Homogenization of reaction-diffusion processes in a two-component porous medium with a non-linear flux-condition on the interface

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Joint work with P. Knabner and M. Neuss-Radu

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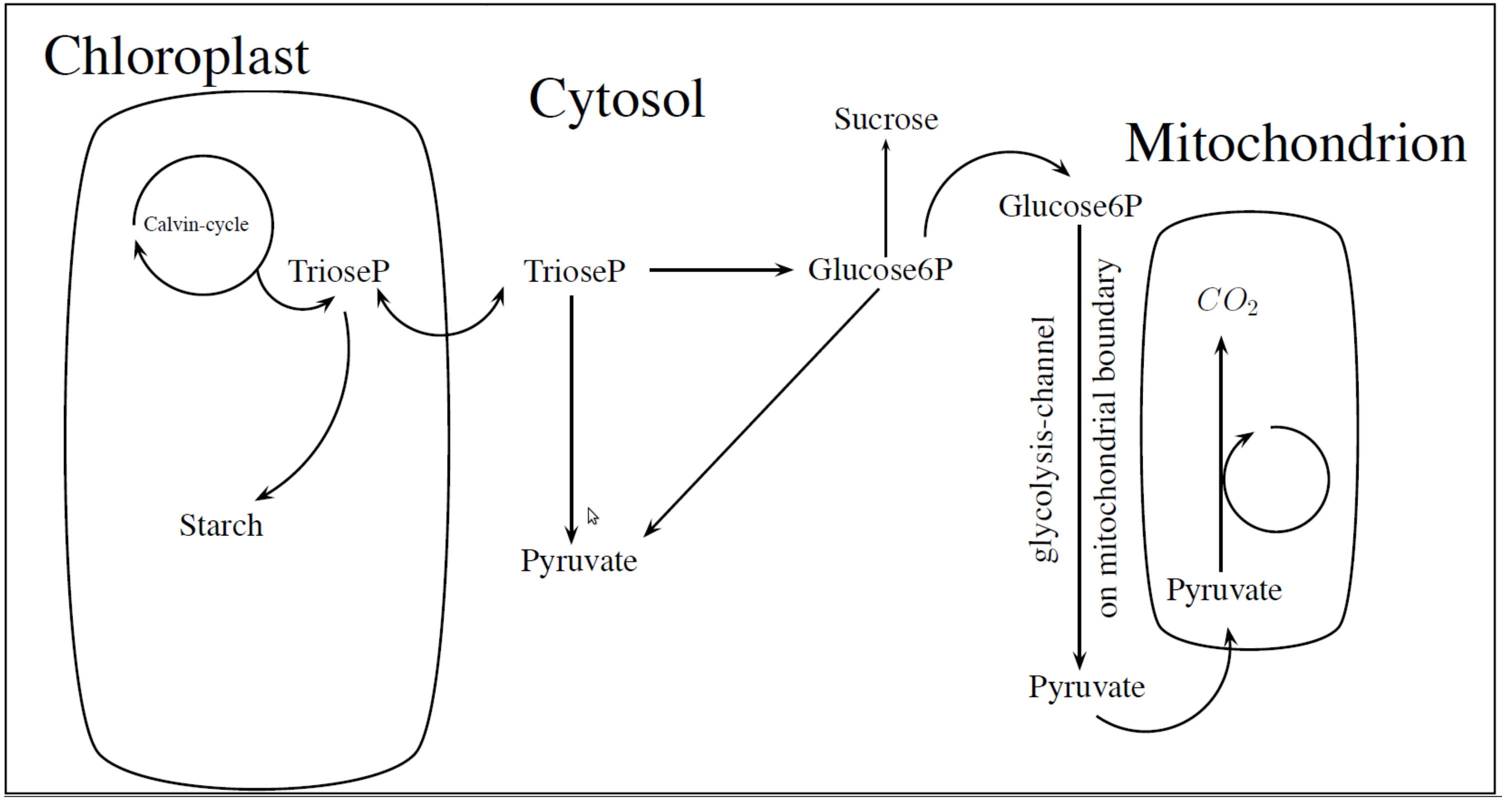
Motivation of the Model

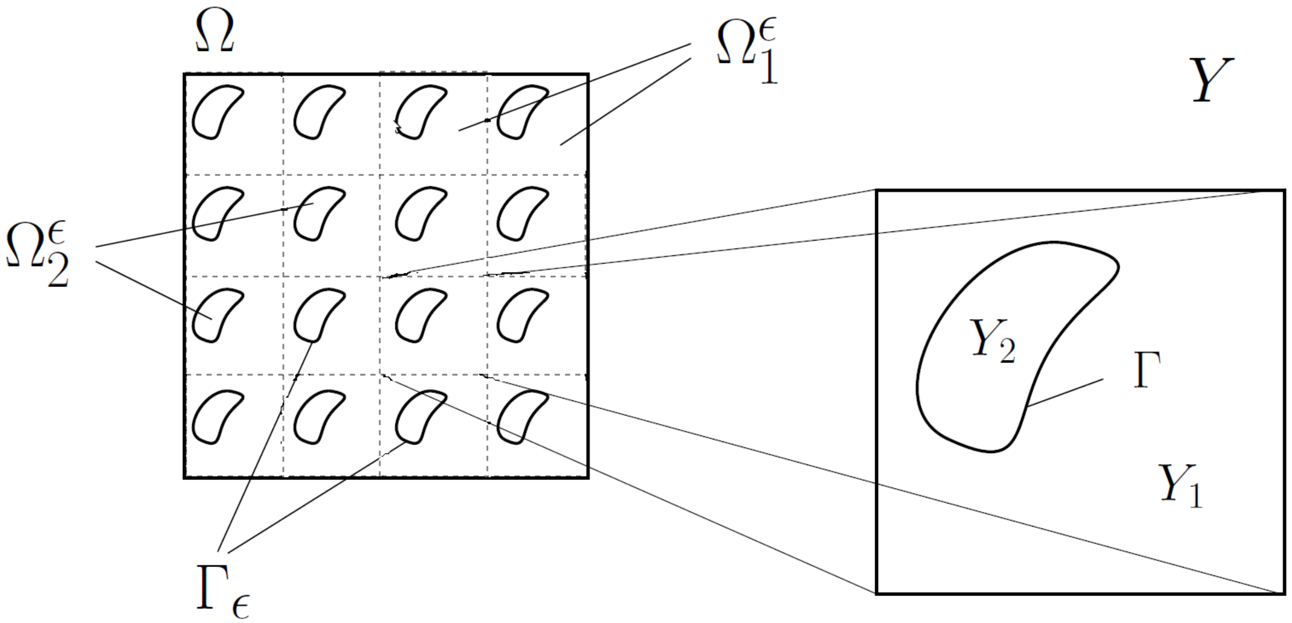
Sketch of the carbohydrate-metabolism in a plant cell:



Microscopic domain

- A porous medium with two periodically distributed components separated by an interface is considered
- Metabolites can be transported through the interface

$\rightarrow$ The flux over the interface is continuous and given by a nonlinear multi-species reaction rate



Notations

- $Y = (0,1)^n$, $\Omega = [a,b]$ with $a, b \in \mathbb{Z}^n$ and $a_i < b_i$
- $Y_2 \subset\subset Y$ open Lipschitz-domain and $Y_1 = Y \setminus \overline{Y_2}$, $\Gamma = \partial Y_2$ and Y_1 is connected
- $\Omega_j^\epsilon = \left\{ x \in \Omega \subset \mathbb{R}^n \middle| x \in \epsilon Y_j^k \text{ for } k \in \mathbb{Z}^n \right\}$ for $j = 1, 2$ and $\epsilon > 0$ with $\epsilon^{-1} \in \mathbb{N}$

  $\rightarrow \Omega_1^\epsilon$ is connected

  $\rightarrow \Omega_2^\epsilon$ is disconnected
- $\Gamma_\epsilon = \left\{ x \in \Omega \middle| x \in \epsilon \Gamma^k \text{ for } k \in \mathbb{Z}^n \right\}$
- $u_i^{j,\epsilon}$ ($j = 1, 2$, $i = 1, \dots m$) denotes the concentration of the i-th species in the domain Ω_j^ϵ

  $\rightarrow u_i^{1,\epsilon}$ and $u_i^{2,\epsilon}$ describe the same species in the different domains Ω_1^ϵ and Ω_2^ϵ

Microscopic equations

Find $u^\epsilon = (u^{1,\epsilon}, u^{2,\epsilon}) = (u_1^{1,\epsilon}, \dots, u_m^{1,\epsilon}, u_1^{2,\epsilon}, \dots, u_m^{2,\epsilon})$, with $u^{1,\epsilon} : (0,T) \times \Omega_1^\epsilon \to \mathbb{R}^m$ and $u^{2,\epsilon} : (0,T) \times \Omega_2^\epsilon \to \mathbb{R}^m$, such that

$$\begin{aligned} \partial_t u_i^{j,\epsilon} - D_i^j \Delta u_i^{j,\epsilon} &= f_i^{j,\epsilon}(t, x, u^{j,\epsilon}) && \text{in } (0,T) \times \Omega_j^\epsilon, \\ -D_i^1 \nabla u_i^{1,\epsilon} \cdot \nu_2 = -D_i^2 \nabla u_i^{2,\epsilon} \cdot \nu_2 &= \epsilon h_i(u^{1,\epsilon}, u^{2,\epsilon}) && \text{on } (0,T) \times \Gamma_\epsilon, \\ -D_i^1 \nabla u_i^{1,\epsilon} \cdot \nu_1 &= 0 && \text{on } (0,T) \times \partial\Omega, \\ u_i^{j,\epsilon}(0) &= u_{i,0}^j && \text{in } L^2(\Omega_j^\epsilon), \end{aligned}$$

where ν_j denotes the outer unit normal on $\partial\Omega_j^\epsilon$.

Multi-Substrate Reactions

- Multi-substrate reactions catalyzed by enzymes are of the form

$$X_1 + \ldots + X_{m_1} \rightleftharpoons Y_1 + \ldots + Y_{m_2}$$

- Reaction-rate under the quasi-steady state assumption

$$v(X, Y) = \frac{a \prod_{k=1}^{m_1} X_k - b \prod_{k=1}^{m_2} Y_k}{p(X, Y)}$$

with $p(X, Y) = \sum_{|\alpha| \le m_1, |\beta| \le m_2} c_{\alpha,\beta} X^\alpha Y^\beta$ and $c_{\alpha,\beta} \ge 0$

Assumptions on the Data

- $f_i^{j,\epsilon}$ and h_i are Lipschitz-continuous
- There exists $M, A > 0$, such that for all $z_i, w_i, v_i \geq M$ holds

$$f_i^{j,\epsilon}(t,x,z) \leq Az_i \ , \quad h_i(v,w) \leq Av_i \ , \quad -h_i(v,w) \leq Aw_i$$

- For $(t,x,z) \in (0,T) \times \Omega \times \mathbb{R}^n$ holds

$$\sum_{i=1}^m f_i^{j,\epsilon}(t,x,z)(z_i)_- \leq C \sum_{i=1}^m |(z_i)_-|^2$$

- For $(v,w) \in \mathbb{R}^n \times \mathbb{R}^n$ holds

$$h_i(v,w)\left[(v_i)_- - (w_i)_-\right] \leq C \sum_{i=1}^m \left(|(v_i)_-|^2 + |(w_i)_-|^2\right)$$

- The initial values satisfy $0 \leq u_{i,0}^j \leq M$

Microscopic Equations - *Variational Formulation*

Find

$$(u^{1,\epsilon}, u^{2,\epsilon}) \in L^2((0,T), H^1(\Omega_1^\epsilon, \mathbb{R}^m)) \times L^2((0,T), H^1(\Omega_2^\epsilon, \mathbb{R}^m))$$

with

$$(\partial_t u^{1,\epsilon}, \partial_t u^{2,\epsilon}) \in L^2((0,T), H^1(\Omega_1^\epsilon, \mathbb{R}^m)') \times L^2((0,T), H^1(\Omega_2^\epsilon, \mathbb{R}^m)'),$$

such that for all $\phi_j \in H^1(\Omega_j^\epsilon)$ and a.e. $t \in (0,T)$ holds

$$\langle \partial_t u_i^{j,\epsilon}, \phi_j \rangle_{\Omega_j^\epsilon} + D_i^j (\nabla u_i^{j,\epsilon}, \nabla \phi_j)_{\Omega_j^\epsilon} = (f_i^{j,\epsilon}(u^{j,\epsilon}), \phi_j)_{\Omega_j^\epsilon} \\ \boldsymbol{-(-1)^j \epsilon (h(u^{1,\epsilon}, u^{2,\epsilon}), \phi_j)_{\Gamma_\epsilon}},$$

with
- $(\cdot,\cdot)_{\Omega_j^\epsilon}$ is the inner-product in $L^2(\Omega_j^\epsilon)$ or $L^2(\Omega_j^\epsilon)^n$, respectively
- $\langle \cdot,\cdot \rangle_{\Omega_j^\epsilon}$ is the duality pairing on $H^1(\Omega_j^\epsilon)' \times H^1(\Omega_j^\epsilon)$.

Existence and Uniqueness

Theorem

- *There exists a unique weak solution* $u^\epsilon = (u^{1,\epsilon}, u^{2,\epsilon})$ *with* $u^{j,\epsilon} \in L^2((0,T), H^1(\Omega_j^\epsilon))^m$ *and* $\partial_t u^{j,\epsilon} \in L^2((0,T), H^1(\Omega_j^\epsilon)')^m$
- *The solution is non-negative*
- *The following a-priori estimates are fulfilled*

$$\begin{aligned}\|u_i^{j,\epsilon}\|_{L^\infty((0,T)\times\Omega_j^\epsilon)} + \|u_i^{j,\epsilon}\|_{L^2((0,T),H^1(\Omega_j^\epsilon))} \\ + \|\partial_t u_i^{j,\epsilon}\|_{L^2((0,T),H^1(\Omega_j^\epsilon)')} + \sqrt{\epsilon}\|u_i^{j,\epsilon}\|_{L^2((0,T),L^2(\Gamma_\epsilon))} \quad &\leq C\end{aligned}$$

Proof:

Solve the Problem for a linear flux-condition on the boundary and use Schaefer's fixed point theorem

Two-scale Convergence ([92Allaire][1], [89Nguetseng][2])

- A sequence $u_\epsilon \subset L^2((0,T)\times\Omega)$ converges in the two-scale sense to $u_0 \in L^2((0,T)\times\Omega\times Y)$, if for every $\phi \in C\left([0,T]\times\overline{\Omega}, C_{per}\left(\overline{Y}\right)\right)$ holds

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right)dxdt = \int_0^T\int_\Omega\int_Y u_0(t,x,y)\phi(t,x,y)dydxdt$$

- A two-scale convergent sequence u_ϵ convergences strongly in the two-scale sense to u_0, if additionally

$$\lim_{\epsilon\to 0}\|u_\epsilon\|_{L^2((0,T)\times\Omega)} = \|u_0\|_{L^2((0,T)\times\Omega\times Y)}$$

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal.,23 (1992), pp. 1482–1518.
[2] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989), pp. 608–623.

Two-scale Convergence on Γ_ϵ ([96Allaire et al.][3], [96Neuss-Radu][4])

- A sequence $u_\epsilon \subset L^2((0,T)\times\Gamma_\epsilon)$ converges in the two-scale sense on the surface Γ_ϵ to a limit $u_0 \in L^2((0,T)\times\Omega\times\Gamma)$, if for every $\phi \in C\left([0,T]\times\overline{\Omega}, C_{per}(\Gamma)\right)$ holds

$$\lim_{\epsilon\to 0} \epsilon \int_0^T \int_{\Gamma_\epsilon} u_\epsilon(t,x)\phi\left(t,x,\frac{x}{\epsilon}\right) d\sigma dt = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y)\phi(t,x,y) d\sigma_y dx dt$$

- A two-scale convergent sequence u_ϵ on Γ_ϵ converges strongly in the two-scale sense on Γ_ϵ, if additionally holds

$$\lim_{\epsilon\to 0} \sqrt{\epsilon}\|u_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} = \|u_0\|_{L^2((0,T)\times\Omega\times\Gamma)}$$

[3] G. Allaire, A. Damlamian, U. Hornung, *Two-scale convergence on periodic surfaces and applications*, Proceedings of the international conference on mathematical modelling of flow through porous media, A. Bourgeat et al., eds., World Scientific, Singapore (1996), pp. 15–25.

[4] M. Neuss-Radu, *Some Extensions of Two-Scale Convergence*, C. R. Acad. Sci. Paris Sér. I Math., 322 (1996), pp. 899–904.

The Unfolding Operator ([02,06,08Cioranescu et al.][5,6,7])

Consider the unfolding operator L_ϵ on three different domains:

$$L_\epsilon : L^2((0,T)\times\Omega) \to L^2((0,T)\times\Omega\times Y)$$
$$L_\epsilon : L^2((0,T)\times\Omega_j^\epsilon) \to L^2((0,T)\times\Omega\times Y_j)$$
$$L_\epsilon : L^2((0,T)\times\Gamma_\epsilon) \to L^2((0,T)\times\Omega\times\Gamma)$$

defined by

$$L_\epsilon u_\epsilon(t,x,y) := u_\epsilon\left(t, \epsilon\left(\left[\frac{x}{\epsilon}\right] + y\right)\right)$$

[5] D. Cioranescu, A. Damlamian, G. Griso, *Periodic unfolding and homogenization*, C. R. Acad. Sci. Paris Sér. 1, 335 (2002), pp. 99–104.
[6] D. Cioranescu, P. Donato, R. Zaki, *The periodic unfolding method in perforated domains*, Port. Math. (N.S.), 63 (2006), pp. 467–496.
[7] D.Cioranescu, A. Damlamian, G. Griso, *The periodic unfolding method in homogenization*, SIAM J. Math. Anal., 40 (2008), pp. 1585–1620.

Unfolding and Two-Scale-Convergence ([96Bourgeat et al.][8])

Lemma

(i) *Let* $\{u_\epsilon\}_{\epsilon>0} \subset L^2((0,T)\times\Omega)$ *be bounded. Then are equivalent*

a) $u_\epsilon \to u_0$ *weakly/strongly in the two-scale sense*
b) $L_\epsilon u_\epsilon \to u_0$ *weakly/strongly in* $L^2((0,T)\times\Omega\times Y)$

(ii) *Let* $\{u_\epsilon\}_{\epsilon>0} \subset L^2((0,T)\times\Gamma_\epsilon)$ *with* $\sqrt{\epsilon}\|u_\epsilon\|_{L^2((0,T)\times\Gamma_\epsilon)} \leq C$. *Then are equivalent*

a) $u_\epsilon \to u_0$ *weakly/strongly in the two-scale sense on* Γ_ϵ
b) $L_\epsilon u_\epsilon \to u_0$ *weakly/strongly in* $L^2((0,T)\times\Omega\times\Gamma)$

[8] A. Bourgeat, S. Luckhaus, A. Mikelić, Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow, SIAM J. Math. Anal., 27 (1996), pp. 1520–1543.

Extensions to the domain Ω

- It exists $F_\epsilon : L^2((0,T),H^1(\Omega_1^\epsilon)) \to L^2((0,T),H^1(\Omega))$ with
$$\|F_\epsilon u_\epsilon\|_{L^2((0,T),H^1(\Omega))} \leq C\|u_\epsilon\|_{L^2((0,T),H^1(\Omega_1^\epsilon))}$$
Write $\tilde{u}_\epsilon = F_\epsilon u_\epsilon$
- For $u_\epsilon \in L^2((0,T)\times\Omega_2^\epsilon)$ denote by $\overline{u_\epsilon}$ the zero extension to Ω
- For $u_\epsilon \in L^2((0,T),H^1(\Omega_j^\epsilon))$ with $\partial_t u_\epsilon \in L^2((0,T),H^1(\Omega_j^\epsilon)')$ define
$$\widetilde{\partial_t u_\epsilon} := \partial_t\left(\chi_{\Omega_j^\epsilon} u_\epsilon\right) \in L^2((0,T),H^1(\Omega)')$$
with the characteristic function $\chi_{\Omega_j^\epsilon}$ on Ω_j^ϵ

Convergence results for $u^{1,\epsilon}$

Theorem

There exists a subsequence of $u_i^{1,\epsilon}$ *and functions* $u_i^{1,0} \in L^2((0,T),H^1(\Omega))$ *with* $\partial_t u_i^{1,0} \in L^2((0,T),H^1(\Omega)')$ *and* $u_i^{1,1} \in L^2\left((0,T)\times\Omega, H^1_{per}(Y)/\mathbb{R}\right)$ *for* $i = 1,\dots,m$*, such that*

$$\tilde{u}_i^{1,\epsilon} \to u_i^{1,0} \qquad \text{strongly in } L^2((0,T),L^2(\Omega))$$

$$\nabla\tilde{u}_i^{1,\epsilon} \to \nabla_x u_i^{1,0} + \nabla_y u_i^{1,1} \qquad \text{in the two-scale sense}$$

$$u_i^{1,\epsilon}|_{\Gamma_\epsilon} \to u_i^{1,0} \qquad \text{strongly in the two-scale sense on } \Gamma_\epsilon$$

$$\widetilde{\partial_t u_i}^{1,\epsilon} \rightharpoonup |Y_1|\partial_t u_i^{1,0} \qquad \text{weakly in } L^2((0,T),H^1(\Omega)')$$

Convergence results for $u^{2,\epsilon}$

Theorem

There exists a function $u_i^{2,0} \in L^2((0,T)\times\Omega)$ *with* $\partial_t u_i^{2,0} \in L^2((0,T),H^1(\Omega)')$, *such that up to a subsequence*

$$\overline{u_i^{2,\epsilon}} \to \chi_{Y_2} u_i^{2,0} \qquad \text{strongly in the two-scale sense}$$

$$\overline{\nabla u_i^{2,\epsilon}} \to 0 \qquad \text{in the two-scale sense}$$

$$u_i^{2,\epsilon}|_{\Gamma_\epsilon} \to u_i^{2,0} \qquad \text{strongly in the two-scale sense on } \Gamma_\epsilon$$

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Problems:

- No good extension operator from $L^2((0,T),H^1(\Omega_2^\epsilon))$ into $L^2((0,T),H^1(\Omega))$

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Problems:
- No good extension operator from $L^2((0,T),H^1(\Omega_2^\epsilon))$ into $L^2((0,T),H^1(\Omega))$
- The “unfolded” equation has an ϵ-scaling in the diffusion term

Convergence results for $u^{2,\epsilon}$

Theorem

There exists a function $u_i^{2,0} \in L^2((0,T)\times\Omega)$ with $\partial_t u_i^{2,0} \in L^2((0,T),H^1(\Omega)')$, such that up to a subsequence

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Problems:
- No good extension operator from $L^2((0,T),H^1(\Omega_2^\epsilon))$ into $L^2((0,T),H^1(\Omega))$
- The “unfolded” equation has an ϵ-scaling in the diffusion term
- We can’t use shifts on the boundary Γ_ϵ resp. Γ (i.e. a standard Kolmogorov-argument is not possible)

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a) Proof: $\overline{\nabla u_i^{2,\epsilon}} \to 0$ in the two-scale sense

- Standard compactness results imply the existence of $\xi_i \in L^2((0,T) \times \Omega \times Y)^n$, such that (subsequence)

$$\overline{\nabla u_i^{2,\epsilon}} \to \chi_{Y_2} \xi_i \quad \text{weakly in the two-scale sense}$$

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- ξ_i can be represented by a gradient in Y_2:

$$\xi_i = \nabla_y p_i \quad \text{in } L^2((0,T) \times \Omega, L^2(Y_2))$$

with $p_i \in L^2((0,T) \times \Omega, H^1(Y_2)/\mathbb{R})$.

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  with $p_i \in L^2((0,T)\times\Omega, H^1(Y_2)/\mathbb{R})$.
- It holds

$$\int_{Y_2} \nabla_y p_i \nabla_y \phi \, dy = 0$$

  for all $\phi \in H^1(Y_2)$

$$\Rightarrow \xi_i = 0 \text{ in } Y_2$$

b) Proof: $u_i^{2,\epsilon}|_{\Gamma_\epsilon} \to u_i^{2,0}$ strongly in the two-scale sense on Γ_ϵ

- The function δu of u for $l \in \mathbb{Z}^n$ is given by

$$\delta u(t,x) = u(t, x + l\epsilon) - u(t,x)$$

- For $U \subset \mathbb{R}^n$ bounded and $h > 0$ set

$$U_h = \{x \in U : \ \mathrm{dist}(x, \partial U) > h\}$$

Lemma

For almost every $t \in (0,T)$ and all $l \in \mathbb{Z}^n$ with $|l\epsilon| < h$ and $0 < h < \frac{1}{2}$ it holds that

$$\|\delta u_i^{2,\epsilon}(t)\|^2_{L^2(\Omega^\epsilon_{2,h})} \leq C\left(\epsilon^2 + \sum_{k=1}^m \left(\|\delta u^2_{k,0}\|^2_{L^2(\Omega^\epsilon_{2,h})} + \|\delta u_k^{1,\epsilon}\|^2_{L^2((0,T),L^2(\Omega^\epsilon_{1,h}))}\right)\right)$$

b) Proof: $u_i^{2,\epsilon}|_{\Gamma_\epsilon} \to u_i^{2,0}$ strongly in the two-scale sense on Γ_ϵ

- For $L_\epsilon u_i^{2,\epsilon} \in L^2((0,T)\times\Omega, H^{1/2}(\Gamma))$ holds with $\xi \in \mathbb{R}^n$

$$\left\| L_\epsilon u_i^{2,\epsilon}(\cdot,\cdot+\xi) - L_\epsilon u_i^{2,\epsilon}(\cdot,\cdot) \right\|_{L^2((0,T)\times\Omega,L^2(\Gamma))}$$

$$\leq C\left(h^2 + \epsilon^2 + \sum_{m\in\{0,1\}^n} \left\| u_i^{2,\epsilon}\left(\cdot,\cdot+\epsilon\left(m+\left[\frac{\xi}{\epsilon}\right]\right)\right) - u_i^{2,\epsilon}(\cdot,\cdot)\right\|^2_{L^2((0,T)\times\Omega^\epsilon_{2,h}}\right)$$

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$$\leq C \left(h^2 + \epsilon^2 + \sum_{m\in\{0,1\}^n} \left\| u_i^{2,\epsilon}\left(\cdot, \cdot + \epsilon\left(m + \left[\frac{\xi}{\epsilon}\right]\right)\right) - u_i^{2,\epsilon}(\cdot,\cdot) \right\|^2_{L^2((0,T)\times\Omega^\epsilon_{2,h}} \right)$$

- This implies for $\delta > 0$, $\vec{\delta} = (\delta, \dots, \delta) \in \mathbb{R}^n$

$$\left\| L_\epsilon u_i^{2,\epsilon}(\cdot + \delta, \cdot + \vec{\delta}) - L_\epsilon u_i^{2,\epsilon}(\cdot,\cdot) \right\|_{L^2(W^\delta, L^2(\Gamma))} \xrightarrow{\delta\to 0} 0$$

uniformly in ϵ with $W^\delta = (0, T-\delta) \times (\Omega \cap (\Omega - \vec{\delta}))$

b) Proof: $u_i^{2,\epsilon}|_{\Gamma_\epsilon} \to u_i^{2,0}$ strongly in the two-scale sense on Γ_ϵ

- For $L_\epsilon u_i^{2,\epsilon} \in L^2((0,T) \times \Omega, H^{1/2}(\Gamma))$ holds with $\xi \in \mathbb{R}^n$

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$$\leq C \left(h^2 + \epsilon^2 + \sum_{m \in \{0,1\}^n} \left\| u_i^{2,\epsilon}\left(\cdot, \cdot + \epsilon\left(m + \left[\frac{\xi}{\epsilon}\right]\right)\right) - u_i^{2,\epsilon}(\cdot,\cdot) \right\|^2_{L^2((0,T)\times\Omega^\epsilon_{2,h}} \right)$$

- This implies for $\delta > 0$, $\vec{\delta} = (\delta, \ldots, \delta) \in \mathbb{R}^n$

$$\left\| L_\epsilon u_i^{2,\epsilon}(\cdot + \delta, \cdot + \vec{\delta}) - L_\epsilon u_i^{2,\epsilon}(\cdot,\cdot) \right\|_{L^2(W^\delta, L^2(\Gamma))} \xrightarrow{\delta \to 0} 0$$

uniformly in ϵ with $W^\delta = (0, T-\delta) \times (\Omega \cap (\Omega - \vec{\delta}))$

- For every measurable subset $A \subset (0,T) \times \Omega$ the set

$$\left\{ \int_A L_\epsilon u_i^{2,\epsilon}(t,x,\cdot)dxdt \right\}_{\epsilon > 0} \subset H^{1/2}(\Gamma)$$

is relatively compact in $L^2(\Gamma)$

b) Proof: $u_i^{2,\epsilon}|_{\Gamma_\epsilon} \to u_i^{2,0}$ strongly in the two-scale sense on Γ_ϵ

Now conclude with the following Theorem

Theorem

Let $F \subset L^2(W, B)$ with a Banach-space B and a $W \subset \mathbb{R}^n$ a cuboid. Then F is relatively compact in $L^2(W, B)$ if and only if for $\delta > 0$ and $\vec{\delta} = (\delta, \dots, \delta) \in \mathbb{R}^n$ holds

(i) *For every measurable set $A \subset W$ the set $\left\{ \int_A u dx \,\middle|\, u \in F \right\}$ is relatively compact in B.*

(ii) $\left\| u(\cdot + \vec{\delta}) - u(\cdot) \right\|_{L^2(W \cap (W - \vec{\delta}), B)} \xrightarrow{\delta \to 0} 0$ *uniformly in F.*

Proof: Same arguments for the 1-dimensional case as in [Simon][9].

$\Longrightarrow \{L_\epsilon u_i^{2,\epsilon}\}_{\epsilon > 0}$ is relatively compact in $L^2((0, T) \times \Omega, L^2(\Gamma))$

[9] J. Simon, *Compact Sets in the Space* $L^p(0, T; B)$, Ann. Mat. Pura Appl., 146 (1987), pp. 65–96.

The Macroscopic Equations

Theorem

The function $u_i^{1,0}$ is a weak solution of

$$
\begin{aligned}
|Y_1|\partial_t u_i^{1,0} - \nabla \cdot \left(D_i^* \nabla u_i^{1,0} \right) &= |Y_1| f_i^1 \left(u^{1,0} \right) + |\boldsymbol{\Gamma}| \boldsymbol{h} \left(\boldsymbol{u}^{1,0}, \boldsymbol{u}^{2,0} \right) && \text{in } (0,T) \times \Omega \\
-D_i^* \nabla u_i^{1,0} \cdot \nu &= 0 && \text{on } (0,T) \times \partial\Omega \\
u_i^{1,0}(0) &= u_{i,0}^1 && \text{in } L^2(\Omega)
\end{aligned}
$$

and for $u_i^{2,0}$ holds the ordinary differential equation

$$
\begin{aligned}
|Y_2|\partial_t u_i^{2,0} &= |Y_2| f_i^2 \left(u^{2,0} \right) - |\boldsymbol{\Gamma}| \boldsymbol{h} \left(\boldsymbol{u}^{1,0}, \boldsymbol{u}^{2,0} \right) && \text{in } (0,T) \times \Omega \\
u_i^{2,0} &= u_{i,0}^2 && \text{in } L^2(\Omega)
\end{aligned}
$$

$\rightarrow$ *The diffusion-term vanishes in the upscaled equations*

The Macroscopic Equations

Theorem

The homogenized matrix $D_i^* \in \mathbb{R}^{n\times n}$ is given by

$$(D_i^*)_{k,l} = \int_{Y_1} D_i^1 \left(\nabla w_k + e_k\right) \cdot \left(\nabla w_l + e_l\right) dy,$$

where the w_k, $k = 1, \ldots, n$, are the unique solutions of the cell problem

$$-\Delta w_i = 0 \quad \text{in } Y_1$$
$$-\nabla w_i \cdot \nu = \nu_i \quad \text{on } \Gamma$$
$$w_i \text{ is } Y\text{-periodic and } \int_{Y_1} w_i dy = 0$$

The Macroscopic Equations

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Theorem

The solution $(u^{1,0}, u^{2,0})$ is unique.

Conclusion and Outlook

- We derived a homogenized model for a two component medium for which one component is disconnected
- We considered nonlinear transmission conditions at the interface between the two media
- The rigorous derivation of the macroscopic model involved the strong two-scale convergence for functions defined on periodic surfaces. To obtain this, we used a compactness criterion of Simon, which we generalized to functions defined on a rectancle in $\mathbb{R}^n$ with values in a Banach-space
- The presented model describes only a part of the carbohydrate metabolism in the plant cell. In an upcoming paper the full carbohydrate metabolism is modelled and analysed.

Thank you for your attention!

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