Westfälische Wilhelms-Universität Münster

Localized Orthogonal Decompositions

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joint work with A. Målqvist (Gothenburg University) and D. Peterseim (Bonn University)

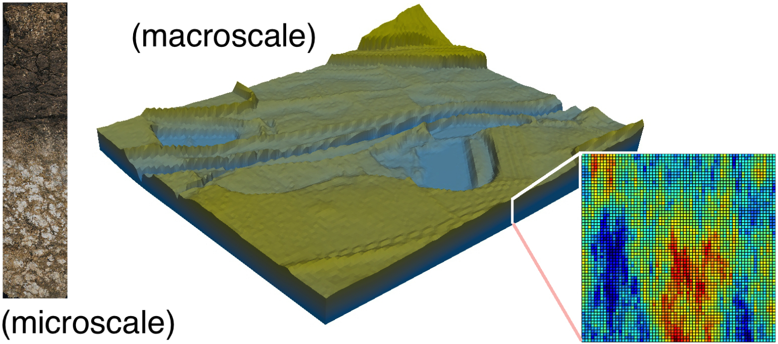
NM2PorousMedia Conference - Dubrovnik, Croatia

29th of September - 3th of October

An Orthogonal Multiscale Decomposition

Multiscale problems

Ex.: simulations related to subsurface flow problems.



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- Rigorosity of numerical approaches:
  If we decompose/simplify the problem,
  is the obtained approximation still reliable?

Example: 1d model problem

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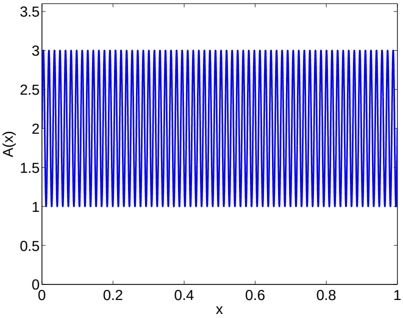

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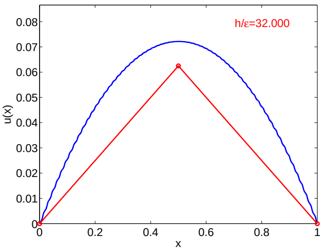

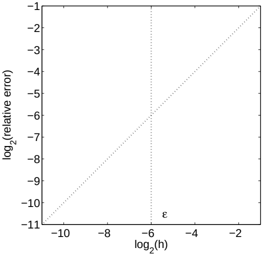

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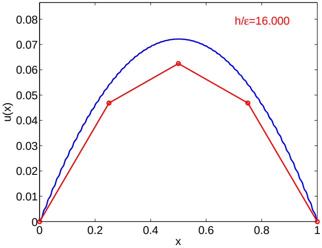

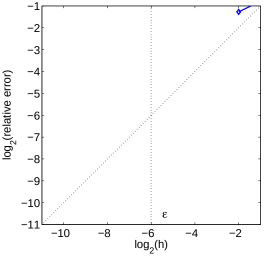

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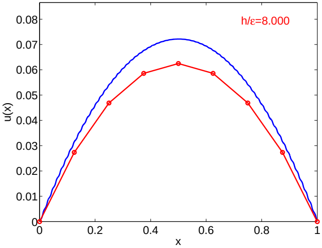

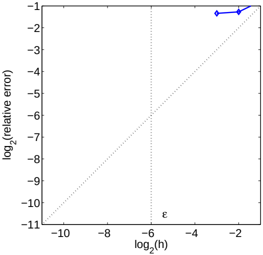

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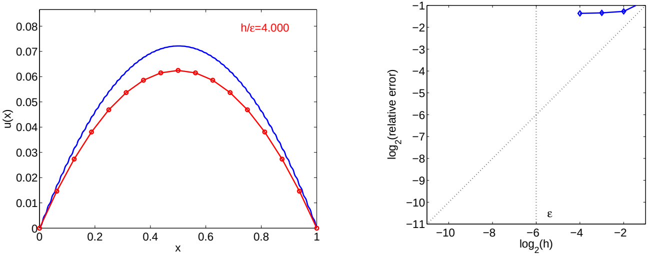

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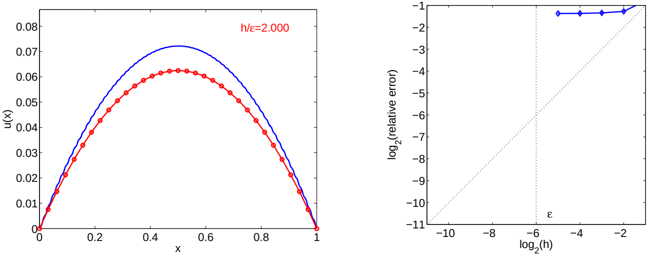

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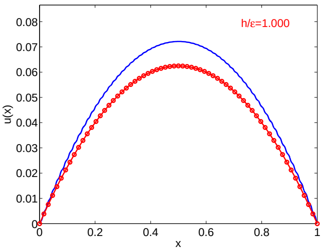

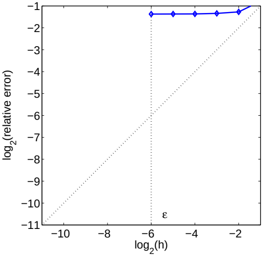

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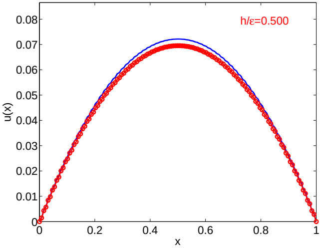

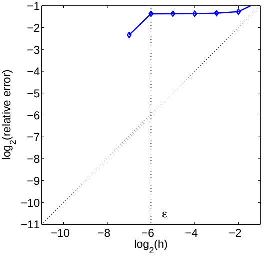

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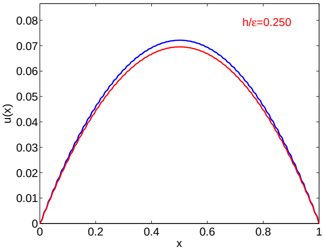

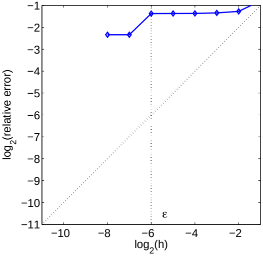

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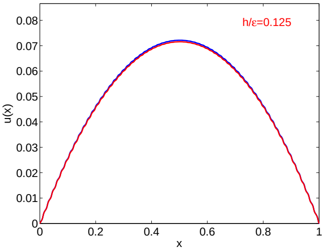

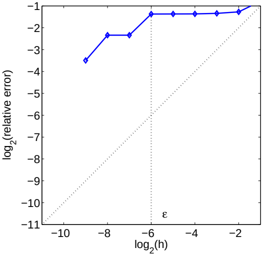

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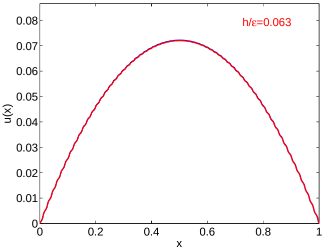

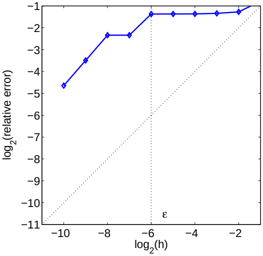

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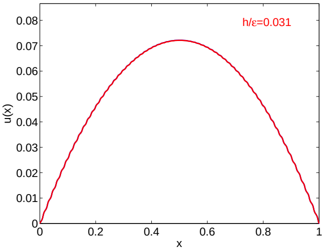

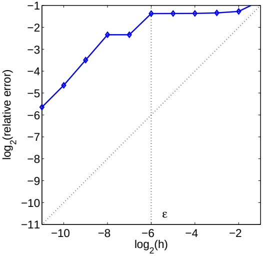

Multiscale Methods

Some known multiscale methods:

- Variational multiscale method: Hughes et al. 95-, Larson-Målqvist 05-, ...
- Multiscale Finite Element Method (MsFEM): Hou-Wu 97-, Efendiev et al. 00- ...
- Residual free bubbles: Brezzi et al. 98-, ...
- Upscaling techniques: Dorobantu-Engquist 98, ...
- Wavelet based homogenization: Durlofsky et al. 98, Nielsen et al. 98
- Heterogeneous Multiscale Method (HMM): Engquist-E 03, Abdulle et al. 05- ...
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- GFEM based on local eigenfunctions: Babuška-Lipton 11
- AL-Basis: Sauter-Grasedyck-Greff 11, ...
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- Localized Orthogonal Decomposition (LOD): Målqvist-Peterseim 11, ..., H. 12- ...

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Define $a(v,w) := (A\nabla v, \nabla w)_{L^2(\Omega)}$.

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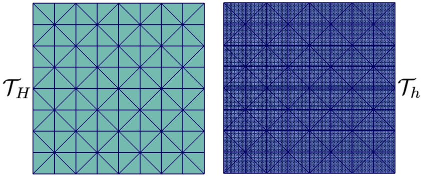

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- Find $V_H^{ms} \subset V_h$ with $\dim V_H^{ms} = \dim V_H$ (i.e. low-dimensional!),
- such that $\inf_{u_H^{ms} \in V_H^{ms}} \|u_H^{ms} - u\|_{H^1(\Omega)} \leq CH$

with C **independent of** the **variations** of A.

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Goal: turn splitting into a multiscale decomposition.

Orthogonal Multiscale Decomposition

Define $P_H : V_h \to V_h^f$ by

$$a(v_h, w_h^f) = a(\, P_H(v_h)\, , w_h^f) \quad \text{for all} \quad w_h^f \in V_h^f.$$

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Original decomposition:

$$V_h = V_H \oplus V_h^f, \qquad \text{where } V_H \perp V_h^f \text{ wrt. } (\cdot,\cdot)_{L^2}.$$

Orthogonal Multiscale Decomposition

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Galerkin orthogonality: $a(u_H^{ms} - u_h, v) = 0.$

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Galerkin approximations:

$$\text{find } u_H^{\text{ms}} \in V_H^{\text{ms}} \text{ with } a(u_H^{\text{ms}}, v) = (f, v) \quad \forall v \in V_H^{\text{ms}},$$

$$\text{finde } u_h \in V_h \text{ with } a(u_h, v) = (f, v) \quad \forall v \in V_h.$$

Galerkin orthogonality: $a(u_H^{\text{ms}} - u_h, v) = 0.$

Orthogonal Multiscale Decomposition

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$$\alpha \|\nabla e_h\|_{L^2(\Omega)}^2 \le a(e_h, e_h) \overset{\text{GO}}{=} a(u_h, e_h) = (f, e_h)$$

Orthogonal Multiscale Decomposition

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$$\alpha \|\nabla e_h\|^2_{L^2(\Omega)} \le a(e_h, e_h) \overset{\text{GO}}{=} a(u_h, e_h) = (f, e_h) = (f, e_h - I_H(e_h)) \le CH\|f\|_{L^2(\Omega)} \|\nabla e_h\|_{L^2(\Omega)}.$$

Orthogonal Multiscale Decomposition

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In conclusion: $\|u_H^{\mathrm{ms}} - u_h\|_{H^1(\Omega)} \le C\alpha^{-1} H \|f\|_{L^2(\Omega)}$, with generic C.

Localized Orthogonal Decomposition

P. Henning and A. Målqvist.
Localized orthogonal decomposition techniques for boundary value problems.
SIAM Journal of Scientific Computing, 36(4):A1609–A1634, 2014.

P. Henning and D. Peterseim.
Oversampling for the Multiscale Finite Element Method.
SIAM Multiscale Model. Simul., 11(4):1149–1175, 2013.

Localization

How to compute $P_H : V_h \rightarrow V_h^f$? For all $w_h^f \in V_h^f$ it holds:

$$a(\, P_H(v_h)\, , w_h^f) = a(v_h, w_h^f)$$

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Advantage?

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Advantage? Solution $P_{H,T}(v_h)$ decays to zero outside of T!

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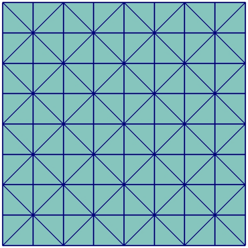
Advantage? Solution $P_{H,T}(v_h)$ decays to zero outside of T!
Replace Ω by small environment $U(T)$!

Coarse grid patches

For $k \in \mathbb{N}$, define the k-layer localization patch $U_k(K)$ by

$$U_0(T) := T,$$
$$U_k(T) := \cup\{K \in \mathcal{T}_H \mid K \cap U_{k-1}(T) \neq \emptyset\} \quad k = 1, 2, \ldots$$



Coarse grid $\mathcal{T}_H$.

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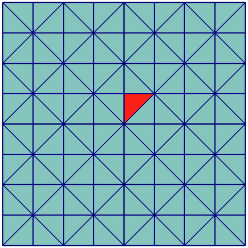

Coarse grid element $T \in \mathcal{T}_H$, $U_0(T) = T$ for k=0.

Coarse grid patches

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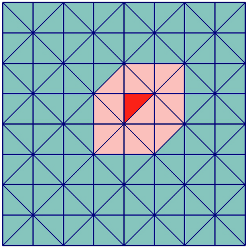

Coarse grid element $T \in \mathcal{T}_H$, $U_1(T)$ for k=1.

Coarse grid patches

For $k \in \mathbb{N}$, define the k-layer localization patch $U_k(K)$ by

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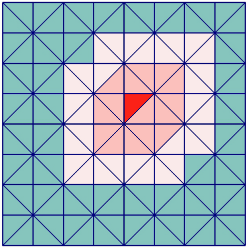

Coarse grid element $T \in \mathcal{T}_H$, $U_2(T)$ for k=2.

Approximations with exponential convergence

Theorem

Let $k \in \mathbb{N}_{>0}$, $U_k(T) \supset T$ *and* $P^k_{H,T}(v_H) \in V^f_h(U_k(T))$ *solve (in parallel)*

$$a(P^k_{H,T}(v_H), w^f_h)_{U_k(T)} = a(v_H, w^f_h)_T \quad \forall w^f_h \in V^f_h(U_k(T))$$

and set

$$P^k_H(v_H) := \sum_{T \in \mathcal{T}_H} P^k_{H,T}(v_H)$$

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then

$$\left\| P_H(v_H) - P^k_H(v_H) \right\|_{H^1(\Omega)} \lesssim k^{d/2} e^{-k} \|\nabla v_H\|_{L^2(\Omega)}.$$

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The choice $k \approx |\ln(H)|$ preserves convergence rates.

Approximations with exponential convergence

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then

$$\left\| P_H(v_H) - P^k_H(v_H) \right\|_{H^1(\Omega)} \lesssim k^{d/2} e^{-k} \|\nabla v_H\|_{L^2(\Omega)}.$$

Instead of $V_H^{\mathrm{ms}} := \mathrm{kern}(P_H|_{V_H})$ use $V_{H,k}^{\mathrm{ms}} := \mathrm{kern}(P^k_H|_{V_H})$.

A priori error estimate

Theorem

Let $V_{H,k}^{\mathrm{ms}} := \mathrm{kern}(P_H^k|_{V_H})$ *and* $k \gtrsim |\ln(H)|$. *Find* $u_{H,k}^{\mathrm{ms}} \in V_{H,k}^{\mathrm{ms}}$ *with*

$$a(u_{H,k}^{\mathrm{ms}}, v) = (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in V_{H,k}^{\mathrm{ms}}.$$

A priori error estimate

Theorem

Let $V^{ms}_{H,k} := \mathrm{kern}(P^k_H|_{V_H})$ *and* $k \gtrsim |\ln(H)|$. *Find* $u^{ms}_{H,k} \in V^{ms}_{H,k}$ *with*

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Then (independent of the variations of A*):*

$$\|u_h - u^{ms}_{H,k}\|_{L^2(\Omega)} + H\|u_h - u^{ms}_{H,k}\|_{H^1(\Omega)} \lesssim H^2,$$

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$$\|u_h - I_H(u^{ms}_{H,k})\|_{L^2(\Omega)} \lesssim H.$$

Numerical experiment

Numerical experiment

Model problem

Let $\Omega :=]0,1[^2$. Find $u \in H^1(\Omega)$ with

$$-\nabla \cdot (A^\epsilon \nabla u) = f \quad \text{in } \Omega,$$
$$u = x_1 \quad \text{on } \partial\Omega.$$

Numerical experiment

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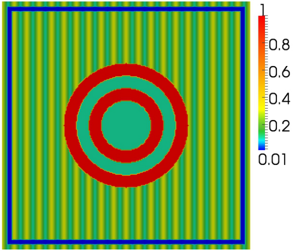
A^ϵ given by



Green/yellow region: $A^\epsilon(x) = \frac{1}{10}(2 + \cos(2\pi \frac{x_1}{\epsilon}))$ for $\epsilon = 0.05$. Isolator (blue region) $A^\epsilon(x) = 0.01$.

Circular layers in the middle: $A^\epsilon = 1$ (red region) and $A^\epsilon = 0.1$ (cyan region).

Numerical experiment

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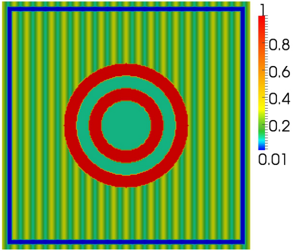

and for $c := (\frac{1}{2}, \frac{1}{2})$ and $r := 0.05$

$$f(x) := \begin{cases} 20 & \text{if } |x - c| \le r \\ 0 & \text{else.} \end{cases}$$

Results

H	k	$\|u_h - u^{ms}_{H,k}\|^{rel}_{L^2(\Omega)}$	$\|u_h - u^{ms}_{H,k}\|^{rel}_{H^1(\Omega)}$
2^{-3}	1	0.01708	0.12064
2^{-3}	2	0.00655	0.07400
2^{-3}	3	0.00557	0.06996
2^{-4}	1		
2^{-4}	2		
2^{-4}	3		
2^{-4}	4		

Table : *Calculations for $h = 2^{-8}$. k: Number of coarse layers in localized patch.*

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2^{-3}	2	0.00655	0.07400
2^{-3}	3	0.00557	0.06996
2^{-4}	1	0.00908	0.09389
2^{-4}	2	0.00159	0.03066
2^{-4}	3	0.00091	0.02269
2^{-4}	4	0.00074	0.02011

Table : *Calculations for $h = 2^{-8}$. k: Number of coarse layers in localized patch.*

Results

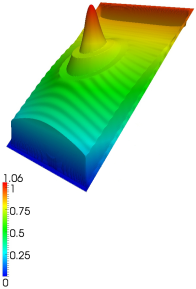

Figure : Fine grid with $h = 2^{-8}$. Reference solution.

Results

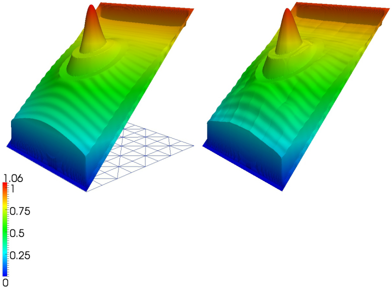

Figure : Fine grid with $h = 2^{-8}$. LOD approximation for $H = 2^{-3}$ and $k = 1$.

Results

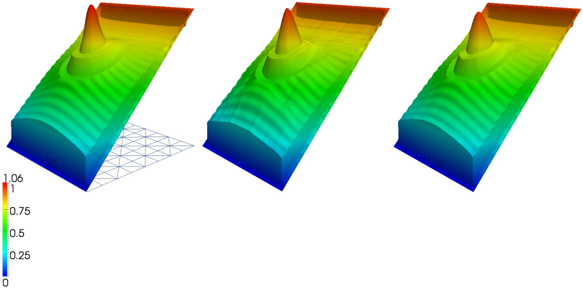

Figure : Fine grid with $h = 2^{-8}$. LOD approximation for $H = 2^{-3}$ and $k = 2$.

Results

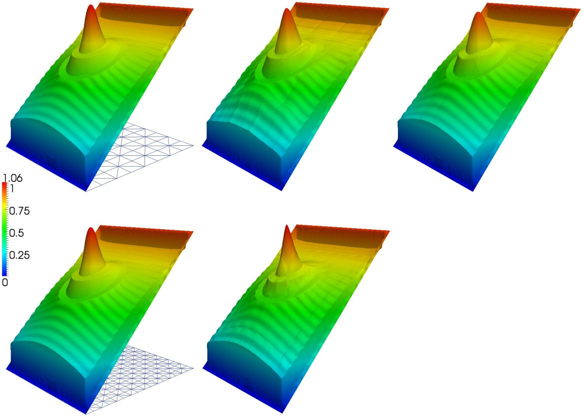

Figure : Fine grid with $h = 2^{-8}$. LOD approximation for $H = 2^{-4}$ and $k = 1$.

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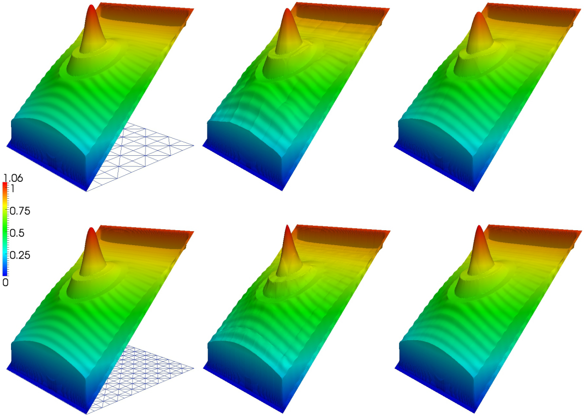

Figure : Fine grid with $h = 2^{-8}$. LOD approximation for $H = 2^{-4}$ and $k = 2$.

Generalizations and further Applications

Time Dependent and nonlinear problems

Wave equation.

A. Abdulle and P. Henning.
Localized orthogonal decomposition method for the wave equation with a continuum of scales.
ArXiv e-print 1406.6325 (submitted), 2014.

Parabolic problems.

A. Målqvist and A. Persson.
Multiscale techniques for parabolic equations.
In preparation, 2014.

Semi-linear problems.

P. Henning, A. Målqvist, and D. Peterseim.
A localized orthogonal decomposition method for semi-linear elliptic problems.
M2AN Math. Model. Numer. Anal., 48:1331–1349, 9 2014.

Eigenvalue problems

Elliptic problems.

A. Målqvist, and D. Peterseim.
Computation of eigenvalues by numerical upscaling.
ArXiv e-print 1212.0090 (submitted), 2013.

Nonlinear Schrödinger equation.

P. Henning, A. Målqvist, and D. Peterseim.
Two-Level discretization techniques for ground state computations of Bose-Einstein condensates.
SIAM J. Numer. Anal. 52 (2014), no. 4, 1525–1550.

Discontinuous Galerkin Methods

DG-LOD and applications by Elfverson et al.

D. Elfverson.
A discontinuous Galerkin multiscale method for convection-diffusion problems.
Preprint (submitted), 2014.

D. Elfverson, V. Ginting, and P. Henning.
On multiscale methods in Petrov-Galerkin formulation.
ArXiv e-print 1405.5758 (submitted), 2014.

D. Elfveron, E. H. Georgoulis and A. Målqvist.
An Adaptive Discontinuous Galerkin Multiscale Method for Elliptic Problems.
SIAM Multi. Model. Simul. 11 (2013), no. 3, 747Ð765.

D. Elfveron, E. H. Georgoulis, A. Målqvist, and D. Peterseim.
Convergence of a Discontinuous Galerkin Multiscale Method.
SIAM J. Numer. Anal. 51 (2013), no. 6, 3351Ð3372.

Other modifications and generalizations

Generalization to arbitrary partitions of unity.

P. Henning, P. Morgenstern, and D. Peterseim.
Multiscale partition of unity.
to appear in: Meshfree Methods for Partial Differential Equations VII, Lect. Notes Comput. Sci. Eng., 100, 2014.

LOD for high contrast by weighted L^2-projection.

R. Scheichl, and D. Peterseim.
Rigorous Numerical Upscaling of Elliptic Multiscale Problems at High Contrast.
in preparation 2014.

Generalizations and further Applications

A DG-LOD application to the Buckley-Leverett.

D. Elfverson, V. Ginting, and P. Henning.
On multiscale methods in Petrov-Galerkin formulation.'
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The Buckley-Leverett system

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Here we have

- S: saturation (of the wetting phase);
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- K: hydraulic conductivity - multiscale!

Proposed Method - Step 1

Preprocessing - DG-LOD projection.

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- $\mathcal{T}_H$: coarse grid, V_H: typical P1 DG-FEM space,
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LOD: approximate V_H^{ms} by some $V_{H,k}^{ms}$ as before (reusable space!).

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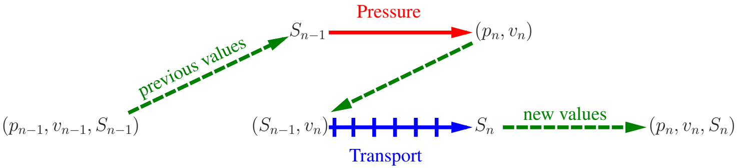

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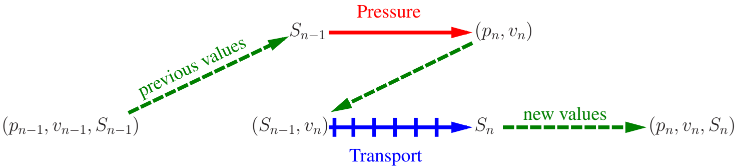

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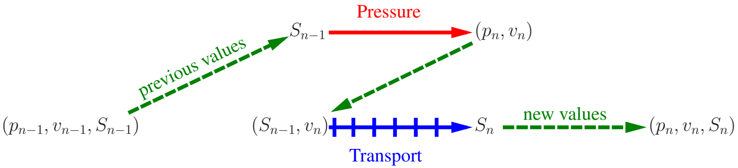

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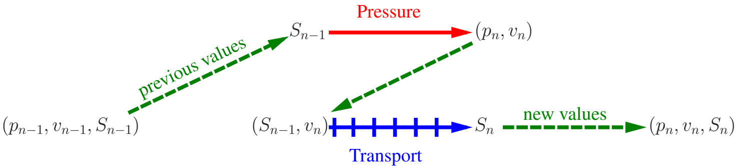

- hyperbolic Buckley-Leverett problem is treated with explicit time stepping;
- flux velocity $\mathbf{v}$ is kept constant for a certain time interval;
- then update $\mathbf{v}$ by solving the elliptic problem with saturation from previous time step.

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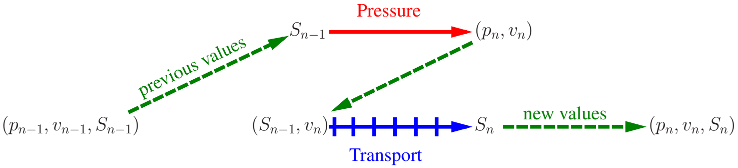

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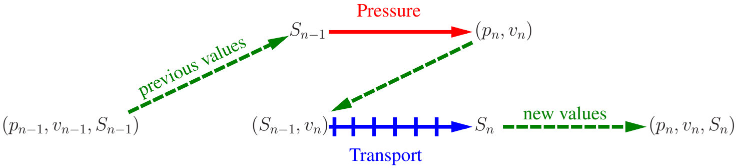

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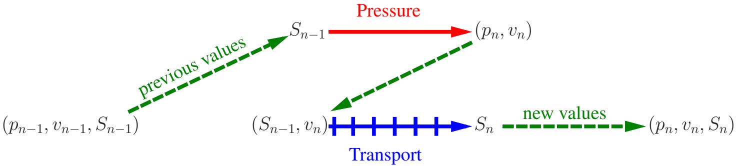

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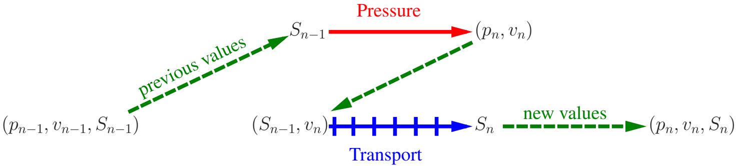

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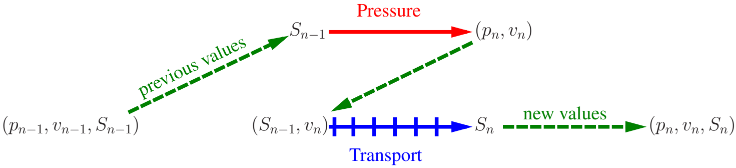

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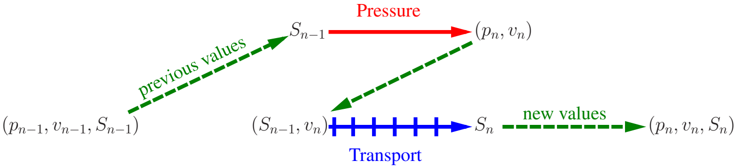

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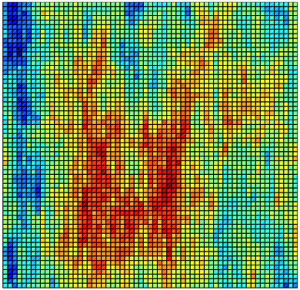
  - is locally mass conservative on coarse grid $\mathcal{T}_H$!

- Set $\mathbf{v}_n := -K\lambda(S_{n-1})\nabla p_n$ and solve $\Theta\partial_t S + \nabla \cdot (f(S)\mathbf{v}_n) = q_w$ on $[t_{n-1}, t_n]$ (explicitly) with your favorite hyperbolic solver on coarse grid.

A test case

- hydraulic conductivity K given by layer 21 of the Society of Petroleum Engineering comparative permeability data

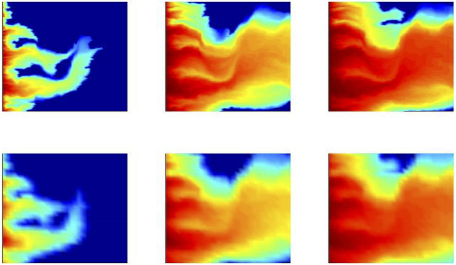
(http://www.spe.org/web/csp):



Conductivity on log-scale with contrast of order $5 \cdot 10^5$.

A test case

- inflow from left to right (initial value for S);
- $q_w = q = 0$;
- pressure boundary condition: $p = 1$ on the left, $p = 0$ on the right, homogeneous Neumann otherwise;



Upper series: full fine scale simulation on $\mathcal{T}_h$ with $h = 2^{-8}$.

Lower series: PG DG-LOD / IMPES multiscale simulation on $\mathcal{T}_H$ with $H = 2^{-5}$.

Thank you for your attention!

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ArXiv e-prints (accepted for publication in Mathematics of Computation, 2013), October 2011.

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Oversampling for the Multiscale Finite Element Method.
Multiscale Model. Simul., 11(4):1149–1175, 2013

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