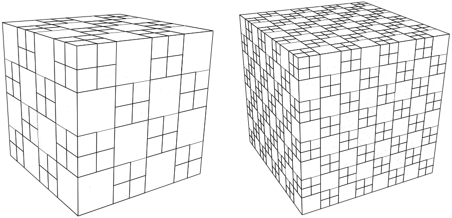
Solver: CG with precond. SSOR,
$n_{iter}^{max} = 144$ for a residual 1.10^{-14}

Min/Max principle

Nsys (#V)	Stencil	erl2	Rate	ergl2	Rate	enerd	Rate
1 800	31	1.0e-02		1.6e-01		4.9e-02	
15 200	35	2.7e-03	1.8	8.5e-02	0.9	1.5e-02	1.6
52 200	36	1.2e-03	1.9	5.7e-02	1.0	7.5e-03	1.7
124 800	37	7.0e-04	1.9	4.3e-02	1.0	4.5e-03	1.7

FVCA6 test case

Checkerboard meshes



#C	Γ_1	Γ_2	Γ_3
36	8.9e-01	1.4e-01	1.6e-01
288	8.9e-01	1.4e-01	1.6e-01
2 304	8.9e-01	1.4e-01	1.6e-01
18 432	8.9e-01	1.4e-01	1.6e-01
147 456	8.9e-01	1.4e-01	1.6e-01

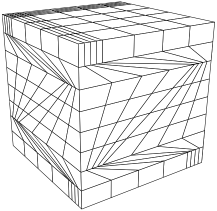
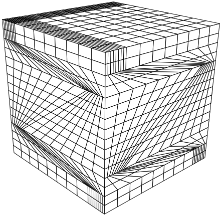
Solver: CG with precond. SSOR,
$n_{iter}^{max} = 128$ for a residual 1.10^{-14}

Min/Max principle

Nsys (#V)	Stencil	erl2	Rate	ergl2	Rate	enerd	Rate
23	11	1.2e-01		5.6e-01		3.6e-01	
311	27	3.5e-02	1.4	3.0e-01	0.7	1.6e-01	0.9
3 119	36	9.0e-03	1.8	1.6e-01	0.9	8.9e-02	0.8
27 743	40	2.3e-03	1.9	7.8e-02	0.9	4.8e-02	0.9
233 663	47	5.6e-04	2.0	3.9e-02	1.0	2.4e-02	0.9

FVCA6 test case

Kershaw meshes

#C	Γ_1	Γ_2	Γ_3
512	6.1e-02	2.4e-03	3.3e-02
4 096	4.6e-02	1.1e-03	2.9e-02
32 768	3.9e-02	3.5e-04	2.7e-02
262 144	4.2e-02	1.5e-04	2.6e-02

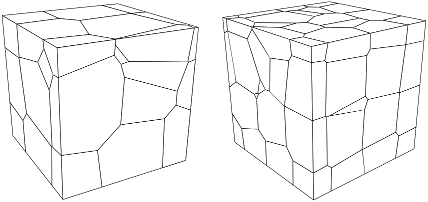
Solver: CG with precond. SSOR,
$n_{iter}^{max} = 267$ for a residual 1.10^{-13}

Min/Max principle

Nsys (#V)	Stencil	erl2	Rate	ergl2	Rate	enerd	Rate
343	19	9.5e-02		7.2e-01		2.2e+00	
3 375	23	4.8e-02	0.9	4.4e-01	0.6	9.7e-01	1.1
29 791	25	2.1e-02	1.1	2.5e-01	0.8	3.1e-01	1.6
250 047	26	7.4e-03	1.5	1.4e-01	0.9	9.0e-02	1.8

FVCA6 test case

Voronoï meshes



#C	Γ_1	Γ_2	Γ_3	warp
29	2.7e-01	3.8e-03	1.1e-01	0.1
66	2.0e-01	2.6e-03	1.4e-01	1.2
130	2.0e-01	3.3e-03	1.2e-01	1.0
228	8.8e-02	2.6e-03	8.6e-02	4.4
356	1.1e-01	2.0e-03	7.5e-02	10.8

Solver: CG with precond. SSOR,
$n_{iter}^{max} = 161$ for a residual 1.10^{-14}

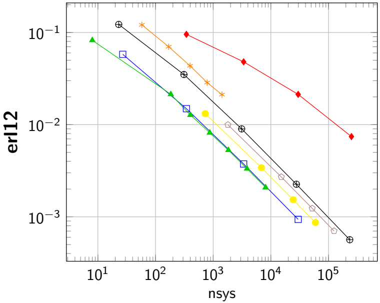
Min/Max principle
$-1.1e^{-2}$ for $\#\mathrm{C} = 228$

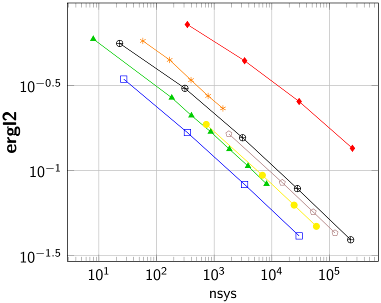
Nsys (#V)	Stencil	erl2	Rate	ergl2	Rate	enerd	Rate
58	31	1.2e-01		5.8e-01		3.9e-01	
169	42	7.0e-02	1.5	4.5e-01	0.7	3.0e-01	0.7
397	51	4.3e-02	1.7	3.4e-01	1.0	2.0e-01	1.4
785	56	2.9e-02	1.8	2.7e-01	0.9	1.4e-01	1.6
1 420	61	2.1e-02	1.5	2.3e-01	0.9	1.2e-01	1.0

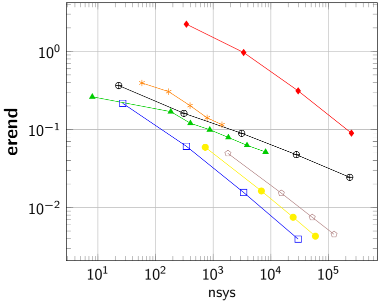
Synthesis

Mesh	Mark	erl12	ergl2	enerd
Hexahedra		1.9	1.0	1.9
Tetrahedra		1.9	1.0	0.8
Prism		1.9	1.0	1.9
Prism-G		1.9	1.0	1.7
Checkerboard		2.0	1.0	0.9
Kershaw		1.5	0.9	1.8
Voronoi		1.5	0.9	1.0

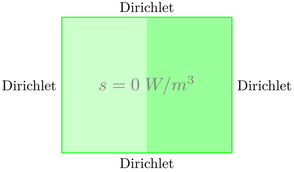
Table: Rate of convergence

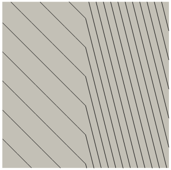






Piecewise linear solution



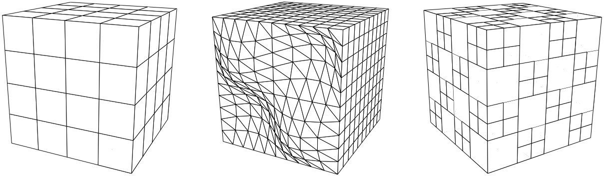


$\lambda_1 = 0.1 W/m/K \qquad x < 0.5\,m$

$\lambda_2 = 1000 W/m/K \qquad x > 0.5\,m$

Mesh	nsys	erl12	ener	enerd
Checkerboard	23	6.6e-17	2.0e-16	2.0e-16
Hexaedral	27	6.6e-17	2.0e-16	2.0e-16
Prism	729	8.6e-16	1.4e-14	1.4e-14

Table: Error on coarse meshes



Outline

CDO for Stokes equations

- Formulations
- Discrete systems
- Analysis of vertex-based schemes
- Numerical results

Formulations

Let $\underline{u}$ be the velocity, p the pressure, $\underline{\omega}$ the vorticity, and $\underline{f}$ the external load

Classical

$$(\underline{u}, p) \quad \begin{cases} -\underline{\Delta}(\underline{u}) + \underline{\mathrm{grad}}(p) & = \underline{f} \\ -\mathrm{div}(\underline{u}) & = 0 \end{cases}$$

(MFV) Droniou & Eymard
(MFD) Beirão da Veiga et al.
(DDFV) Krell & Manzini
(HFV) Di Pietro & Lemaire

Two-field Curl

$$(\underline{u}, p) \quad \begin{cases} \underline{\mathrm{curl}}\,\underline{\mathrm{curl}}(\underline{u}) + \underline{\mathrm{grad}}(p) & = \underline{f} \\ -\mathrm{div}(\underline{u}) & = 0 \end{cases}$$

(FE) Bramble & Lee
(FE) Abboud et al.
(FV) Eymard et al.

Three-field Curl

$$(\underline{u}, \underline{\omega}, p) \quad \begin{cases} -\underline{\omega} + \underline{\mathrm{curl}}(\underline{u}) & = 0 \\ \underline{\mathrm{curl}}(\underline{\omega}) + \underline{\mathrm{grad}}(p) & = \underline{f} \\ -\mathrm{div}(\underline{u}) & = 0 \end{cases}$$

(FE) Nédélec; Dubois
(Spectral) Bernardi & Chorfi
(MSEM) Kreeft & Gerritsma
(DDFV) Delcourte & Omnes

Discrete systems

Vertex-based Pressure

Two-field curl formulation

- Pressure DoFs $\in \mathcal{V}$
- Velocity DoFs $\in \mathcal{E}$ (circulation)

System size: $\#\mathrm{V} + \#\mathrm{E}$

Cell-based Pressure

Three-field curl formulation

- Pressure DoFs $\in \widetilde{\mathcal{V}}$ (1:1 with $\mathcal{C}$)
- Velocity DoFs $\in \mathcal{F}$ (flux)
- Vorticity DoFs $\in \mathcal{E}$

System size: $\#\mathrm{C} + \#\mathrm{F} + \#\mathrm{E}$

Main Features

$\rightarrow$ Exact mass and momentum balances on polyhedral meshes

$\rightarrow$ 1^{st} order CV rate for smooth solutions

- on velocity, vorticity, and pressure gradient

$\rightarrow$ Robust treatment of external load with large divergence-free or curl-free part

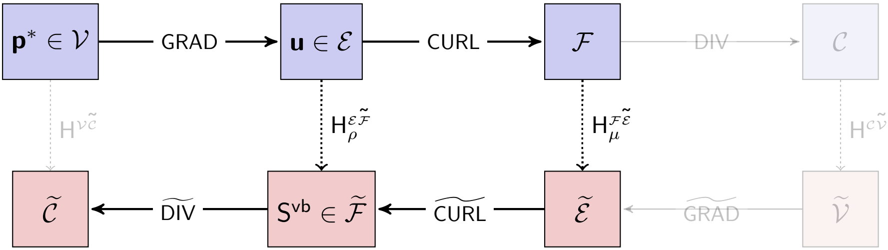
Vertex-based pressure schemes

Two-field Curl

$\underline{u},\ p^* := \frac{p}{\rho}$ ($\rho \equiv 1$ and $\mu \equiv 1$)

$$\begin{cases} \underline{\mathrm{curl}}(\mu\, \underline{\mathrm{curl}}(\underline{u})) + \rho\, \underline{\mathrm{grad}}(p^*) = \rho \underline{f} \\ -\mathrm{div}(\rho \underline{u}) = 0 \end{cases}$$

- Pressure DoFs: $\mathbf{p}^*$ at primal vertices
- Velocity DoFs: $\mathbf{u}$ at primal edges
- $\mathsf{H}_{\rho}^{\mathcal{E}\tilde{\mathcal{F}}}$ and $\mathsf{H}_{\mu}^{\mathcal{F}\tilde{\mathcal{E}}}$



Exact mass balance

Exact momentum balance

$$\begin{cases} \widetilde{\mathsf{CURL}} \cdot \mathsf{H}_{\mu}^{\mathcal{F}\tilde{\mathcal{E}}} \cdot \mathsf{CURL}(\mathbf{u}) + \mathsf{H}_{\rho}^{\mathcal{E}\tilde{\mathcal{F}}} \cdot \mathsf{GRAD}(\mathbf{p}^*) & = \mathsf{S}^{\mathrm{vb}}(\underline{f}) \\ -\widetilde{\mathsf{DIV}} \cdot \mathsf{H}_{\rho}^{\mathcal{E}\tilde{\mathcal{F}}}(\mathbf{u}) & = 0_{\tilde{\mathcal{C}}} \end{cases}$$

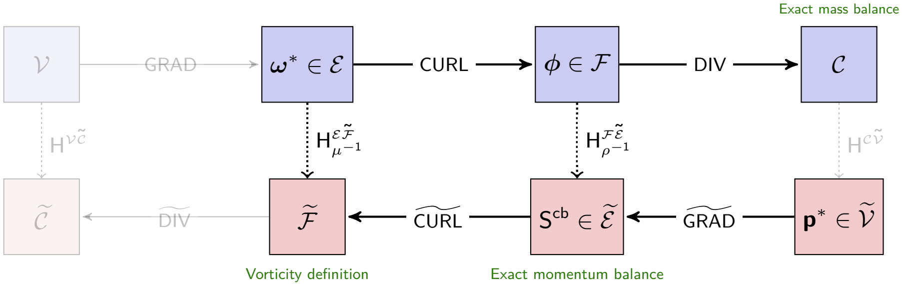
Cell-based pressure schemes

Three-field Curl

$\underline{\phi} := \rho \underline{u}$, $\underline{\omega}^* := \mu \underline{\omega}$, $p^* := \frac{p}{\rho}$

$$\begin{cases} -\mu^{-1}\underline{\omega}^* + \underline{\mathrm{curl}}(\rho^{-1}\underline{\phi}) = \underline{0} \\ \rho^{-1}\,\underline{\mathrm{curl}}(\underline{\omega}^*) + \underline{\mathrm{grad}}(p^*) = \underline{f} \\ -\mathrm{div}(\underline{\phi}) = 0 \end{cases}$$

- Pressure DoFs: $\mathbf{p}^*$ at dual vertices
- Velocity DoFs: $\boldsymbol{\phi}$ at primal faces
- Vorticity DoFs: $\boldsymbol{\omega}^*$ at primal edges
- $\mathsf{H}^{\mathcal{E}\tilde{\mathcal{F}}}_{\mu^{-1}}$ and $\mathsf{H}^{\mathcal{F}\tilde{\mathcal{E}}}_{\rho^{-1}}$



$$\begin{cases} -\mathsf{H}^{\mathcal{E}\tilde{\mathcal{F}}}_{\mu^{-1}}(\boldsymbol{\omega}^*) + \widetilde{\mathrm{CURL}} \cdot \mathsf{H}^{\mathcal{F}\tilde{\mathcal{E}}}_{\rho^{-1}}(\boldsymbol{\phi}) & = 0_{\tilde{\mathcal{F}}} \\ \mathsf{H}^{\mathcal{F}\tilde{\mathcal{E}}}_{\rho^{-1}} \cdot \mathrm{CURL}(\boldsymbol{\omega}^*) + \widetilde{\mathrm{GRAD}}(\mathbf{p}^*) & = \mathsf{S}^{\mathrm{cb}}(\underline{f}) \\ -\mathrm{DIV}(\boldsymbol{\phi}) & = 0_{\mathcal{C}} \end{cases}$$

Analysis of vertex-based schemes

Stability & well-posedness hinge on Poincaré inequalities + Hodge stability

→ Mesh regularity: Assume there exists a shape-regular simplicial submesh

→ • $\|\mathbf{a}\|^2_{2,\mathcal{X}_c} = \sum_{\mathrm{x}\in \mathrm{X}_c} h_c^3 (\frac{\mathbf{a}_\mathrm{x}}{|\mathrm{x}|})^2$ where $\mathbf{a} \in \mathcal{X} \in \{\mathcal{V}, \mathcal{E}, \mathcal{F}\}$ and $\mathrm{X} \in \{\mathrm{V}, \mathrm{E}, \mathrm{F}\}$

• $\|\mathbf{a}\|^2_{2,\mathcal{X}}$ collects local contributions

1 Discrete Poincaré-Wirtinger inequality

There exists $C_P^{(0)} > 0$ s.t. $\forall \mathbf{p} \in \mathcal{V}$ verifying $\left[\!\left[\mathbf{p}, \mathsf{H}^{\mathcal{V}\tilde{\mathcal{C}}}(\mathbf{1})\right]\!\right]_{\mathcal{V}\tilde{\mathcal{C}}} = 0$

$$\|\mathbf{p}\|_{2,\mathcal{V}} \leq C_P^{(0)} \|\mathsf{GRAD}(\mathbf{p})\|_{2,\mathcal{E}}$$

2 Discrete Poincaré inequality for CURL

There exists $C_P^{(1)} > 0$ s.t. $\forall \mathbf{u} \in \mathcal{E}$ verifying $\left[\!\left[\mathbf{u}, \mathsf{H}_\alpha^{\mathcal{E}\tilde{\mathcal{F}}}(\mathbf{v})\right]\!\right]_{\mathcal{E}\tilde{\mathcal{F}}} = 0$ where $\mathbf{v} \in \mathrm{Ker}\,\mathsf{CURL}$

$$\|\mathbf{u}\|_{2,\mathcal{E}} \leq C_P^{(1)} \|\mathsf{CURL}(\mathbf{u})\|_{2,\mathcal{F}}$$

Sketch

→ Consider conforming reconstructions on polyhedral meshes (Christiansen)
- use compatibility property with the differential operators

→ Use the continuous Poincaré inequalities

Error estimates

$\mathbb{P}_0$-consistency of $\mathsf{H}_\alpha^{\mathcal{X}\tilde{\mathcal{Y}}}$ → $\lfloor \alpha,\ _{\mathcal{X}\tilde{\mathcal{Y}}} \rceil(\underline{A}) = 0,\ \forall \underline{A} \in [\mathbb{P}_0(\mathrm{C})]^3$, $\lfloor \alpha,\ _{\mathcal{X}\tilde{\mathcal{Y}}} \rceil(\underline{A}) := \mathsf{R}_{\tilde{\mathcal{Y}}}(\alpha\,\underline{A}) - \mathsf{H}_\alpha^{\mathcal{X}\tilde{\mathcal{Y}}} \cdot \mathsf{R}_{\mathcal{X}}(\underline{A})$ → 1^{st} order CV on smooth solutions

Let $(\underline{u}, p)$ be the exact solution and $\underline{\omega} = \underline{\mathrm{curl}}(\underline{u})$, $\underline{g} = \underline{\mathrm{grad}}(p)$
Let $(\mathbf{u}, \mathbf{p})$ solve the discrete system and $\boldsymbol{\omega} = \mathrm{CURL}(\mathbf{u})$, $\mathbf{g} = \mathrm{GRAD}(\mathbf{p})$

Pressure Gradient	$\Vert \mathsf{R}_{\mathcal{E}}(\underline{g}) - \mathbf{g} \Vert_\rho$	$\lesssim E$
Vorticity	$\Vert \mathsf{R}_{\mathcal{F}}(\underline{\omega}) - \boldsymbol{\omega} \Vert_\mu$	$\lesssim E + \Vert \lfloor \mu,\ _{\mathcal{F}\tilde{\mathcal{E}}} \rceil(\underline{\omega}) \Vert_{(\mu)^{-1}}$
Velocity	$\Vert \mathsf{R}_{\mathcal{E}}(\underline{u}) - \mathbf{u} \Vert_\rho$	$\lesssim E + \Vert \lfloor \mu,\ _{\mathcal{F}\tilde{\mathcal{E}}} \rceil(\underline{\omega}) \Vert_{(\mu)^{-1}} + \Vert \lfloor \rho,\ _{\mathcal{E}\tilde{\mathcal{F}}} \rceil(\underline{u}) \Vert_{(\rho)^{-1}}$

where $\Vert \bullet \Vert_\alpha := \left[\!\left[\bullet, \mathsf{H}_\alpha^{\mathcal{X}\tilde{\mathcal{Y}}}(\bullet) \right]\!\right]_{\mathcal{X}\tilde{\mathcal{Y}}}$, $\Vert \bullet \Vert_{(\alpha)^{-1}} := \left[\!\left[(\mathsf{H}_\alpha^{\mathcal{X}\tilde{\mathcal{Y}}})^{-1}(\bullet), \bullet \right]\!\right]_{\mathcal{X}\tilde{\mathcal{Y}}}$ for $\mathsf{H}_\alpha^{\mathcal{X}\tilde{\mathcal{Y}}} \in \{\mathsf{H}_\rho^{\mathcal{E}\tilde{\mathcal{F}}}, \mathsf{H}_\mu^{\mathcal{F}\tilde{\mathcal{E}}}\}$

$\mathsf{S}^{\mathrm{vb}}(\rho, \underline{f})$

- (PL) $\mathsf{H}_\rho^{\mathcal{E}\tilde{\mathcal{F}}} \cdot \mathsf{R}_{\mathcal{E}}(\underline{f})$ → $E := \Vert \lfloor \rho,\ _{\mathcal{E}\tilde{\mathcal{F}}} \rceil(\underline{\mathrm{curl}}\,\underline{\omega}) \Vert_{(\rho)^{-1}}$

  Better when $\underline{\mathrm{curl}}(\underline{\omega}) \ll \underline{\mathrm{grad}}(p)$ ⇒ Large curl-free part

- (DL) $\mathsf{R}_{\tilde{\mathcal{F}}}(\rho \underline{f})$ → $E := \Vert \lfloor \rho,\ _{\mathcal{E}\tilde{\mathcal{F}}} \rceil(\underline{\mathrm{grad}}\, p) \Vert_{(\rho)^{-1}}$

  Better when $\underline{\mathrm{grad}}(p) \ll \underline{\mathrm{curl}}(\underline{\omega})$ ⇒ Large divergence-free part

3D Taylor–Green test case

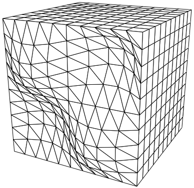
Analytical solutions for $p, \underline{u}$ and $\underline{\omega}$ are a product of sin and cos

Vertex-based Pressure Scheme

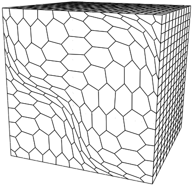
- BCs: Fully Natural $\underline{u} \cdot \underline{\nu}$ and $\underline{\omega}^* \times \underline{\nu}$
- $\mathrm{H}_{\rho}^{\varepsilon\tilde{\mathcal{F}}}$ and $\mathrm{H}_{\mu}^{\mathcal{F}\tilde{\varepsilon}}$ built with the Discrete Geometric Approach (Codecasa et al.)
- Algorithm: Uzawa - Augmented Lagrangian (Preconditioned CG as inner solver)
- System size: $\mathrm{N_{sys}} = \#\mathrm{E} + \#\mathrm{V}$

→ Influence of the Primal external load (PL) and Dual external load (DL)

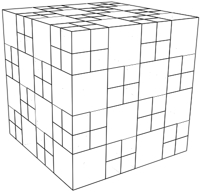
→ Mesh sequences from FVCA VI benchmark



Prismatic meshes

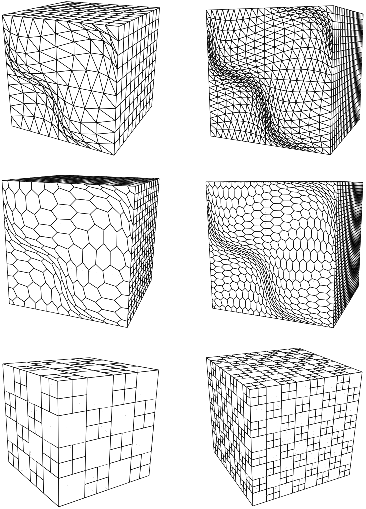


Polyhedral meshes



Polyhedral meshes with hanging nodes

Convergence rates



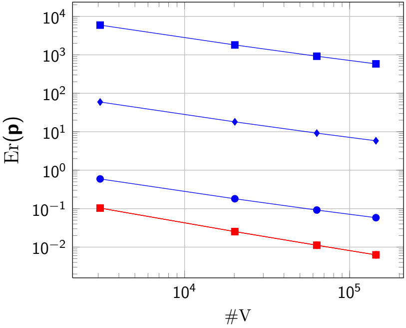
Load	$\mathrm{Er}(\mathbf{p})$	$\mathrm{Er}(\mathbf{u})$	$\mathrm{Er}(\boldsymbol{\omega})$
(PL)	1.8	2.0	1.9
(DL)	2.0	2.0	1.9
(PL)	1.7	1.8	1.7
(DL)	2.1	1.8	1.7
(PL)	1.8	1.1	1.0
(DL)	2.2	1.1	1.0

External load

$$\underline{f} = \underbrace{\chi_{\underline{u}}\,\underline{\mathrm{curl}}(\underline{\Psi}_{\underline{u}})}_{\text{divergence-free}} + \underbrace{\chi_p\,\underline{\mathrm{grad}}(\theta_p)}_{\text{curl-free}}$$

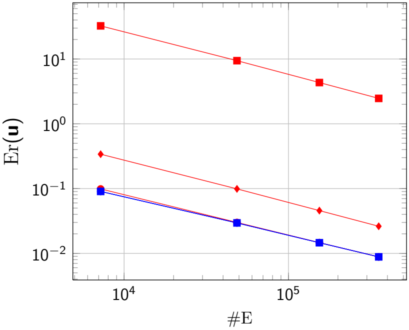
The Hodge-Helmholtz decomposition of $\underline{f}$ is not used in the discretization

$\chi_{\underline{u}} \gg \chi_p$ large divergence-free
(DL) $\mathsf{R}_{\widetilde{\mathcal{F}}}(\underline{f})$ is better suited



$\chi_p = 1$ and $\chi_{\underline{u}} \in \{1, 10^2, 10^4\}$

$\chi_p \gg \chi_{\underline{u}}$ large curl-free
(PL) $\mathsf{H}^{\mathcal{E}\tilde{\mathcal{F}}}\mathsf{R}_{\mathcal{E}}(\underline{f})$ is better suited



$\chi_p \in \{1, 10^2, 10^4\}$ and $\chi_{\underline{u}} = 1$

Conclusions

→ Benefits of CDO approach

- physical fidelity (conservation, body forces, ...)
- unified vision of several schemes
- identify discrete Hodge operator as key design tool

→ Extensions

- convection and Navier–Stokes (in progress), linear elasticity
- high-order methods: Hybrid High-Order HHO methods (Di Pietro & AE, 2014)
- multiscale problems

→ References

- Elliptic problems (Bonelle & AE, M2AN, 2014)
- Stokes equations (Bonelle & AE, HAL preprint, 2014)