The effect of sorption on linear stability
for the solutal Horton-Rogers-Lapwood problem

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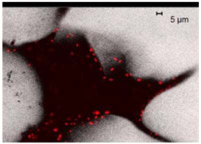
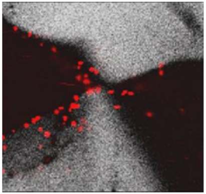
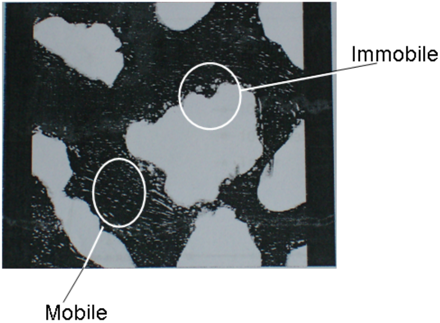
Outline

Immobilization of solute







MIM principles

1. The diffusion in porous media usual is complicated by immobilization of admixture.
2. Adsorbed solute - immobile phase (cannot move).
3. Free solute - mobile phase (can move).

Transport equation with immobile phase (M. Th. van Genuchten *et al.*, 1976).

$$\frac{\partial}{\partial t}(C_{tot}) = -\mathbf{V}\nabla C_m + D\triangle C_m$$

$$\frac{\partial}{\partial t}C_{im} = \alpha(C - K_d C_{im})$$

D – diffusivity, $\mathbf{V}$ – fluid velocity, α – mass transport coefficient,

K_d – distribution coefficient

$C_{tot} = C_m + C_{im}$ – volumic density of solute.

Darcy-Boussinesq approximation with MIM

1. The porous media is saturated by incompressible fluid.
2. The density of mixture linearly depends on mobile concentration.
3. The density variations are taken into account only in buoyancy term.

Equations of solutal convection in MIM model

$$\frac{\partial}{\partial t}(C_{tot}) = -\mathbf{V}\nabla C_m + D\triangle C_m$$

$$\frac{\partial}{\partial t}C_{im} = \alpha(C_m - K_d C_{im})$$

$$\frac{\eta}{\kappa}\mathbf{V} + \gamma\rho\beta_c C_m g = -\nabla p$$

$$div\mathbf{V} = 0$$

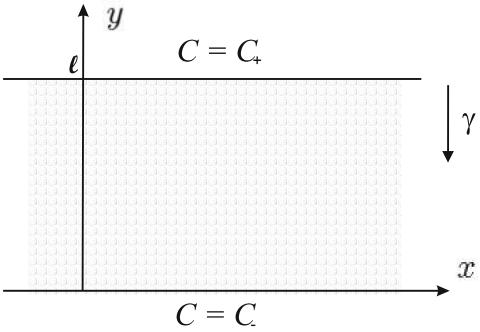
η – fluid viscosity, κ – permeability,

ρ – fluid density, g – gravity acceleration,

β_c – concentrational expansion coefficient, p – pressure,

Solutal HRL problem

Horton-Rogers-Lapwood problem configuration (C. W. Horton and F. T. Rogers (1945), E. R. Lapwood (1948))



Dimensionless equations and parameters

Equations for solutal convection

$$\frac{\partial}{\partial t}C = -\mathbf{V}\nabla C + \triangle C$$

$$\mathbf{V} + \gamma Rp_c C = -\nabla p$$

$$div\mathbf{V} = 0$$

$$Rp_c = \frac{C_0 g \ell \kappa \rho \beta_c}{D\eta}$$

Scales

$$[L] = \ell,\ [t] = \frac{\ell^2}{D},\ [V] = \frac{D}{\ell},\ [p] = \frac{D\eta}{\kappa},\ [C] = C_+ - C_- = C_0$$

Basic solution and pertubation equations

Basic solution - mechanical equilibrium

$$\mathbf{V} = 0, \ \mathrm{C} = \mathrm{y}$$

Perturbation equations in terms of stream function ($V_x = -\partial_y \psi, \ V_y = \partial_x \psi$)

$$\partial_t c - \partial_y \psi \partial_x c + \partial_x \psi \partial_y c + \partial_x \psi = \triangle c,$$

$$\triangle \psi = -Rp \partial_x c$$

$$c - \text{perturbation of concentration}$$

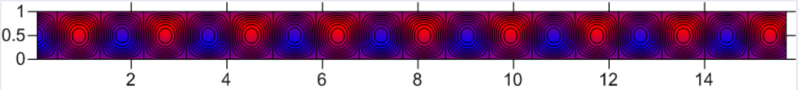
Solution of linear stability problem

Neutral perturbations

$$c,\ \psi \sim \exp\left[ikx\right]\sin\left(\pi y\right)$$

$$Rp = \frac{\left(k^2+\pi^2\right)^2}{k^2}$$

The critical perturbation($Rp = \pi^2,\ k = \pi$)



The effect of external flow

Basic solution - horizontal seepage (M. Prats, 1966)

$$\mathbf{V} = (\mathrm{Pe}, 0)\,,\ \mathrm{C} = \mathrm{y}$$

$$\mathrm{Pe} = \frac{\mathrm{V}\ell}{\mathrm{D}} - \text{dimensionless speed of external filtration flow}$$

Perturbation equations in terms of stream function ($V_x = -\partial_y \psi,\ V_y = \partial_x \psi$)

$$\partial_t c - \partial_y \psi \partial_x c + \partial_x \psi \partial_y c + \partial_x \psi + Pe\partial_x c = \triangle c,$$

$$\triangle \psi = -Rp\partial_x c$$

$$c - \text{perturbation of concentration}$$

Solution of linear stability problem

Neutral perturbations

$$c, \ \psi \sim \exp\left[ikx - i\omega t\right]\sin\left(\pi y\right)$$

$$Rp = \frac{\left(k^2 + \pi^2\right)^2}{k^2}, \ \omega = kPe$$

The critical perturbation ($Rp = \pi^2$, $k = \pi$)

Concentration

Stream function

Dimensionless equations and parameters

Equations for solutal convection

$$\partial_t (C+Q) = \triangle C - \mathbf{V}\cdot\nabla C,$$

$$\nabla\cdot\mathbf{V} = 0,$$

$$\mathbf{V} = -\nabla p + RpC\gamma,$$

$$\partial_t Q = aC - bQ,$$

$$C - \text{mobile solute concentration}$$

$$Q - \text{immobile solute concentration}$$

$$a = \frac{\alpha \mathrm{D}}{\ell^2},\ b = \frac{\alpha \mathrm{K_d D}}{\ell^2} -$$

dimensionless adsorption and desorption rates

Sorption without external flow

The case of $Pe = 0$: Basic solution - mechanical equilibrium

$$\mathbf{V} = 0, \ \mathrm{C} = \mathrm{y}$$

Linear perturbation equations steady neutral perturbations $(\partial_t = 0)$

$$\partial_x \psi = \triangle c,$$

$$\triangle \psi = -Rp \partial_x c$$

$$q = \frac{a}{b} c$$

$c, \ q$ – perturbations of mobile and immobile concentration

Sorption with external flow

Basic solution - horizontal seepage

$$\mathbf{V} = (\mathrm{Pe}, 0)\,,\ \mathrm{C} = \mathrm{y}$$

$$\mathrm{Pe} = \frac{\mathrm{V}\ell}{\mathrm{D}} - \text{dimensionless speed of external filtration flow}$$

Linear perturbation equations, oscillatory neutral perturbations ($\partial_t = -i\omega$)

$$-i\omega(c+q) + \partial_x\psi + Pe\partial_x c = \triangle c,$$

$$\triangle\psi = -Rp\partial_x c$$

$$-i\omega q = ac - bq$$

Solution of linear stability problem

Neutral perturbations

$$c,\ \psi,\ q \sim \exp\left[ikx - i\omega t\right]\sin\left(\pi y\right)$$

$$Rp = \frac{\left(k^2+\pi^2\right)^2}{k^2} + \frac{\pi^2 n^2 + k^2}{k^2}\frac{\omega^2 a}{b^2+\omega^2}$$

$$\omega^3 - \omega^2 kPe + \omega b\left(a+b\right) - b^2 kPe = 0$$

Limit case analysis

Limit cases

Low external flow rate:

$$kPe \ll a+b$$

$$\omega = \frac{bkPe}{a+b},$$

$$k_{min} = \frac{4\left(a+b\right)^2}{4\left(a+b\right)^2 + aPe^2}\pi,$$

$$Rp_{min} = 2\pi^2\left(2 + \frac{Pe^2 a}{\left(a+b\right)^2}\right)$$

Hight external flow rate:

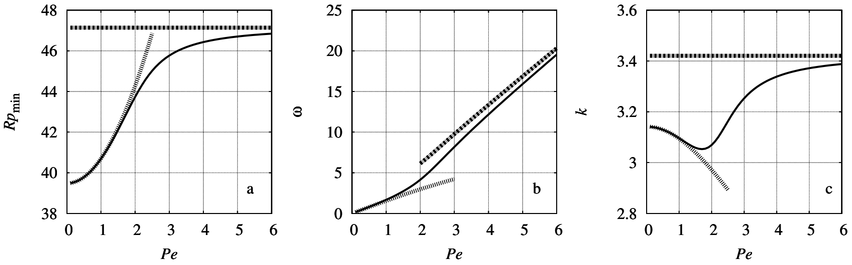
$$kPe \gg a+b$$

$$\omega = kPe$$

$$k_{min} = \pi\sqrt{B},\ B = \sqrt{1 + \frac{a^2}{\pi^2}},$$

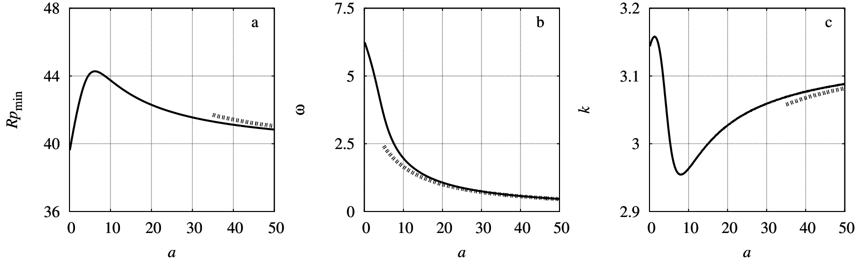
$$Rp_{min} = \pi^2\frac{(1+a+B)(1+B)}{B}$$

Stability map. Peclet number sensitivity



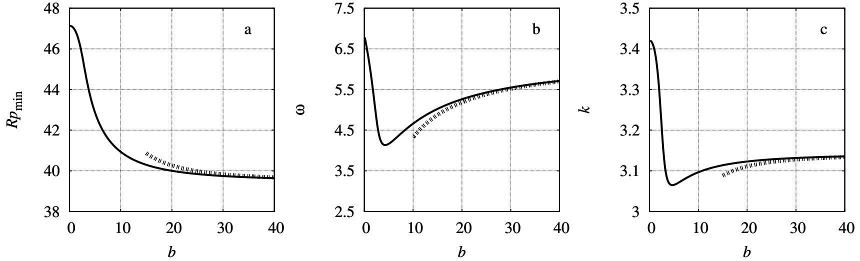
$a = 4$, $b = 4$ (dush lines - limit cases)

Stability map. Adsorption rate sensitivity



$Pe = 2,\ b = 4$(dush line - high adsorption rate case
$a \gg b \Rightarrow kPe \sim b \ll a \Rightarrow$low external flow rate)

Stability map. Desorption rate sensitivity



$Pe = 2$, $a = 4$(dush line - high adsorption rate case
$b \gg a \Rightarrow kPe \sim a \ll b \Rightarrow$low external flow rate)

Dimensionless equations and parameters

Basic solution - horizontal seepage

$$\mathbf{V} = \left(Pe\left\{S + Acos\Omega t\right\}, 0\right),\ \mathbf{C} = y$$

S – strength of steady flow

A – modulated flow amplitude

Linear perturbation equations

$$\partial_t\left(c + q\right) + \partial_x \psi + Pe\left\{S + Acos\Omega t\right\}\partial_x c = \triangle c,$$

$$\triangle \psi = -Rp\partial_x c$$

$$\partial_t q = ac - bq$$

Amplitude equation

Find the solution in form $c, \psi, q \sim \exp ikx \sin \pi y$

$$\partial_{tt} q + \partial_t q \left[b + a - \gamma + ikPe\left(S + A\cos\Omega t\right)\right] + b\left[ikPe\left(S + A\cos\Omega t\right) - \gamma\right] q = 0$$

$$\gamma = \frac{k^2 Rp}{\pi^2 n^2 + k^2} - \pi^2 n^2 - k^2.$$

The classical case without sorption (D. V. Lyubimov and V. S. Teplov (1998))

$$c, \psi, q \sim \exp\left[\gamma t - ikPe\left(St + \frac{A}{\Omega}\sin\Omega t\right)\right]$$

$$Rp = \frac{\left[\pi^2 n^2 + k^2\right]^2}{k^2}$$

No impact to stability

Limit case

The case of low external flow rate $kPe \ll a+b$

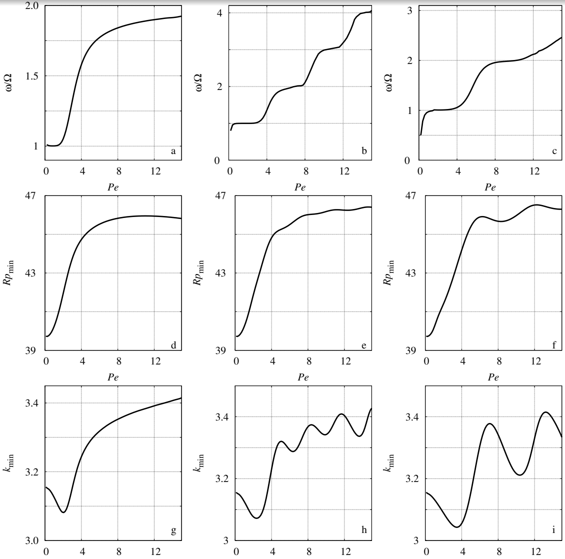
$$Rp = \frac{\left[\pi^2 n^2 + k^2\right]^2}{k^2} + \left[\pi^2 n^2 + k^2\right] Pe^2 a \left[\frac{S^2}{(b+a)^2} + \frac{A^2}{2\left\{\Omega^2 + (b+a)^2\right\}}\right]$$

Critical perturbations :

$$k_{min} = \frac{8\pi}{8 + aPe^2\left[\frac{2S^2}{(b+a)^2} + \frac{A^2}{\Omega^2+(b+a)^2}\right]}$$

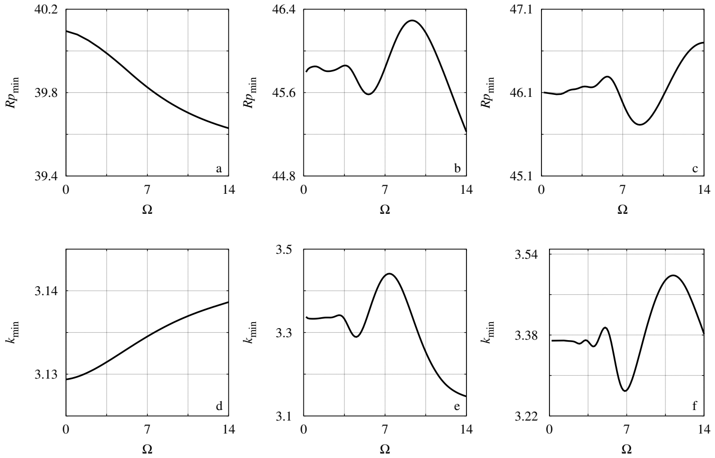
$$Rp_{min} = 2\pi^2\left(2 + \frac{Pe^2 a}{2}\left[\frac{2S^2}{(b+a)^2} + \frac{A^2}{\Omega^2 + (b+a)^2}\right]\right)$$

Stability maps. Peclet number sensitivity



$a = 4, b = 4;$
$S = 0, A = 1.$
Modullation frequencies:
a,d,g - $\Omega = 1;$
b,e,h - $\Omega = 4;$
c,f,i - $\Omega = 7.$

Stability maps. Modulation frequency sensitivity



$a = 4, b = 4, S = 0, A = 1$. Peclet number: a,d - $Pe = 1$; b,e - $Pe = 7$; c,f - $Pe = 10$.

References

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Summary

- The coupling consideration of external flow and sorption within HRL problem lead to dependence of critical parameters on flow strength
- The principal possibility of deposit (immobile fraction) distribution control by the flow parameters was demonstrated
- The modulation of flow lead to "parametric" - like instability and improves the control

Questions?

Thank you!!!