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Flow in Fractal Fractured-porous Media: Macroscopic Model with Super-memory, Appearance of Non-linearity and Instability

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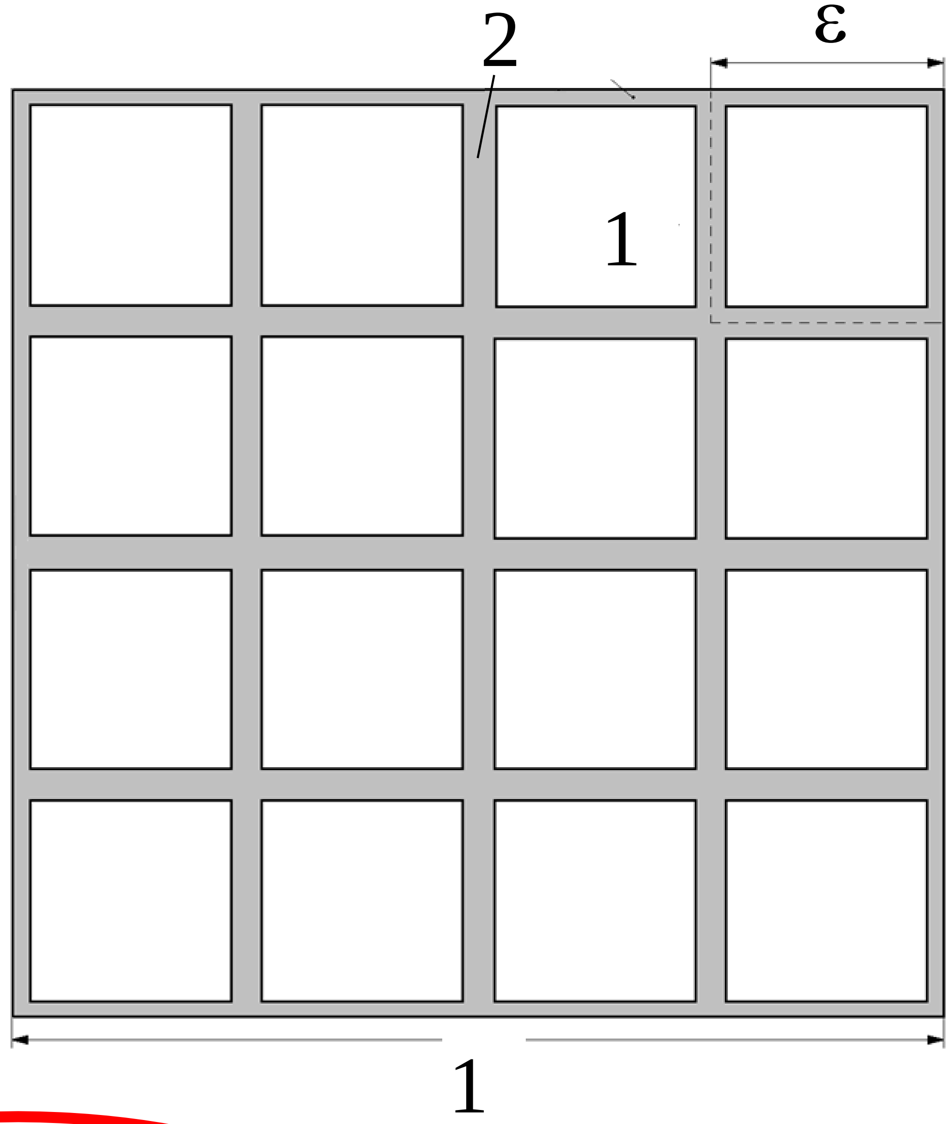
Rasoulzadeh M., Panfilov M., and Kuchuk F.
Effect of memory accumulation in three-scale fractured-porous media.
International Journal of Heat and Mass Transfer, v. 76: 171 – 183 (2014)

Essence of the problem

ε^2- two-scale medium

$$\frac{\text{Diff. coef}_1}{\text{Diff. coef}_2} \sim \varepsilon^2$$



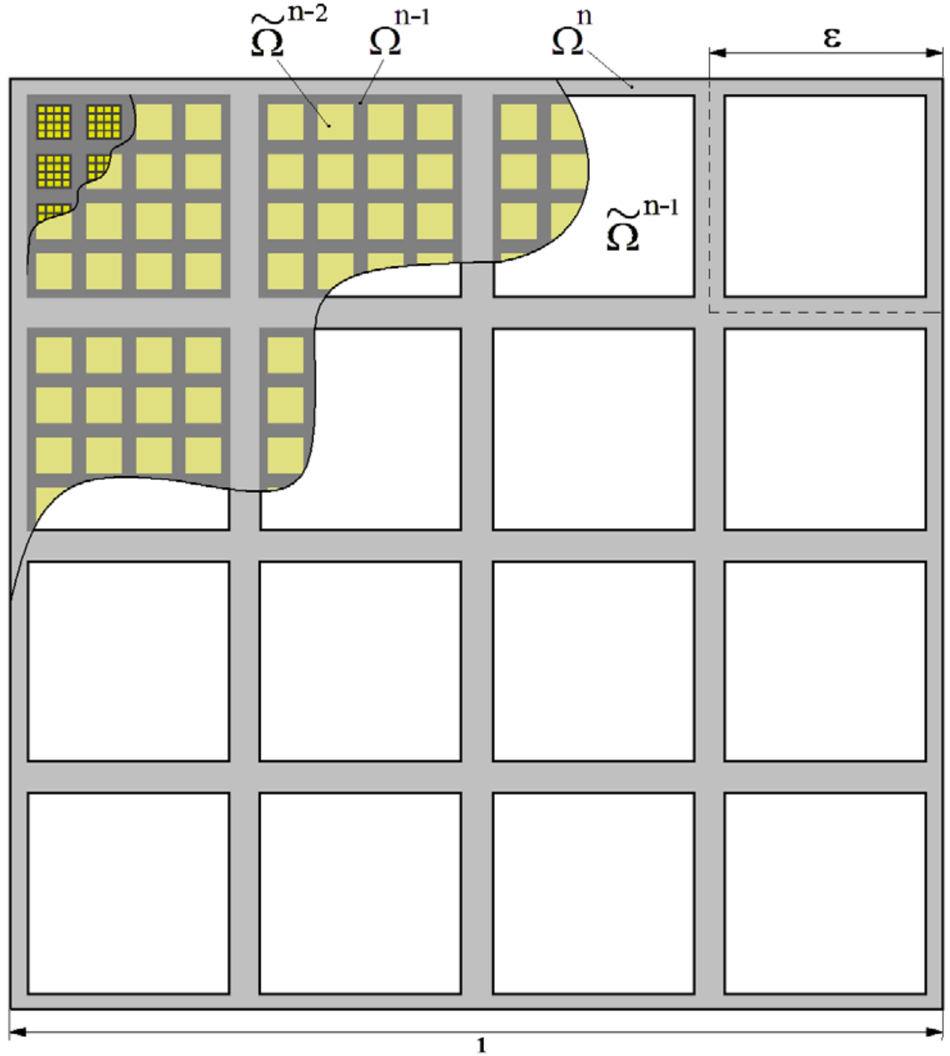
Microscopic equation:

$$b\left(x/\varepsilon\right)\partial_t p = \nabla\cdot\left(\mathrm{a}\left(x/\varepsilon\right)\nabla p\right)$$

Macroscopic model:

$$B\partial_t P = \nabla\cdot\left(A\nabla P\right) + B^1\partial_t\left(\int_0^t K\left(t-\tau\right)\partial_\tau P d\tau\right)$$

ε^2- multi-scale medium



Microscopic equation:

$$b\left(x/\varepsilon\right)\partial_t p = \nabla\cdot\left(\mathrm{a}\left(x/\varepsilon\right)\nabla p\right)$$

Macroscopic model:

???

OBJECTIVES

- to develop the effective model for infinite number of scales

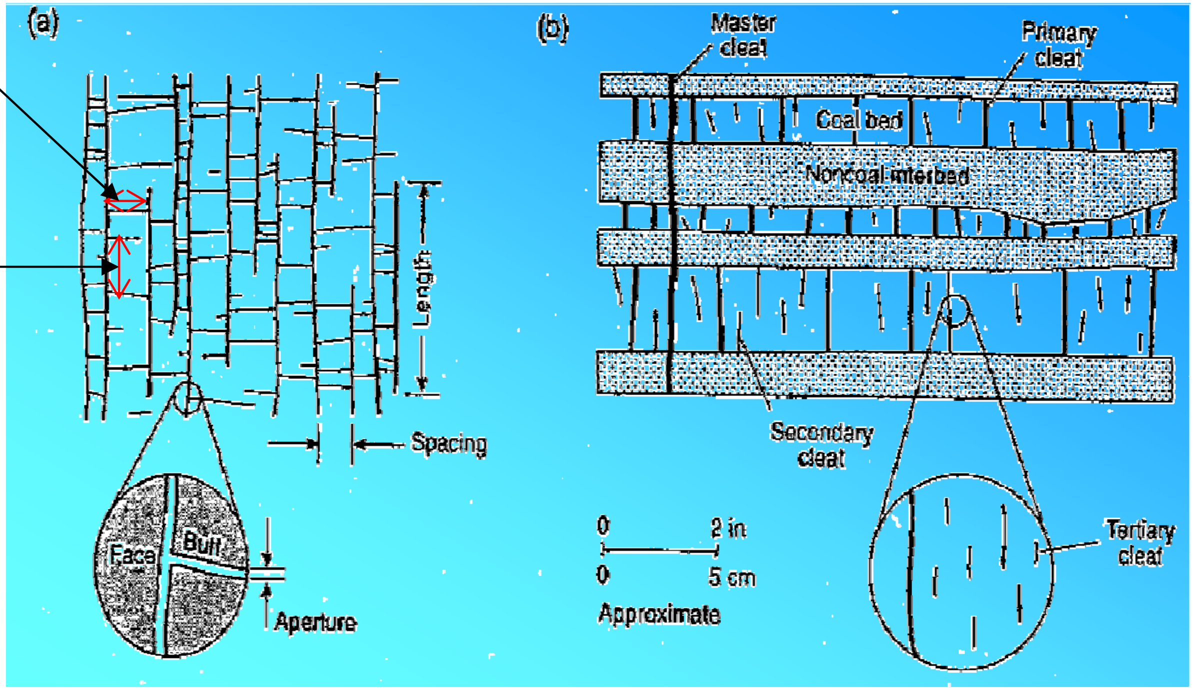
- to develop an algorithm of calculating the effective parameters

APPLICATIONS

Coalbed methane

Largeur = 1,6 cm

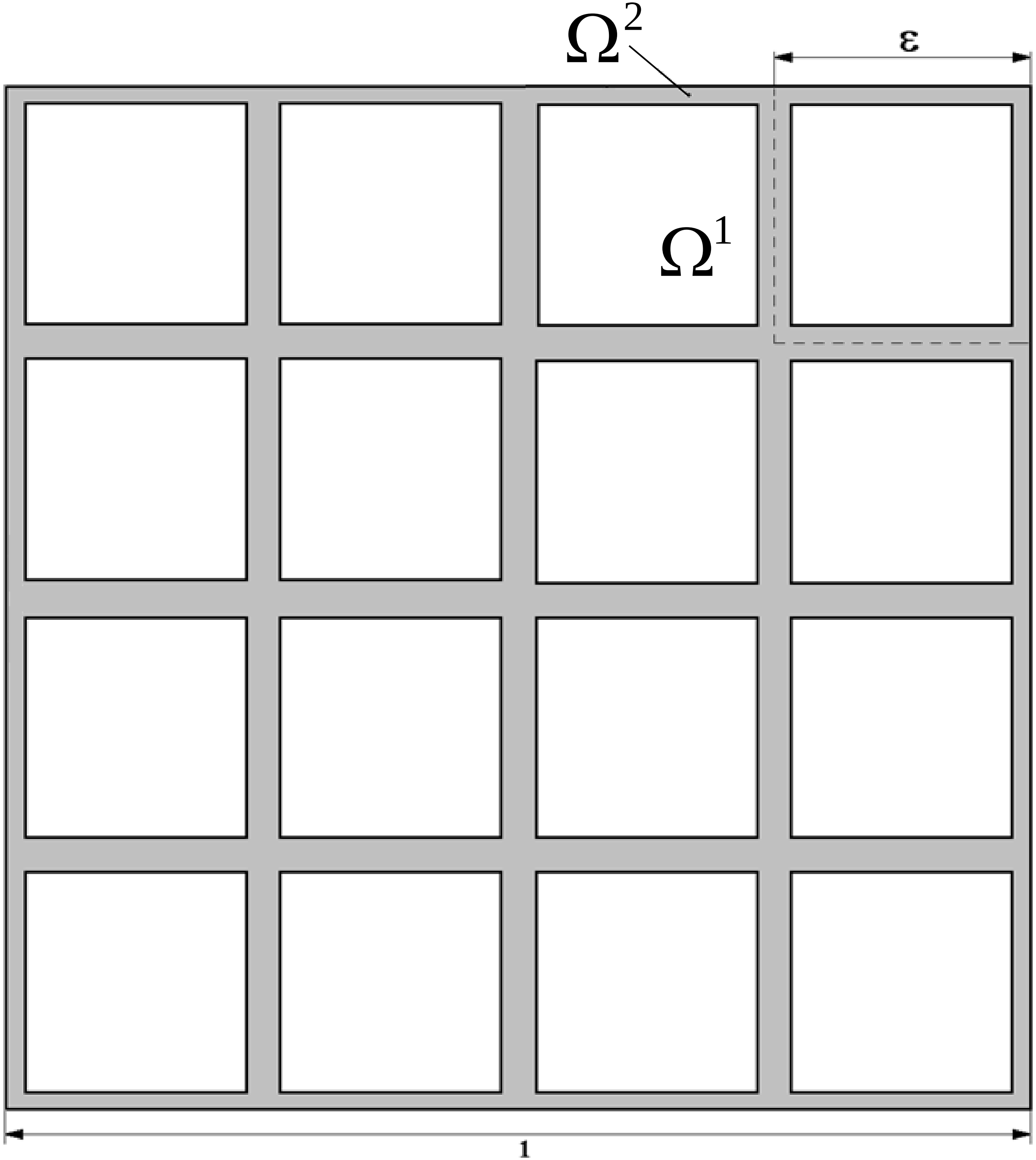
2 * *l* = 2,2 cm



ε^2- multiscale self-similar media

ε^2 - 2-scale medium:

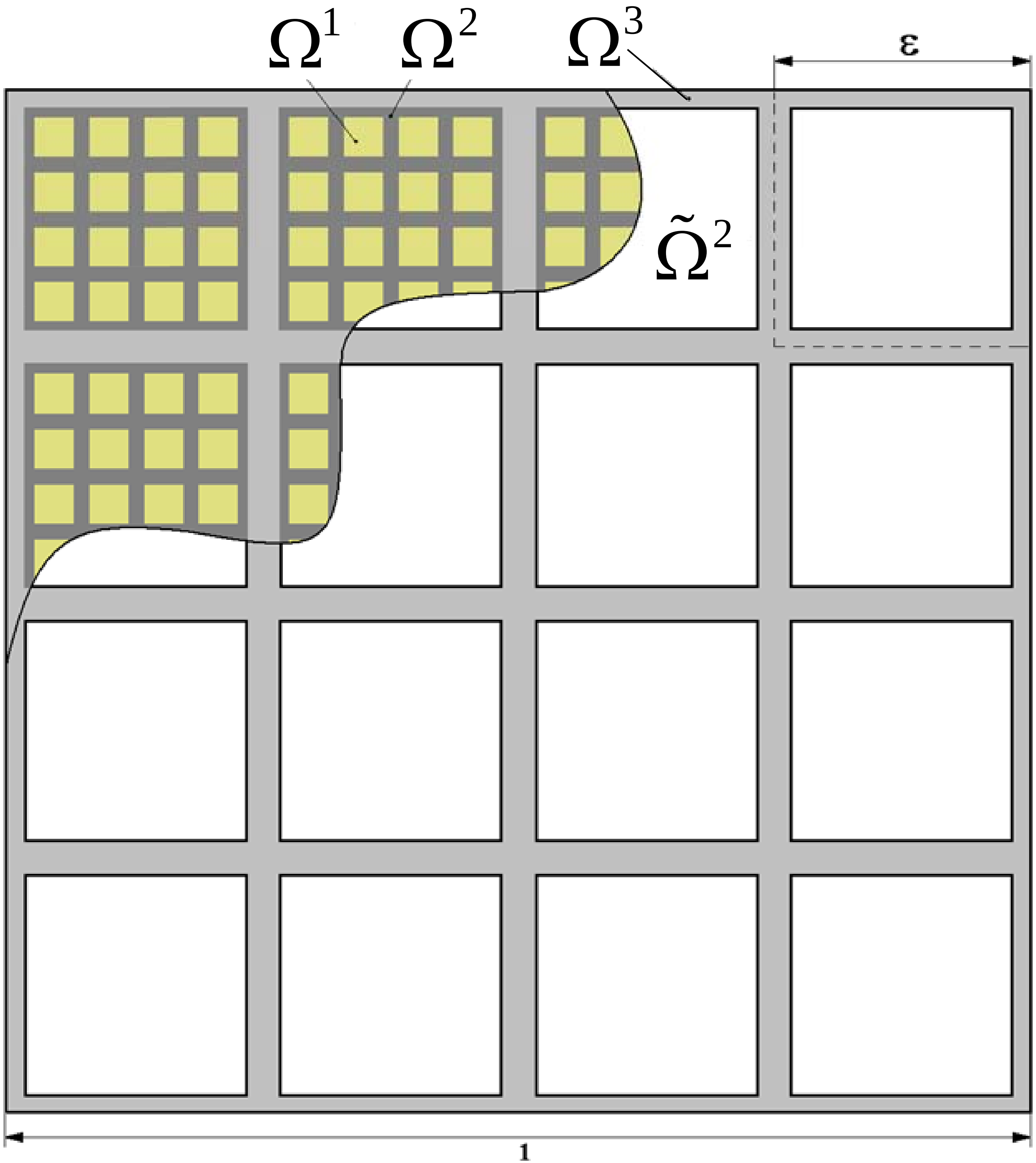
$$\frac{a_1}{a_2} \sim \varepsilon^2$$



ε^2 - 3-scale medium:

$$\frac{a_1}{a_2} \sim \varepsilon^2$$

$$\frac{a_2}{a_3} \sim \varepsilon^2$$

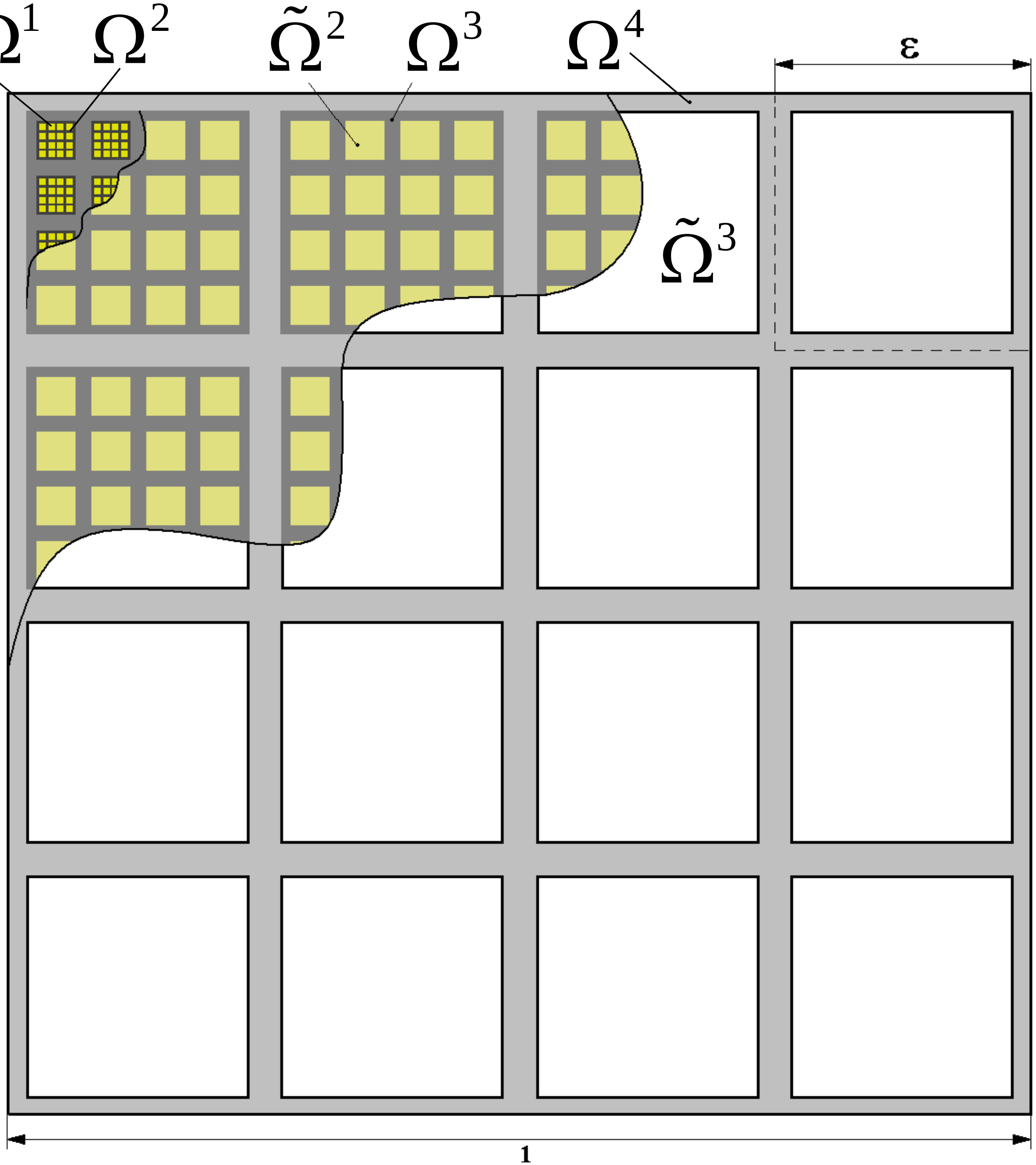


ε^2 - 4-scale medium:

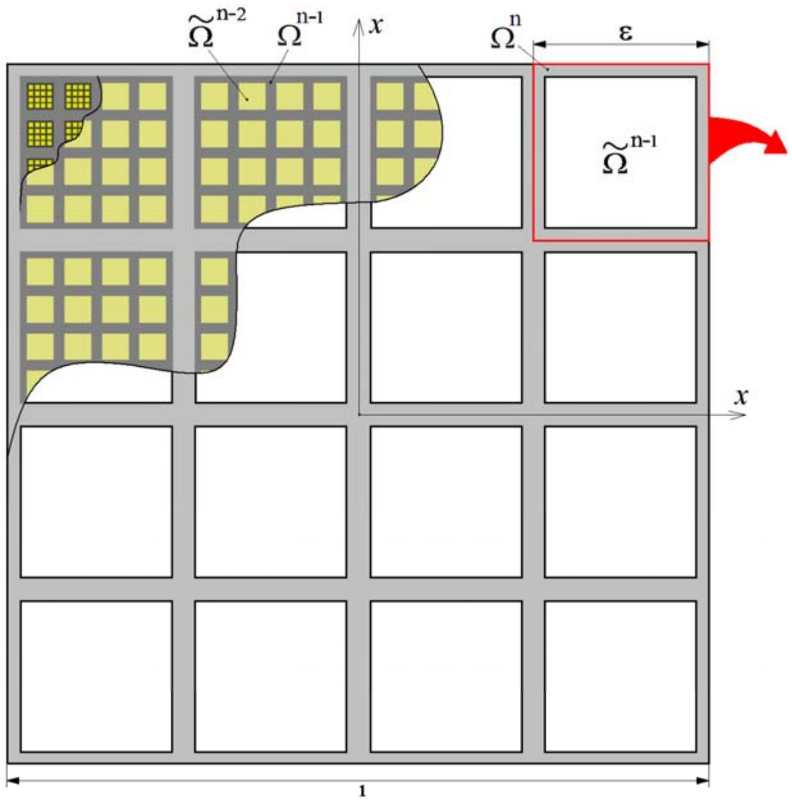
$$\frac{a_1}{a_2} \sim \varepsilon^2$$

$$\frac{a_2}{a_3} \sim \varepsilon^2$$

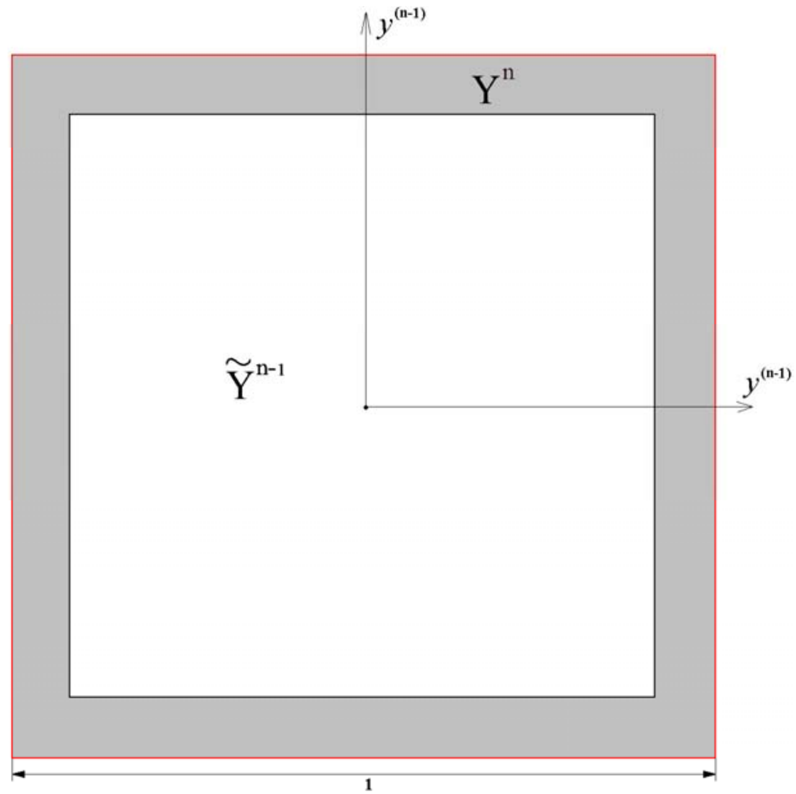
$$\frac{a_3}{a_4} \sim \varepsilon^2$$



Local coordinates for scale *n*



Medium



Periodicity cell at scale *n*

$$y^{n-1} = \frac{x}{\varepsilon}, \qquad y^{n-2} = \frac{x}{\varepsilon^2}, \quad \dots \quad y^i = \frac{x}{\varepsilon^{n-i}} = \frac{y^{i-1}}{\varepsilon}, \quad \dots \quad y^1 = \frac{x}{\varepsilon^{n-1}}$$

For domain Ω^i: the « slow » variable is $y^{(i)}$
the « fast » variable is $y^{(i-1)}$

Transport Equations

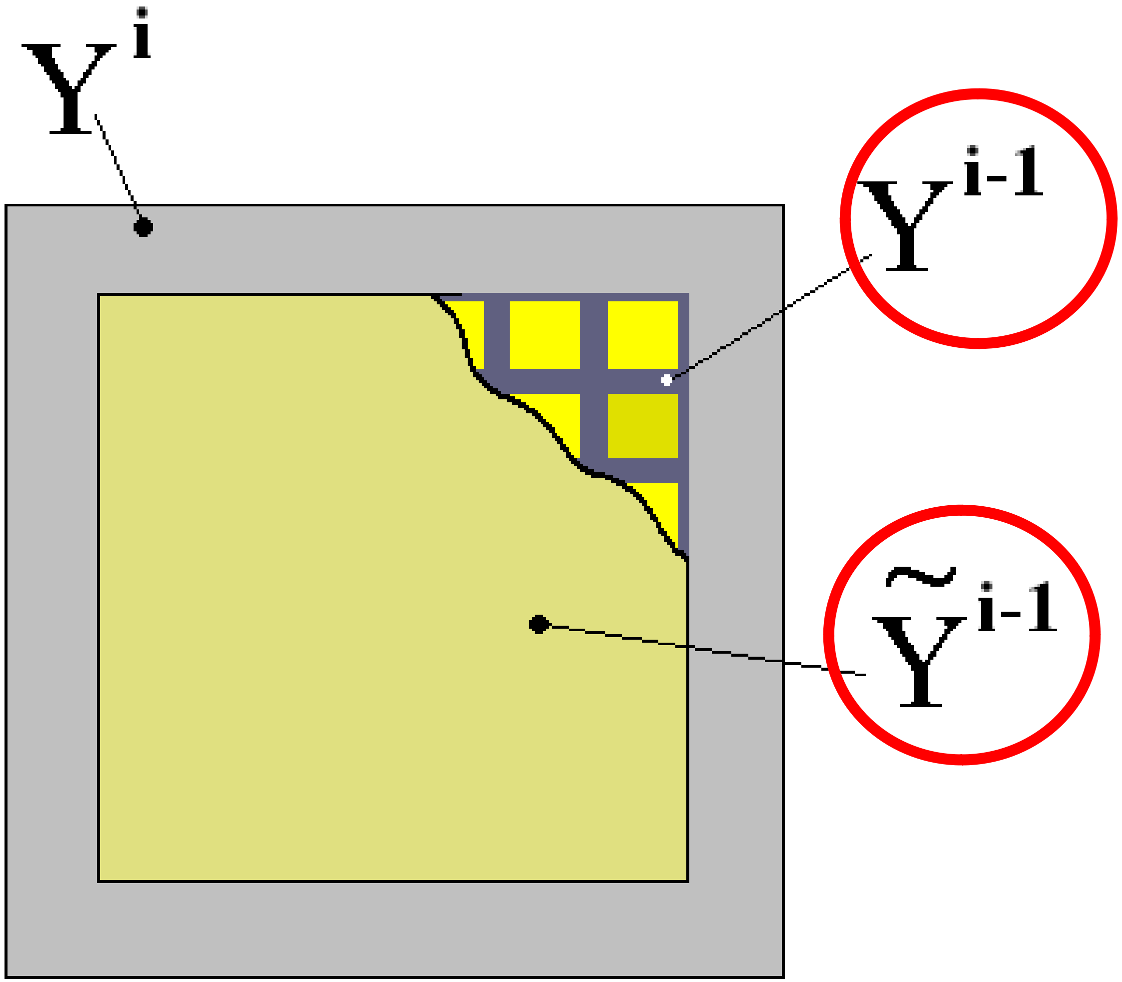
$$\begin{cases} b^{(i)}\dfrac{\partial p}{\partial t} = \nabla\cdot\left(a^{(i)}\nabla p\right), & x\in\Omega^i \\ a^{(i)}\dfrac{\partial p}{\partial n}\bigg|_{\Gamma_-^{i+1,i}} = a^{(i+1)}\dfrac{\partial p}{\partial n}\bigg|_{\Gamma_+^{i+1,i}}, & \\ \left[p\right]_\Gamma = 0 & \end{cases}$$

Condition of high heterogeneity:

$$\frac{a^{(i)}}{a^{(i-1)}} \sim \varepsilon^2 << 1$$

$$\frac{b^{(i)}}{b^{(i-1)}} \sim 1$$

Two types of domains



Method of homogenization

RECURRENT TWO-SCALE HOMOGENIZATION

- Splitting into the series of recurrent two-scale homogenizations

- Assumption about the general form of the averaged equations for any scale;

- Two-scale homogenization at an arbitrary scale

- Closure of the recursion: two-scale homogenization at the lowest scale

Main assumptions:

(1) boundary layers between two scales may be neglected ;

(2) contact condition may be accepted in the natural form:

$$A \frac{\partial P^{i-1}}{\partial n}\bigg|_{\Gamma_{-}^{i,i-1}} = a^{i} \frac{\partial p^{i}}{\partial n}\bigg|_{\Gamma_{+}^{i,i-1}} ; \qquad P^{i-1}\Big|_{\Gamma_{-}^{i,i-1}} = p^{i}\Big|_{\Gamma_{+}^{i,i-1}}$$

RECURRENT TWO-SCALE HOMOGENIZATION

Averaged Medium $\tilde{\Omega}^n$

$\begin{cases} \text{Medium } \Omega^n \\ \text{Averaged Medium } \tilde{\Omega}^{n-1} \end{cases}$

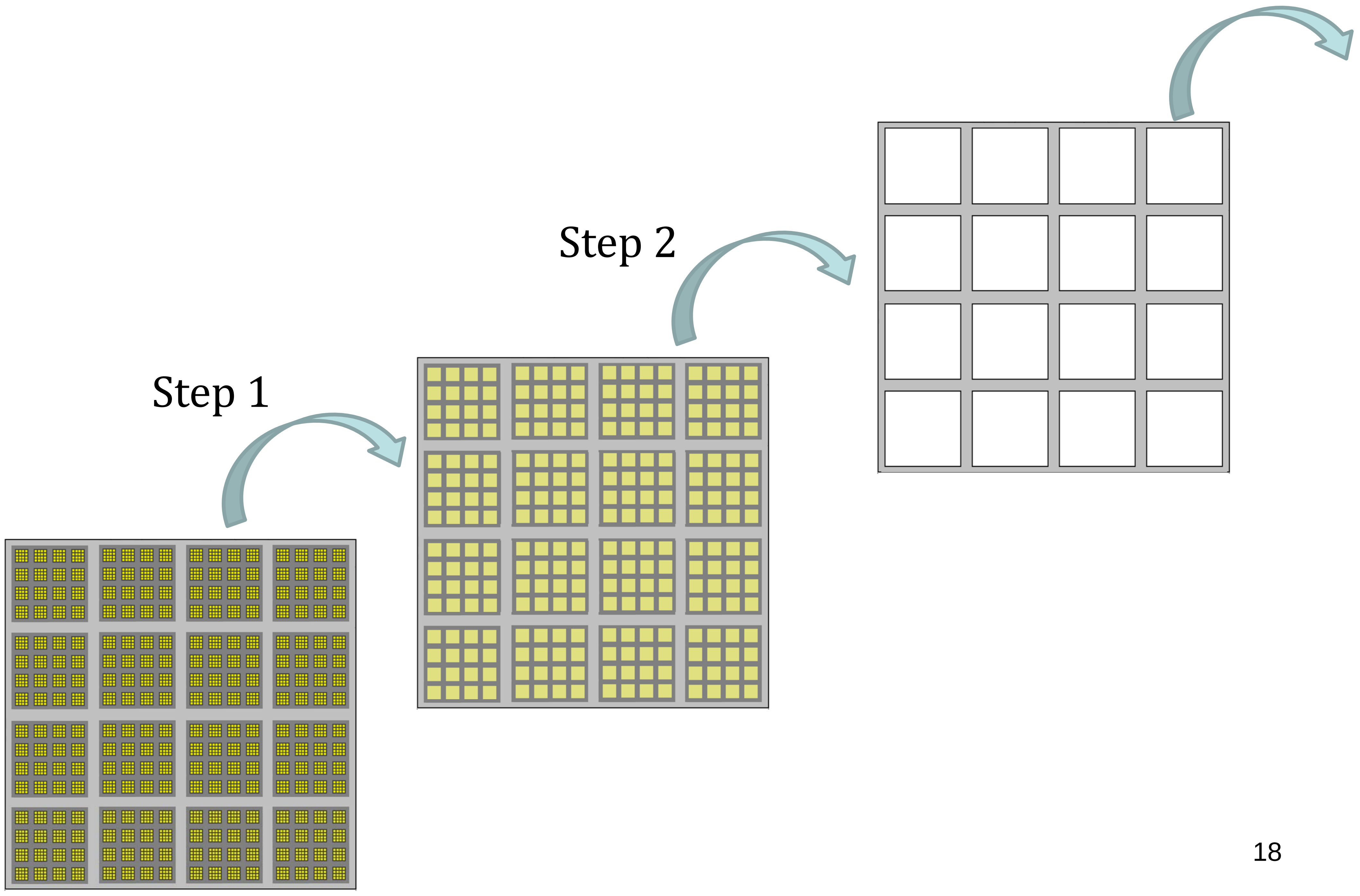
...

$\begin{cases} \text{Medium } \Omega^4 \\ \text{Averaged Medium } \tilde{\Omega}^{3} \end{cases}$

$\begin{cases} \text{Medium } \Omega^3 \\ \text{Averaged Medium } \tilde{\Omega}^{2} \end{cases}$

$\begin{cases} \text{Medium } \Omega^2 \\ \text{Medium } \Omega^{1} \end{cases}$

RECURRENT TWO-SCALE HOMOGENIZATION



ASSUMPTION ABOUT THE GENERAL FORM OF THE AVERAGED EQUATIONS

First step of homogenization (1 → 2):

$$\begin{cases} P^1 = P^2 - K^{(1)} * \dfrac{\partial P^2}{\partial t}, \\ \\ b\,(1-\theta)\,\dfrac{\partial P^2}{\partial t} - \nabla_2 \cdot \left(A \nabla_2 P^2 \right) = -b\theta \dfrac{\partial P^1}{\partial t} \end{cases}$$

By the analogy, for any step i-1 → i:

$$\begin{cases} \tilde{P}^{i-1} = P^i - \dfrac{\partial P^i}{\partial t} * \mathbb{K}^{(i-1)}, \\ \\ b\,(1-\theta)\,\dfrac{\partial P^i}{\partial t} - \nabla_i \cdot \left(A \nabla_i P^i \right) = -b\theta \dfrac{\partial \tilde{P}^{i-1}}{\partial t} * \mathbb{L}^{(i-1)} \end{cases}$$

ASSUMPTION ABOUT THE GENERAL FORM OF THE AVERAGED EQUATIONS

The last equation may be presented in the closed form w.r.t.

$$b(1-\theta)\frac{\partial P^i}{\partial t} - \varepsilon^2 \nabla_i \cdot \left(A \nabla_i P^i \right) = -b\theta \frac{\partial}{\partial t}\left(P^i - \frac{\partial P^i}{\partial t} * \mathbb{K}^{(i-1)} \right) * \mathbb{L}^{(i-1)}$$

Kernels $\mathbb{K}^{(i)}$ **and** $\mathbb{L}^{(i)}$ **:**

$$\mathbb{K}^1 = 1, \qquad \mathbb{L}^1 = \delta(t)$$

The objective is to determine them for any *i*

Arbitrary step "*i*"

Formulation of the problem at scale *i*

$$\begin{cases} b\,(1-\theta)\,\dfrac{\partial P^{i-1}}{\partial t} - \varepsilon^2 \nabla_i\cdot\left(A\nabla_i P^{i-1}\right) = \\ \qquad -b\theta\dfrac{\partial}{\partial t}\left(P^{i-1} - \dfrac{\partial P^{i-1}}{\partial t} * \mathbb{K}^{(i-2)}\right) * \mathbb{L}^{(i-2)}, \qquad y^{(i)} \in \widetilde{\Omega}^{i-1}/\varepsilon^{n-i} \\ \\ b\dfrac{\partial p^i}{\partial t} = \nabla_i\cdot\left(a\nabla_i p^i\right), \quad y^{(i)} \in \Omega^i/\varepsilon^{n-i} \\ \\ \varepsilon^2 A\nabla_i P^{i-1}\cdot\vec{n}\big|_{\Gamma_-^{i,i-1}} = a\nabla p^i\cdot\vec{n}\big|_{\Gamma_+^{i,i-1}}, \qquad P^{i-1}\big|_{\Gamma_-^{i,i-1}} = p^i\big|_{\Gamma_+^{i,i-1}} \end{cases}$$

Homogenization …

ASYMPTOTIC TWO-SCALE HOMOGENIZATION AT EACH STEP

$$y^{(i)} \to \left(y^{(i)}, y^{(i-1)} \right)$$

$y^{(i)} =$ slow variable

$y^{(i-1)} =$ faste variable: $y^{(i-1)} = y^{(i)} / \varepsilon$

$$\frac{\partial}{\partial y^{(i)}} \to \frac{\partial}{\partial y^{(i)}} + \frac{1}{\varepsilon} \frac{\partial}{\partial y^{(i-1)}}$$

Asymptotique expansion w.r.t. ε

Additional condition of periodicity w.r.t. $y^{(i-1)}$

Condition of existence of periodic solutions → Averaged equations

First approximation → Cell problems

Upscaled equations

$$
\begin{cases}
b\,(1-\theta)\,\dfrac{\partial P^i}{\partial t} - \nabla_i \cdot \left(A \nabla_i P^i\right) = \\
\qquad = -b\theta\,(1-\theta)\,\dfrac{\partial \tilde{P}^{i-1}}{\partial t} + b\theta^2\,\mathbb{L}^{(i-2)} * \mathbb{K}^{(i-2)'} * \dfrac{\partial \tilde{P}^{i-1}}{\partial t}, \\
\tilde{P}^{i-1} = P^i - K^{(i-1)} * \left[\dfrac{\partial P^i}{\partial t} - \alpha\,\mathbb{L}^{(i-2)} * \mathbb{K}^{(i-2)'} \dfrac{\partial P^i}{\partial t}\right]
\end{cases}
$$

Comparison with the general form

Result of recurrent homogenization:

$$\begin{cases} \tilde{P}^{i-1} = P^i - K^{(i-1)} * \left[\dfrac{\partial P^i}{\partial t} - \alpha \, \mathbb{L}^{(i-2)} * \mathbb{K}^{(i-2)\prime} \dfrac{\partial P^i}{\partial t} \right] \\ b\,(1-\theta)\, \dfrac{\partial P^i}{\partial t} - \nabla_i \cdot \left(A \nabla_i P^i \right) = \\ \qquad = -b\theta\,(1-\theta)\, \dfrac{\partial \tilde{P}^{i-1}}{\partial t} + b\theta^2\, \mathbb{L}^{(i-2)} * \mathbb{K}^{(i-2)\prime} * \dfrac{\partial \tilde{P}^{i-1}}{\partial t}, \end{cases}$$

General form:

$$\begin{cases} \tilde{P}^{i-1} = P^i - \dfrac{\partial P^i}{\partial t} * \mathbb{K}^{(i-1)} \\ b\,(1-\theta)\, \dfrac{\partial P^i}{\partial t} - \nabla_i \cdot \left(A \nabla_i P^i \right) = -b\theta \dfrac{\partial \tilde{P}^{i-1}}{\partial t} * \mathbb{L}^{(i-1)} \end{cases}$$

Determination of kernels $\mathbb{K}^{(i)}$ and $\mathbb{L}^{(i)}$

$$\mathbb{K}^{(i)} = \begin{cases} K^{(i)} * \left(\delta(t) - \alpha \mathbb{L}^{(i-1)} * \mathbb{K}^{(i-1)'}\right), & i \geq 2 \\ K^{(i)}, & i = 1 \end{cases}$$

$$\mathbb{L}^{(i)} = \begin{cases} (1-\theta)\delta(t) - \theta\, \mathbb{L}^{(i-1)} * \mathbb{K}^{(i-1)'}, & i \geq 2 \\ \delta(t), & i = 1 \end{cases}$$

$$K^{(i)}(t) \equiv \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} T_k^{(i)}(t),$$

$$T_k^{(i)\prime} - \alpha\, T_k^{(i)\prime} * \mathbb{L}^{(i-1)} * \mathbb{K}^{(i-1)'} = -T_k^{(i)} \mu_k \frac{A}{b\,(1-\theta)}, \qquad T_k^{(i)}\Big|_{t=0} = 1$$

Four coupled recurrent equations

Physical meaning of functions

$$\mathbb{K}^{(i)}(t) \qquad K^{(i)}(t) \qquad T_k^{(i)}(t) \qquad \mathbb{L}^{(i)}(t)$$

$\mathbb{K}^{(i)}(t)$ - **"effective exchange kernels"**
cumulated history of the exchange between all the preceding couples of media

$K^{(i)}(t)$ - **"local exchange kernels"**
local mass exchange between two media i and $i-1$

$T_k^{(i)}(t)$ - **"local spectrum (of local kernels)"**
the Fourier-components of the kernel $K^{(i)}$

$\mathbb{L}^{(i)}(t)$ are liner functions of the effective kernels $\mathbb{K}^{(i)}$

Infinite number of scales

Stabilization w.r.t. i when $i \to \infty$

If the limit behaviour exists, then the iterative system of averaged models should prouve a stabilization w.r.t. i

$\mathbb{K}^{(i-1)}, \mathbb{L}^{(i-1)}, K^{(i-1)}$ should tend respectively to $\mathbb{K}^{(i)}, \mathbb{L}^{(i)}, K^{(i)}$

or

$$\mathbb{K}^{(i)}, \mathbb{L}^{(i)}, K^{(i)} \to \underbrace{\mathbb{K}, \mathbb{L}, K}_{\text{Limit kernels}} \quad \text{as} \quad i \to \infty.$$

Pressures P^i and $\tilde{P}^{i-1}$ will also tend to the couple of limit pressures P and $\tilde{P}$

Limit homogenized model

Pressures P^i and $\tilde{P}^{i-1}$ tend to the couple of limit pressures P and $\tilde{P}$

$$\begin{cases} b(1-\theta)\dfrac{\partial P}{\partial t} - \nabla_i \cdot (A\nabla_i P) = -b\theta \mathbb{L} * \dfrac{\partial \tilde{P}}{\partial t}, \\ \tilde{P} = P - \mathbb{K} * \dfrac{\partial P}{\partial t} \end{cases}$$

Effective kernel:

$$(1-\theta)\mathbb{K} + \theta \int_0^t \mathbb{K}(\tau)\mathbb{K}(t-\tau)d\tau = K$$

Appearance of non-linearity in the effective and local kernels

$$(1-\theta)\mathbb{K}+\theta\int_0^t \mathbb{K}(\tau)\mathbb{K}(t-\tau)d\tau = K$$

or in terms of the Laplace transform:

$$\overline{\mathbb{K}}=\frac{1}{\theta}\left(\theta-1+\sqrt{(1-\theta)^2+4\theta\overline{K}}\right)$$

$$\overline{K}=\frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\theta-1+\sqrt{(1-\theta)^2+4\theta\overline{K}}}{s\left(\theta-1+\sqrt{(1-\theta)^2+4\theta\overline{K}}\right)+\dfrac{\alpha A\mu_k\overline{K}}{b}}$$

θ is the volume of a block / the volume of a fracture

Approximation for the kernels

exacte: $$\overline{K} = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\theta - 1 + \sqrt{(1-\theta)^2 + 4\theta\overline{K}}}{s\left(\theta - 1 + \sqrt{(1-\theta)^2 + 4\theta\overline{K}}\right) + \frac{\alpha A\mu_k\overline{K}}{b}}$$

If in the right-hand side: $K(t) \approx \delta(t)$, then $\bar{K}(s) = 1$

Then

$$\overline{K} \approx \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\theta - 1 + \sqrt{(1-\theta)^2 + 4\theta}}{s\left(\theta - 1 + \sqrt{(1-\theta)^2 + 4\theta}\right) + \frac{\alpha A\mu_k}{b}} = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{s + \frac{A\mu_k}{b(1-\theta)}}$$

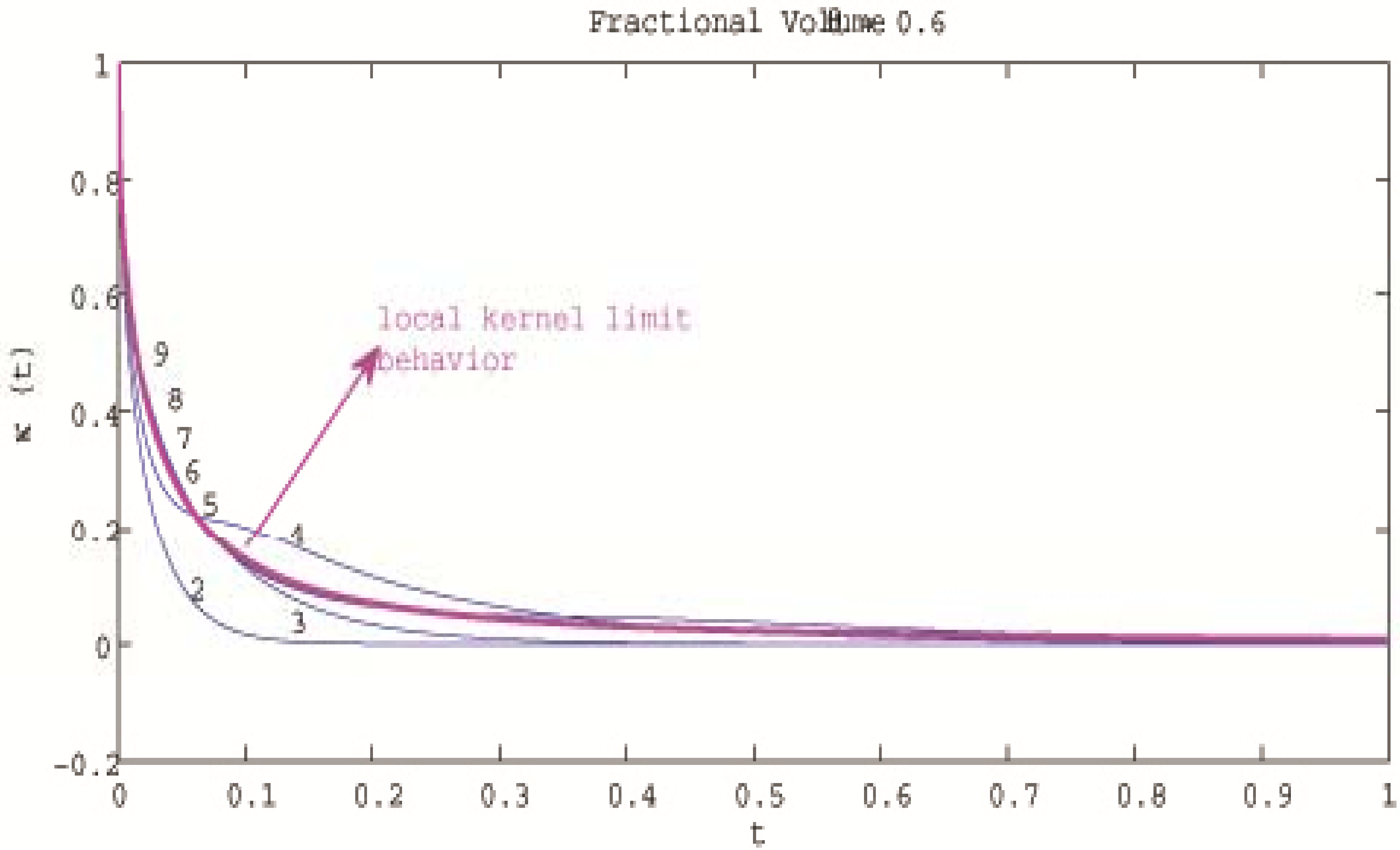
or

$$K(t) \approx \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left\{-\frac{A\mu_k}{b(1-\theta)}t\right\}$$

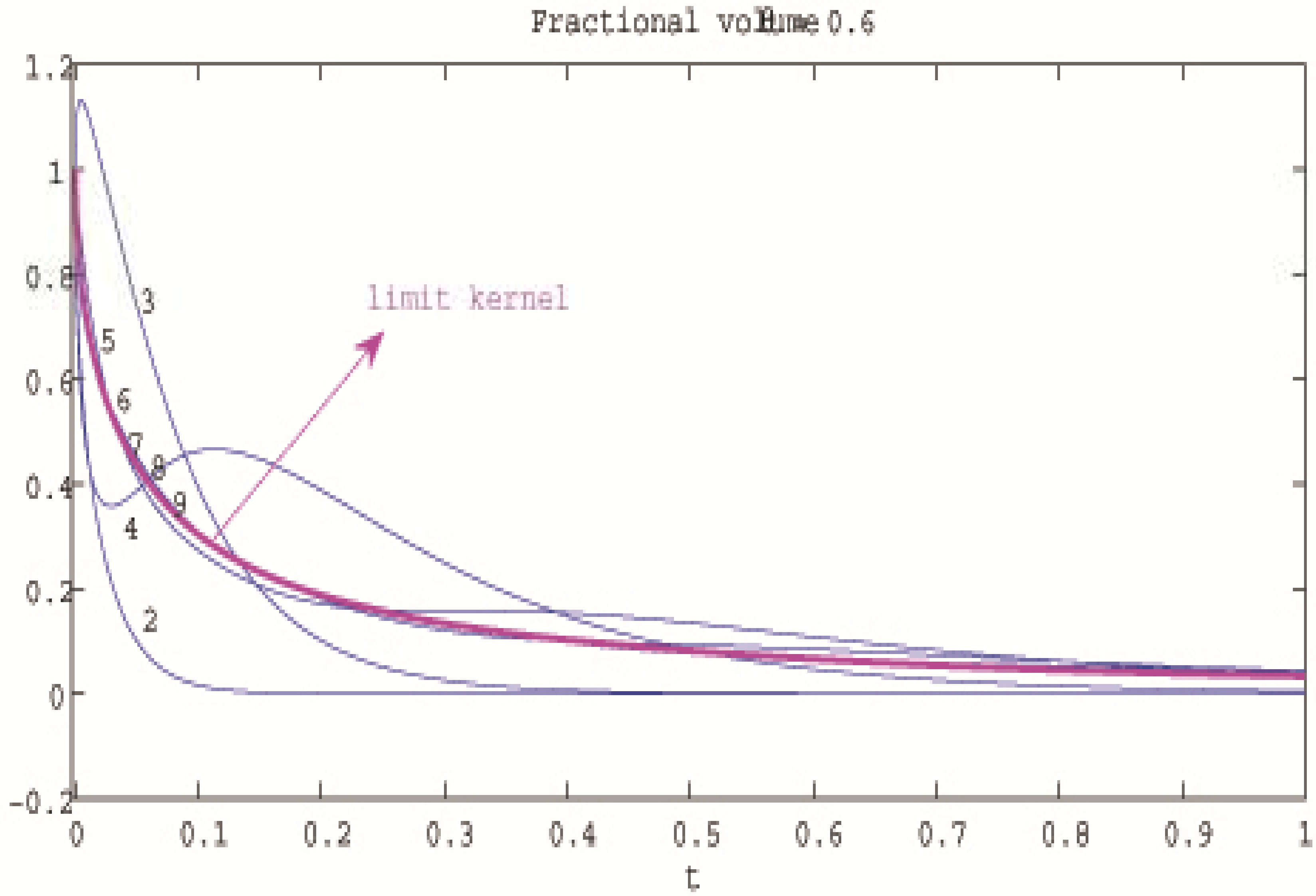
$$\mathbb{K}(s) = \frac{1}{\theta}\left(\theta - 1 + \sqrt{(1-\theta)^2 + 4\theta\frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{s + \frac{A\mu_k}{b(1-\theta)}}}\right)$$

Results of simulation for iterative kernels

Local kernels $K^{(i)}(t)$

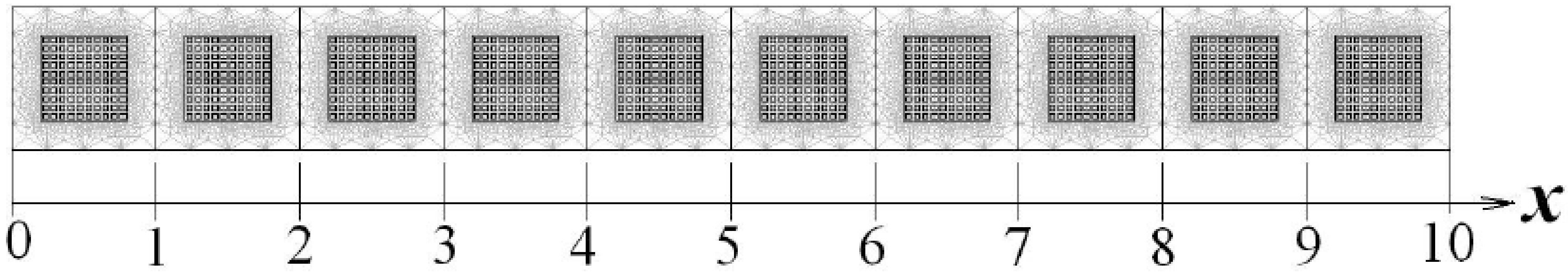


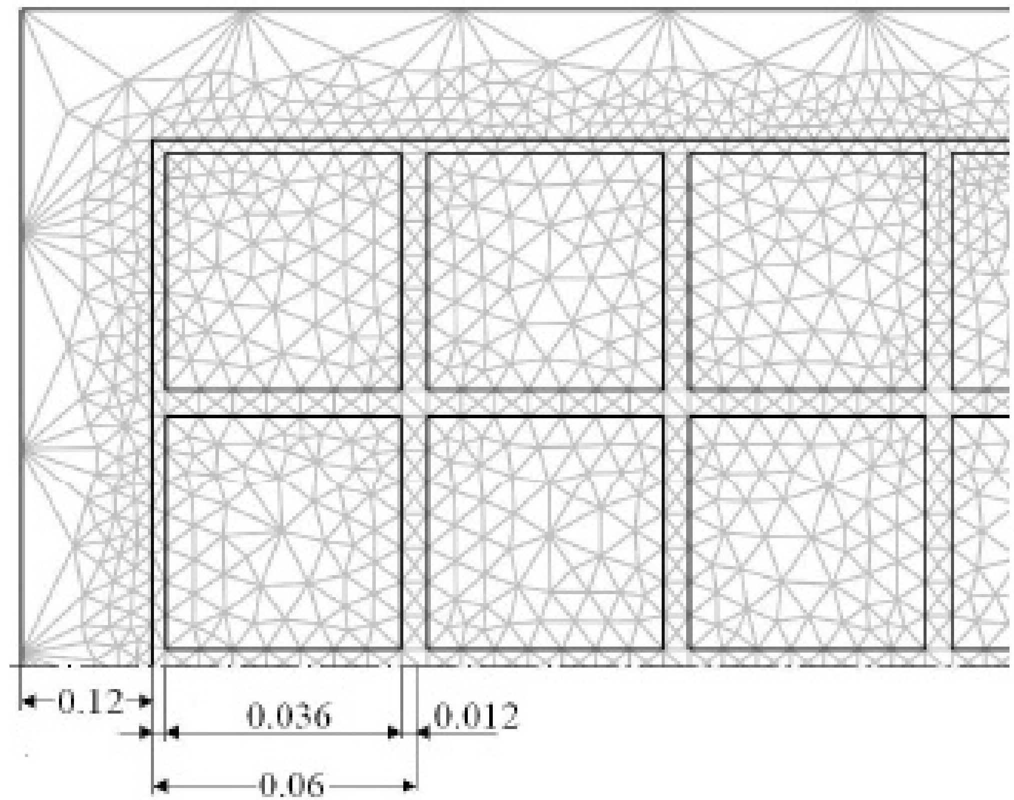
Effective kernels $\mathbb{K}^{(i)}$



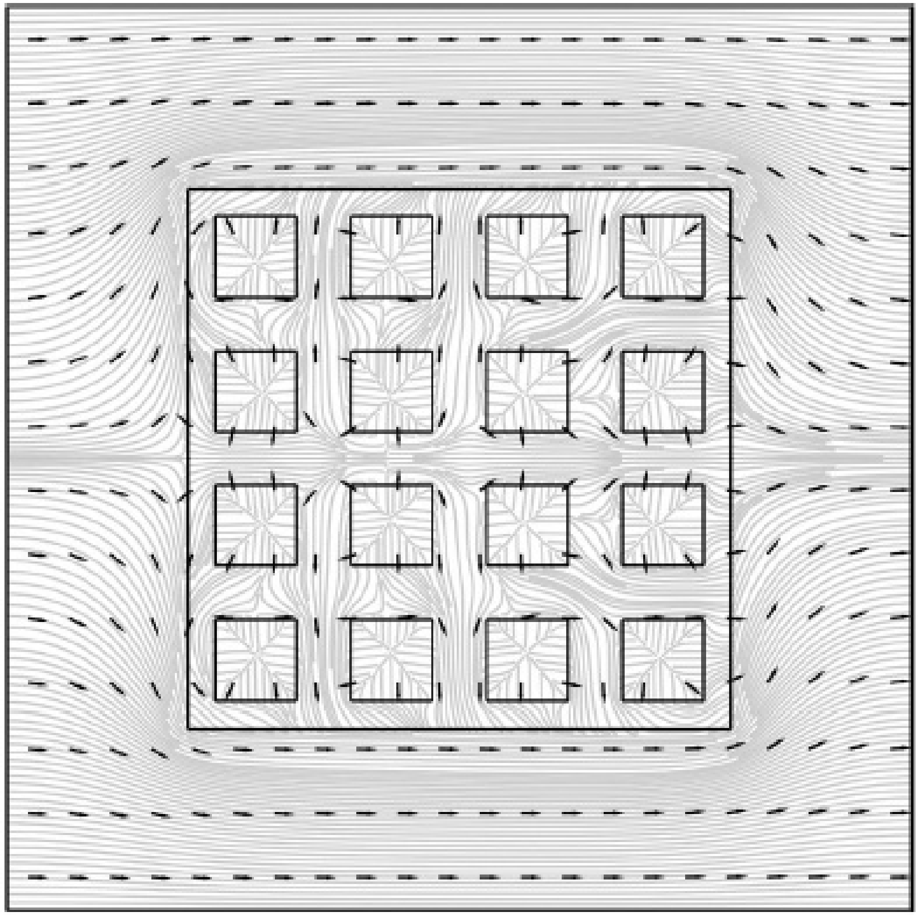
Comparison with microscale numerical simulations

Medium and mesh (Comsol Multiphysics)

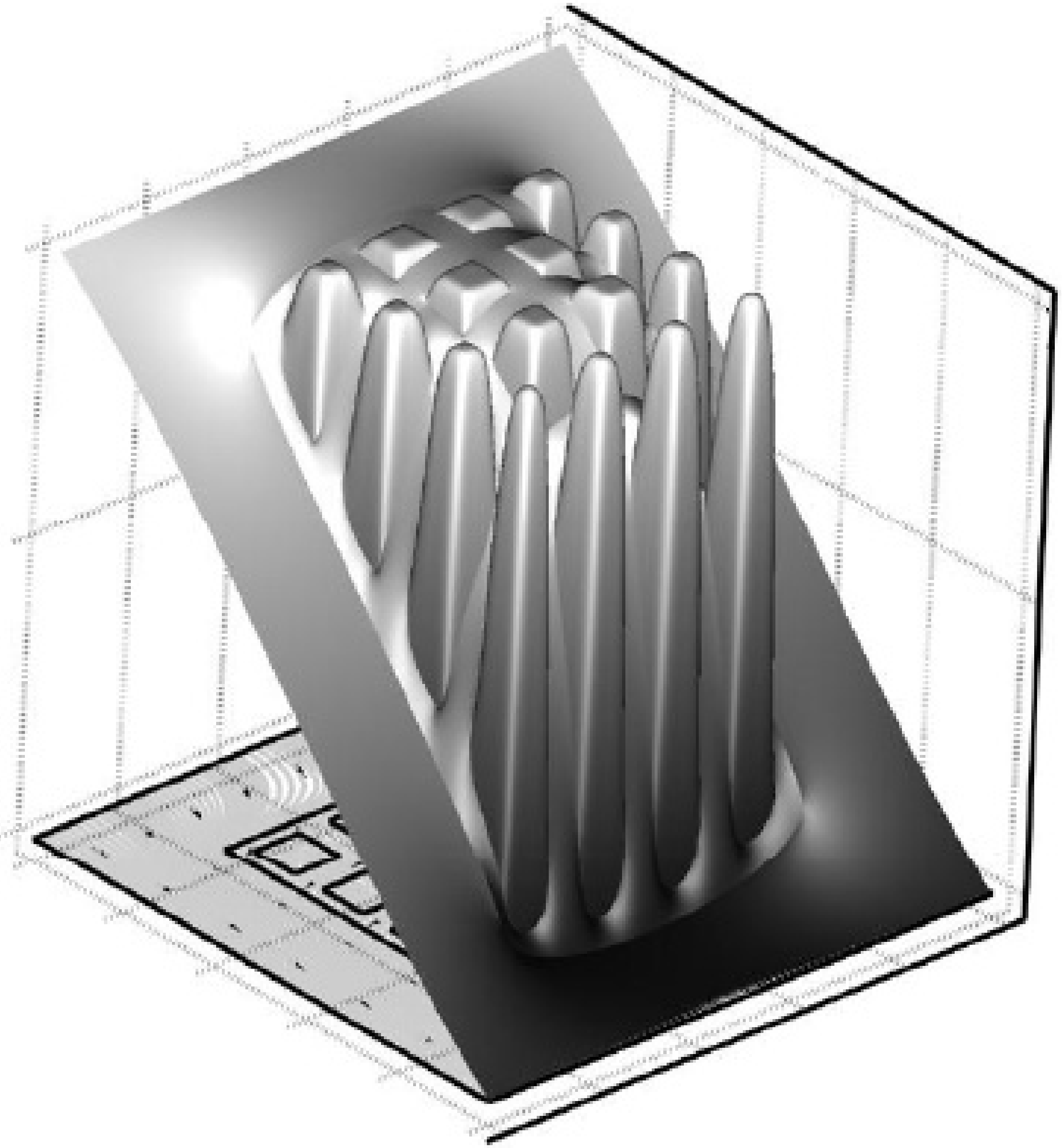




Results of simulation for 3-scale medium

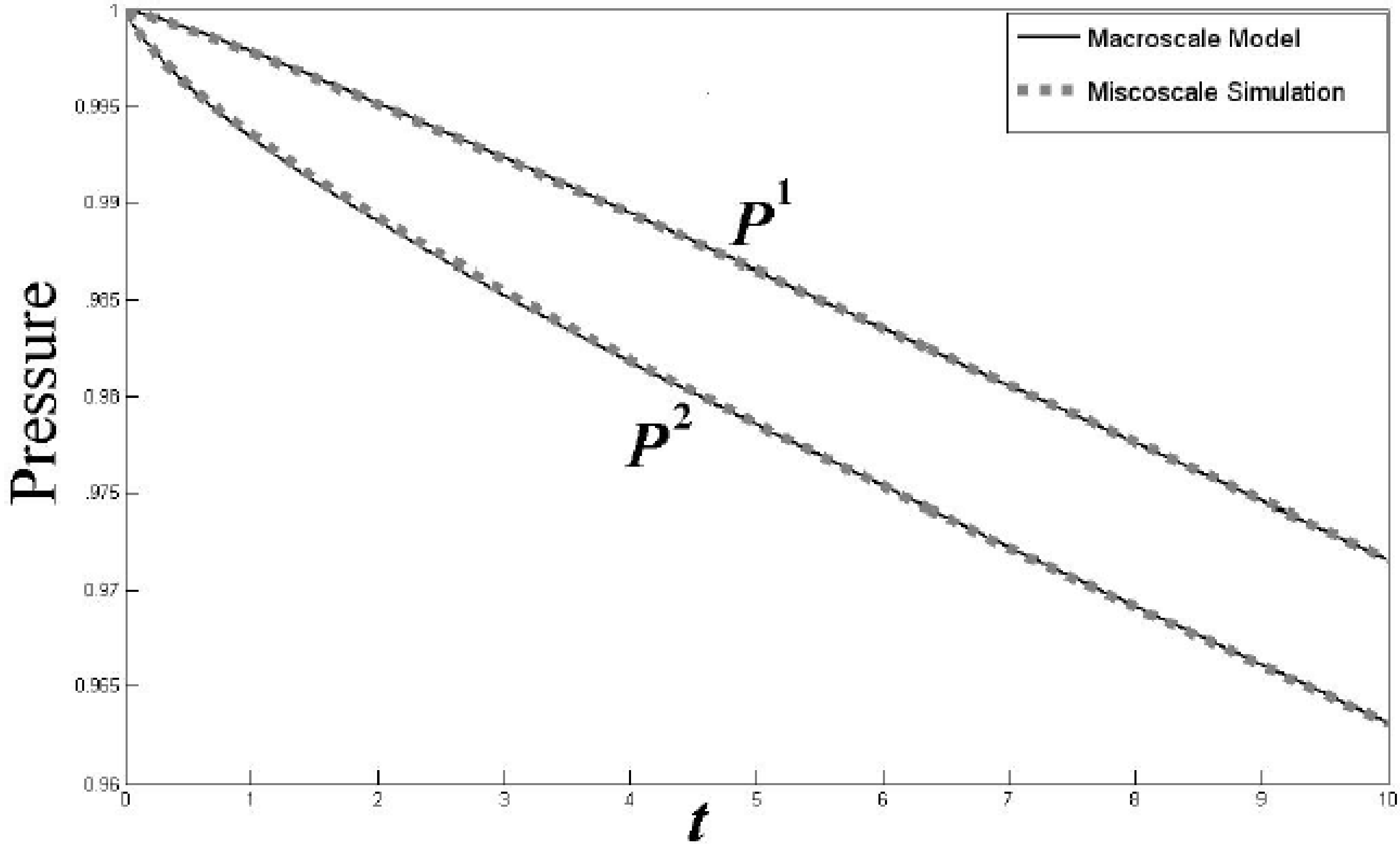


Streamlines

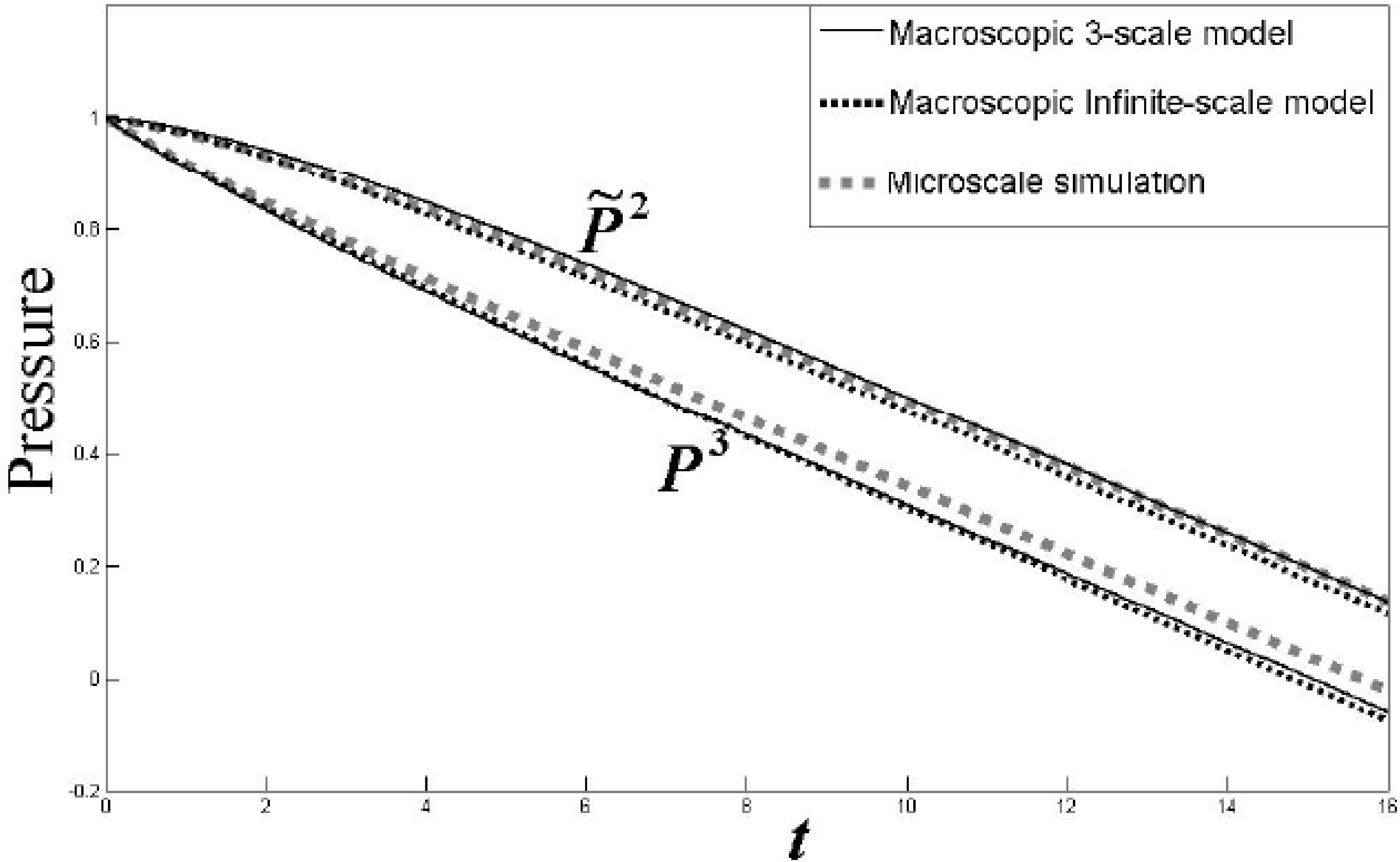


Pressure

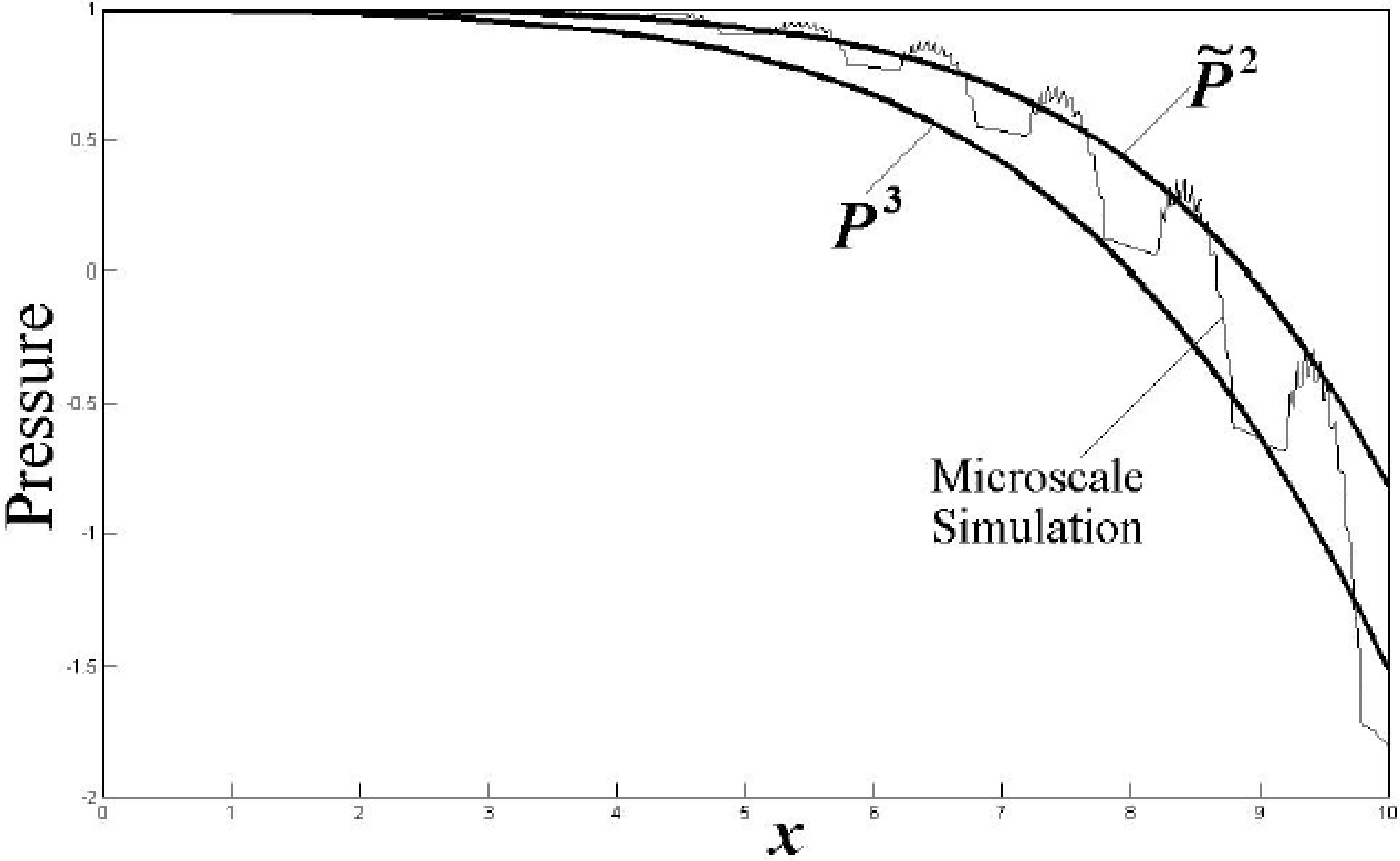
Testing the numerical code: 2-scale medium



Three-scale medium



Three-scale medium



Effect of memory accumulation

The difference in pressure between fractures and blocks

- For two-scale medium :

$$P^1 - P^2 = -\frac{\partial P^2}{\partial t} * K^{(1)}$$

- For three-scale medium :

$$\widetilde{P}^2 - P^3 = -\frac{\partial P^3}{\partial t} * K^{(2*)} - \frac{\theta}{1-\theta}\frac{\partial P^1}{\partial t} * K^{(1)} * K^{(2*)}$$

↑ Delay between fracture 3 and block 2

↑ Influence of medium 1 on the interaction between 3 and 2

Memory accumulation

Case of thin fractures

Two-scale medium: fractional derivatives

$$B^2 \partial_t P^2 - \nabla \cdot \left(A \nabla P^2 \right) = -\sqrt{\pi} B^1 \lambda D_t^{-1/2} P^2$$

$$D_t^{\alpha} P \equiv \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \left(\int_0^t \frac{1}{(t-\tau)^{\alpha}} \frac{\partial P}{\partial \tau} d\tau \right)$$

For infinite number of scale, it is impossible to obtain the macroscopic model, as the medium is non self-similar

Three-scale medium

$$\begin{cases} \mathcal{P}_b = \mathcal{P}_f - \dfrac{\widehat{B}^{(2)}}{\widehat{A}^{(2)}} \dfrac{\partial \mathcal{P}_f}{\partial t} * \mathbb{K}^2, \\ \\ \widehat{B}^{(3)} \dfrac{\partial \mathcal{P}_f}{\partial t} = \nabla \cdot \left(\widehat{A}^{(3)} \nabla \mathcal{P}_f \right) - \widehat{B}^{(2)} \dfrac{\partial \mathcal{P}_b}{\partial t} * \mathbb{L}^{(2)} \end{cases}$$

$$\mathbb{K}^2 = \mathcal{K} - \gamma\lambda \frac{\widehat{B}^{(1)}}{\widehat{B}^{(2)}} \left(\mathcal{K} * \frac{1}{\sqrt{t}} \right)$$

$$\mathcal{K}(t) = \langle G(x, t-\tau) \rangle_2 = \frac{1}{\theta^{(2)}} \sum_{k > k_*} \frac{\langle r_k \rangle_2^2}{\langle r_k^2 \rangle_2} \left\{ e^{\rho(1-\alpha_k)t} \left[\cos(\rho\nu_k t) - \frac{1}{\nu_k} \sin(\rho\nu_k t) \right] + \right.$$

$$\left. + \frac{\beta_k}{\nu_k} \int_0^t \frac{\sin\left(\rho\nu_k (t-\tau)\right)}{\sqrt{\tau}} e^{\rho(1-a_k)(t-\tau)} d\tau \right\} +$$

$$+ \frac{1}{\theta^{(2)}} \frac{\langle r_{k_*} \rangle_2^2}{\langle r_{k_*}^2 \rangle_2} \left\{ e^{\rho t/2} (1 - \rho t) + \beta_{k_*} \rho \int_0^t \frac{t-\tau}{\sqrt{\tau}} e^{\rho(t-\tau)/2} d\tau \right\} +$$

$$+ \frac{1}{\theta^{(2)}} \sum_{1 < k < k_*} \frac{\langle r_k \rangle_2^2}{\langle r_k^2 \rangle_2} \left\{ \frac{\nu_k + 1}{2\nu_k} e^{\omega_k^{(1)} t} + \frac{\nu_k - 1}{2\nu_k} e^{\omega_k^{(2)} t} + \right.$$

$$\left. + \frac{\beta_k}{2\nu_k} \int_0^t \frac{1}{\sqrt{\tau}} \left(e^{\omega_k^{(1)}(t-\tau)} - e^{\omega_k^{(2)}(t-\tau)} \right) d\tau \right\}$$

Conclusion

1. Appearance of the effect of memory accumulation (“super-memory”)

2. Appearance of the nonlinearity in the structure of the integral kernels

3. Good approximation of the 3-scale and more media by the limit infinite-scale model

Instability

Numerically it was observed that the limit model becomes unstable below a threshold value of the permeability ratio and volume fraction of blocks

Instability

Numerically it was observed that the limit model becomes unstable below a threshold value of the permeability ratio and volume fraction of blocks

Thank you for your attention