Coupling Stokes and Darcy equations: modeling and numerical methods

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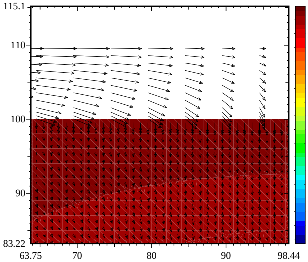
Motivation

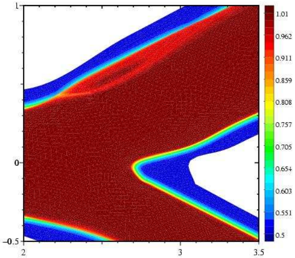
- We want to model the filtration of a *free fluid* through a *porous medium*:

Environmental application



[simulation by E. Miglio]

Biomedical simulations



[simulation by P. Zunino]

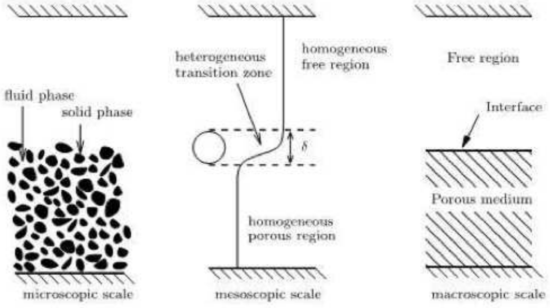
Industrial applications: filters, porous foams, fuel cells...

- We must introduce a suitable modeling framework and set up effective numerical methods.

Modeling filtration (i)

To model the filtration process three different approaches are generally considered:

I - **Micro**scale II - **Meso**scale III - **Macro**scale

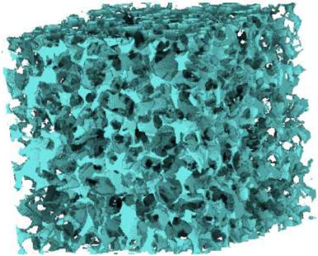


[Chandesris and Jamet, *Transp. Porous Med.*, 2009]

Modeling filtration (ii)

I - **Micro**scale approach

The geometry of the porous medium must be taken into account and the Navier-Stokes equations must be solved in the whole domain.



[R. Hilfer, Lecture Notes in Physics 554, p. 203-241 (2000)]

Modeling filtration (iii)

II - Mesoscale approach

A transition region is considered between the porous medium and the free fluid [Whitaker and Ochoa-Tapia (1995); Jackson et al (2012)].

Penalization approach [Iliev et al. (2004,2007); Angot (1999)]:

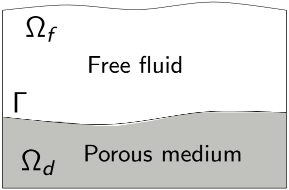
$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{n\nu}{\mathsf{K}}\mathbf{u} + \frac{C_F}{\sqrt{\mathsf{K}}}|\mathbf{u}|\mathbf{u} &= \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned} \qquad \text{in } \Omega_f \cup \Omega_d$$

The penalization terms are set to zero in Ω_f.

Modeling filtration (iv)

III - **Macro**scale approach

Sharp interface Γ with continuity conditions across it.



- Free flow: **Navier-Stokes** equations

$$\left.\begin{array}{rcl} -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_f, p_f) + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f & = & \mathbf{f}_f \\ \operatorname{div}\mathbf{u}_f & = & 0 \end{array}\right. \quad \text{in } \Omega_f$$

where $\boldsymbol{\sigma}(\mathbf{u}_f, p_f) = 2\mu\nabla^s\mathbf{u}_f - p_f\mathbf{I}$ is the Cauchy stress tensor.

- Porous media flow: **Darcy** equation

$$\begin{array}{r} \mathbf{u}_d + \frac{\mathbf{K}}{\mu}(\nabla p_d - \mathbf{f}_d) = \mathbf{0} \\ \operatorname{div}\mathbf{u}_d = 0 \end{array} \;\Leftrightarrow\; -\operatorname{div}\left(\frac{\mathbf{K}}{\mu}(\nabla p_d - \mathbf{f}_d)\right) = 0 \quad \text{in } \Omega_d$$

Modeling filtration (v)

Continuity conditions (I)

1. Continuity of normal velocities (due to incompressibility)

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_d \cdot \mathbf{n} \quad \text{on } \Gamma$$

2. Continuity of normal stresses

$$-\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = p_d \quad \text{on } \Gamma$$

(pressures may be discontinuous across Γ: $p_f - p_d \neq 0$ on Γ);

3. Beavers–Joseph–Saffman condition

$$-\boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = \frac{\alpha_d \mu}{\sqrt{\boldsymbol{\tau} \cdot \mathbf{K} \cdot \boldsymbol{\tau}}} \mathbf{u}_f \cdot \boldsymbol{\tau}$$

[Beavers and Joseph (1967); Saffman (1971);
Jäger and Mikelić (1996,2000,2001) Miglio et al (2002); Layton et al (2003)]

Modeling filtration (vi)

Continuity conditions (II)

1. Continuity of normal velocities (due to incompressibility)

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_d \cdot \mathbf{n} \quad \text{on } \Gamma$$

2. Continuity of pressures

$$\boxed{p_f = p_d \quad \text{on } \Gamma}$$

[Ene and Sanchez-Palencia (1975); Levy and Sanchez-Palencia (1975)]

Modeling filtration (vii)

- The microscale approach is very demanding: careful reconstruction of the geometry and solution of the Navier-Stokes equations.
- The macroscale approach allows using ad-hoc models in each subdomain.
  Once the coupling conditions are set, a global well-posed problem may be set up and non-overlapping domain decomposition methods can be used to solve the corresponding interface problem (*substructuring techniques*).
- The mesoscale approach requires solving a more expensive problem in $\Omega_f \cup \Omega_d$ than the macroscale approach.

Coupling through a sharp interface:

domain decomposition methods without overlap
(macroscale setting)

Weak form of the Stokes-Darcy problem

Find $\mathbf{u}_f \in H^1(\Omega_f)$, $p_f \in L^2(\Omega_f)$, $p_d \in H^1(\Omega_d)$:

$$\int_{\Omega_f} 2\mu\, \nabla^s \mathbf{u}_f : \nabla^s \mathbf{v} + \int_\Gamma \frac{\alpha_d \mu}{\sqrt{\boldsymbol{\tau} \cdot \mathbf{K} \cdot \boldsymbol{\tau}}} (\mathbf{u}_f)_\tau \mathbf{v}_\tau - \int_{\Omega_f} p_f \operatorname{div} \mathbf{v} + \int_\Gamma p_d\, (\mathbf{v} \cdot \mathbf{n}) = \int_{\Omega_f} \mathbf{f}_f \cdot \mathbf{v}$$

$$\int_{\Omega_f} q_f \operatorname{div} \mathbf{u}_f = 0$$

$$\int_{\Omega_d} \nabla q_d \cdot \frac{\mathbf{K}}{\mu} \nabla p_d - \int_\Gamma (\mathbf{u}_f \cdot \mathbf{n}) q_d = \int_{\Omega_d} \frac{\mathbf{K}}{\mu} \mathbf{f}_d \cdot \nabla q_d$$

Algebraic form

A suitable conforming finite element approximation of the coupled problem leads to the linear system:

$$\left(\begin{array}{ccc|c} (\mathrm{A}_f + \bar{\alpha}\mathrm{M}_\Gamma)_{II} & (\mathrm{A}_f)_{I\Gamma} & \mathrm{D}_I^T & 0 \\ (\mathrm{A}_f)_{\Gamma I} & (\mathrm{A}_f)_{\Gamma\Gamma} & \mathrm{D}_\Gamma^T & \mathrm{C}_\Gamma^T \\ \mathrm{D}_I & \mathrm{D}_\Gamma & 0 & 0 \\ \hline 0 & -\mathrm{C}_\Gamma & 0 & \mathrm{A}_d \end{array}\right) \begin{pmatrix} (\mathbf{u}_f)_I \\ (\mathbf{u}_f)_n \\ p_f \\ p_d \end{pmatrix} = \mathbf{b}$$

with $(\mathbf{u}_f)_n$ vector of the nodal values of $\mathbf{u}_f \cdot \mathbf{n}$ on Γ

An equivalent interface equation

The Stokes-Darcy problem can be formulated in terms of the solution λ (normal velocity across Γ) of the interface problem

$$S_f \lambda + S_d \lambda = \chi \quad \text{on } \Gamma$$

- S_f *"fluid"* operator:

$$S_f : \lambda \text{ (normal velocities on } \Gamma) \xrightarrow[\textit{Stokes}]{\textit{solve}} -\mathbf{n}\cdot\boldsymbol{\sigma}(\mathbf{u}_f, p_f)\cdot\mathbf{n} \text{ (normal stresses on } \Gamma).$$

- S_d *"porous medium"* operator:

$$S_d : \lambda \text{ (normal velocities on } \Gamma) \xrightarrow[\textit{Darcy}]{\textit{solve}} p_{d|\Gamma} \text{ (pressure } p_d \text{ on } \Gamma) \;.$$

Discrete interface equation for the normal velocity:

$$(\boldsymbol{\Sigma}_f + \boldsymbol{\Sigma}_d)(\mathbf{u}_f)_n = \boldsymbol{\chi}_s + \boldsymbol{\chi}_d$$

[MD, Quarteroni (2003)]

Dimensional analysis of the Stokes-Darcy problem

The Stokes-Darcy problem can be characterized in terms of three dimensionless numbers:

- the Reynolds number

$$\mathrm{Re} = \frac{U_f X_f \rho}{\mu}$$

- the Euler number

$$\mathrm{Eu} = \frac{P_f}{\rho U_f^2}$$

- the Darcy number [Bear (1991)]

$$\mathrm{Da} = \frac{K}{X_f^2}$$

[MD (in preparation)]

Dimensionless Schur complement system

The Schur complement system with respect to $(\mathbf{u}_f)_n$ becomes:

$$\boxed{(\boldsymbol{\Sigma}_s + \boldsymbol{\Sigma}_d)(\mathbf{u}_f)_n = \boldsymbol{\chi}_s + \boldsymbol{\chi}_d}$$

with

$$\begin{aligned} \boldsymbol{\Sigma}_f &= (\mathrm{Re}\,\mathrm{Eu}\,)^{-1}\hat{\boldsymbol{\Sigma}}_f \\ \boldsymbol{\Sigma}_d &= (\mathrm{Re}\,\mathrm{Eu}\,\mathrm{Da}\,)^{-1}\hat{\boldsymbol{\Sigma}}_d \end{aligned}$$

and

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}_f &= (\mathrm{A}_f)_{\Gamma\Gamma} - (\mathrm{A}_f)_{\Gamma I}(\mathrm{A}_f + \alpha \mathrm{Da}^{-1/2}\mathrm{M}_\Gamma)^{-1}_{II}(\mathrm{A}_f)_{I\Gamma} \\ &\quad + \left((\mathrm{A}_f)_{\Gamma I}(\mathrm{A}_f + \alpha \mathrm{Da}^{-1/2}\mathrm{M}_\Gamma)^{-1}_{II}\mathrm{D}_I^T - \mathrm{D}_\Gamma^T\right) \\ &\qquad \cdot \left(\mathrm{D}_I(\mathrm{A}_f + \alpha \mathrm{Da}^{-1/2}\mathrm{M}_\Gamma)^{-1}_{II}\mathrm{D}_I^T\right)^{-1} \\ &\qquad \cdot \left(\mathrm{D}_I(\mathrm{A}_f + \alpha \mathrm{Da}^{-1/2}\mathrm{M}_\Gamma)^{-1}_{II}(\mathrm{A}_f)_{I\Gamma} - \mathrm{D}_\Gamma\right) \\ \hat{\boldsymbol{\Sigma}}_d &= \mathrm{C}_\Gamma^T\mathrm{A}_d^{-1}\mathrm{C}_\Gamma \end{aligned}$$

If we multiply by (Re Eu Da), we obtain:

$$(\mathrm{Da}\,\hat{\boldsymbol{\Sigma}}_f + \hat{\boldsymbol{\Sigma}}_d)(\mathbf{u}_f)_n = \hat{\chi}_f + \hat{\chi}_d$$

Remark that we have independence of Re .

We can prove that

$$cond((\mathrm{Da}\,\hat{\boldsymbol{\Sigma}}_f + \hat{\boldsymbol{\Sigma}}_d)) \le c_1 \frac{k_+}{k_-} \cdot \frac{\mathrm{Da}\,k_- + h}{\mathrm{Da}\,k_+ + \frac{k_-}{k_+} h} \cdot \frac{1}{h}$$

	Da :	1	10^{-3}	10^{-6}	10^{-12}
Re $= 1$	$h = 1/4$	8.1586	55.4938	303.7483	305.7232
	$h = 1/8$	12.9894	43.9005	617.5967	634.4864
	$h = 1/16$	23.4144	32.1862	1.1558e+03	1.2809e+03
	$h = 1/32$	44.8932	23.3303	1.8151e+03	2.5736e+03
Re $= 10^2$	$h = 1/4$	8.1586	55.4938	303.7483	305.7232
	$h = 1/8$	12.9894	43.9005	617.5967	634.4864
	$h = 1/16$	23.4144	32.1862	1.1558e+03	1.2809e+03
	$h = 1/32$	44.8932	23.3303	1.8151e+03	2.5736e+03

Dirichlet-Neumann type preconditioners (i)

It can be proved that

$$cond((\text{Da}\,\hat{\boldsymbol{\Sigma}}_f)^{-1}(\hat{\boldsymbol{\Sigma}}_f + \hat{\boldsymbol{\Sigma}}_d)) \leq C \frac{1 + \dfrac{1}{k_- \text{Da}}}{1 + \dfrac{k_-}{k_+}\dfrac{h^2}{k_+ \text{Da}}}$$

from which we can see that

- if $\text{Da} \gg h^2$, then $cond((\text{Da}\,\hat{\boldsymbol{\Sigma}}_f)^{-1}(\hat{\boldsymbol{\Sigma}}_f + \hat{\boldsymbol{\Sigma}}_d)) \sim 1$
- if $\text{Da} \ll h^2$, then $cond((\text{Da}\,\hat{\boldsymbol{\Sigma}}_f)^{-1}(\hat{\boldsymbol{\Sigma}}_f + \hat{\boldsymbol{\Sigma}}_d)) \sim h^{-2}$

PCG iterations, $tol = 10^{-10}$, **Re** $= 1$

Da :	1	10^{-2}	10^{-3}	10^{-5}	10^{-9}	10^{-12}	10^{-14}
$h = 1/4$	8	8	9	9	9	9	9
$h = 1/8$	16	15	15	21	21	21	20
$h = 1/16$	27	23	16	41	46	47	46
$h = 1/32$	40	33	20	45	80	80	81

no prec.

Da :	1	10^{-2}	10^{-3}	10^{-5}	10^{-9}	10^{-12}	10^{-14}
$h = 1/4$	5	9	9	10	10	10	10
$h = 1/8$	5	11	17	24	24	24	24
$h = 1/16$	5	11	19	56	64	65	65
$h = 1/32$	5	11	19	74	131	135	135

$P = (\text{Da}\,\hat{\boldsymbol{\Sigma}}_f)^{-1}$

Dirichlet-Neumann type preconditioners (ii)

It can be proved that

$$cond(\hat{\mathbf{\Sigma}}_d^{-1}(\hat{\mathbf{\Sigma}}_f + \hat{\mathbf{\Sigma}}_d)) \leq C \frac{1 + \dfrac{k_+}{k_-}\dfrac{k_+\mathrm{Da}}{h^2}}{1 + k_-\mathrm{Da}}$$

from which we can see that

- if Da $\gg h^2$, then $cond(\hat{\mathbf{\Sigma}}_d^{-1}(\hat{\mathbf{\Sigma}}_f + \hat{\mathbf{\Sigma}}_d)) \sim h^{-2}$
- if Da $\ll h^2$, then $cond(\hat{\mathbf{\Sigma}}_d^{-1}(\hat{\mathbf{\Sigma}}_f + \hat{\mathbf{\Sigma}}_d)) \sim 1$

PCG iterations, $tol = 10^{-10}$, **Re** $= 1$

Da :	1	10^{-2}	10^{-3}	10^{-5}	10^{-9}	10^{-12}	10^{-14}
$h = 1/4$	8	8	9	9	9	9	9
$h = 1/8$	16	15	15	21	21	21	20
$h = 1/16$	27	23	16	41	46	47	46
$h = 1/32$	40	33	20	45	80	80	81

no prec.

Da :	1	10^{-2}	10^{-3}	10^{-5}	10^{-9}	10^{-12}	10^{-14}
$h = 1/4$	10	8	8	5	2	1	1
$h = 1/8$	24	20	18	6	2	1	1
$h = 1/16$	60	52	44	10	2	1	1
$h = 1/32$	149	135	114	17	2	1	1

$P = (\hat{\mathbf{\Sigma}}_d)^{-1}$

Comments

- Using $(\hat{\boldsymbol{\Sigma}}_f)^{-1}$ requires solving a Stokes problem:

$$\mathbf{y} = \hat{\boldsymbol{\Sigma}}_f^{-1}\mathbf{w} \quad \Leftrightarrow \quad y = \mathbf{u}_f \cdot \mathbf{n} \text{ on } \Gamma \quad \text{where}$$

$$\begin{aligned} Stokes(\mathbf{u}_f, p_f) &= 0 \quad \text{in } \Omega_f \\ -\mathbf{n}\cdot\boldsymbol{\sigma}(\mathbf{u}_f, p_f)\cdot\mathbf{n} &= \mathbf{w} \quad \text{on } \Gamma \end{aligned}$$

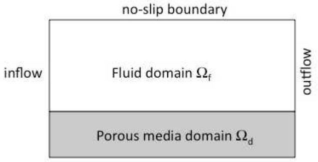
- Using $(\hat{\boldsymbol{\Sigma}}_d)^{-1}$ requires solving a Darcy problem:

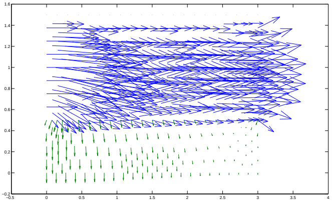
$$\mathbf{y} = \hat{\boldsymbol{\Sigma}}_d^{-1}\mathbf{w} \quad \Leftrightarrow \quad y = -\frac{\mathbf{K}}{\mu}\frac{\partial p_d}{\partial n} \text{ on } \Gamma \quad \text{where}$$

$$\begin{aligned} -\text{div}\left(\frac{\mathbf{K}}{\mu}\nabla p_d\right) &= 0 \quad \text{in } \Omega_d \\ p_d &= w \quad \text{on } \Gamma \end{aligned}$$

Numerical results (i)

We consider the following setting proposed in [Hanspal et al, 2009]:





- $\Omega_f = 0.015 \times 0.005$ m, $\Omega_d = 0.015 \times 0.0025$ m
- Inflow with max horizontal velocity 0.1 m/s, Re $= 500$

no prec.

K:	10^{-10} m^2	10^{-12} m^2	10^{-14} m^2
Da :	$1.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-7}$	$1.3 \cdot 10^{-9}$
$h = 1/4$	31	31	32
$h = 1/8$	68	76	76
$h = 1/16$	93	154	159

$P = (\text{Da}\,\hat{\mathbf{\Sigma}}_f)^{-1}$

K:	10^{-10} m^2	10^{-12} m^2	10^{-14} m^2
Da :	$1.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-7}$	$1.3 \cdot 10^{-9}$
$h = 1/4$	36	37	36
$h = 1/8$	98	106	107
$h = 1/16$	168	$>$250	$>$250

$P = (\hat{\mathbf{\Sigma}}_d)^{-1}$

K:	10^{-10} m^2	10^{-12} m^2	10^{-14} m^2
Da :	$1.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-7}$	$1.3 \cdot 10^{-9}$
$h = 1/4$	6	3	2
$h = 1/8$	10	4	3
$h = 1/16$	17	5	3

Numerical results (ii)

Following [Zunino (2002)] we consider an artery with

- radius = 0.3 cm
- wall thickness = 0.03 cm
- blood density $\rho_b = 1.04$ g/cm^3
- blood (kinematic) viscosity $\nu_b = 0.033$ cm^2/s
- wall permeability $\mathbf{K} = 2 \cdot 10^{-14}$ cm^2

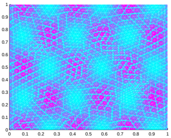
This corresponds to

- Re $= 272.73$
- Da $= 2.78 \cdot 10^{-14}$

Dofs ($\mathbf{u}_f + p_f + p_d$)					h	No prec.	PCG with $\hat{\mathbf{\Sigma}}_d$
259	+	73	+	138	1/4	32	1
973	+	259	+	495	1/8	76	1
3769	+	973	+	1869	1/16	147	2
14833	+	3769	+	7257	1/32	226	2

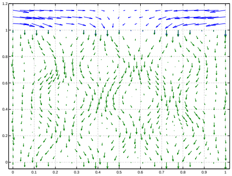
Numerical results (iii)

- $\Omega_f = (0,10) \times (10,12)$, $\Omega_d = (0,10) \times (0,10)$
- $\text{Re} = 1000$
- Variable permeability $10^{-8} \leq \mathbf{K} \leq 10^{-6}$ m^2 ($\text{Da} = 3.3 \cdot 10^{-10}$)

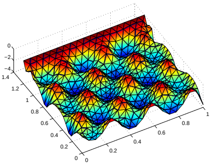


Iterations (PCG with $\hat{\boldsymbol{\Sigma}}_d$)	Relative residual ($tol = 10^{-10}$)
3	$8.9 \cdot 10^{-14}$
3	$3.6 \cdot 10^{-12}$
3	$6.7 \cdot 10^{-11}$

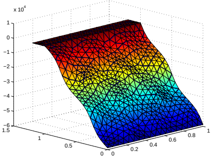
Velocity field



Vertical velocity



Hydrodynamic pressure



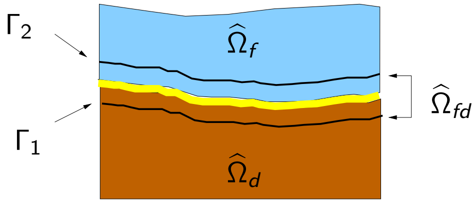
Coupling through a transition region:

interface control domain decomposition methods
(mesoscale setting)

Joint work with: **Paola Gervasio** (Università di Brescia)
Alfio Quarteroni (EPF Lausanne)

The ICDD method for the Stokes-Darcy problem (i)

Similarly to the mesoscale approach, we introduce a transition region overlapping Ω_f and Ω_d:



Stokes

$$-\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_f, p_f) = \mathbf{f}_f \quad \text{in } \widehat{\Omega}_f$$
$$\text{div } \mathbf{u}_f = 0 \quad \text{in } \widehat{\Omega}_f$$
$$\Phi_f(\mathbf{u}_f, p_f) = \boldsymbol{\lambda}_1 \quad \text{on } \Gamma_1$$

Darcy

$$-\text{div}\left(\frac{\mathsf{K}}{\mu}(\nabla p_d - \mathbf{f}_d)\right) = 0 \quad \text{in } \widehat{\Omega}_d$$
$$\Phi_d(p_d) = \lambda_2 \quad \text{on } \Gamma_2$$

$\boldsymbol{\lambda}_1,\ \lambda_2$ control interface functions

The ICDD method for the Stokes-Darcy problem (ii)

Consider controls corresponding to the traces of the Stokes velocity and the Darcy pressure on the interfaces:

$$\mathbf{u}_f = \boldsymbol{\lambda}_1 \text{ on } \Gamma_1 \quad \text{and } p_d = \lambda_2 \text{ on } \Gamma_2.$$

We set the minimization problem:

$$\inf_{\boldsymbol{\lambda}_1, \lambda_2} J(\boldsymbol{\lambda}_1, \lambda_2)$$

with

$$J(\boldsymbol{\lambda}_1, \lambda_2) = \frac{1}{2}\|\mathbf{u}_f - \mathbf{u}_d\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|p_f - p_d\|^2_{L^2(\Gamma_2)}$$

[MD, Gervasio, Quarteroni (submitted 2014)]

The ICDD method for the Stokes-Darcy problem (iii)

The corresponding optimality system becomes:

- state problems:

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}_f, p_f) &= \mathbf{f}_f \quad \text{in } \widehat{\Omega}_f \\ \text{div } \mathbf{u}_f &= 0 \quad \text{in } \widehat{\Omega}_f \\ \mathbf{u}_f &= \boldsymbol{\lambda}_1 \quad \text{on } \Gamma_1 \end{aligned}$$

$$\begin{aligned} -\text{div}\left(\frac{\mathbf{K}}{\mu}(\nabla p_d - \mathbf{f}_d)\right) &= 0 \quad \text{in } \widehat{\Omega}_d \\ p_d &= \lambda_2 \quad \text{on } \Gamma_2 \end{aligned}$$

- pseudo-adjoint problems:

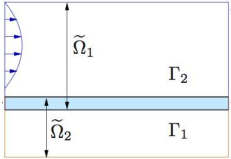
$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{w}_f, q_f) &= \mathbf{0} \quad \text{in } \widehat{\Omega}_f \\ \text{div } \mathbf{w}_f &= 0 \quad \text{in } \widehat{\Omega}_f \\ \mathbf{w}_f &= \mathbf{u}_f - \mathbf{u}_d \quad \text{on } \Gamma_1 \end{aligned}$$

$$\begin{aligned} -\text{div}\left(\frac{\mathbf{K}}{\mu}\nabla \psi_d\right) &= 0 \quad \text{in } \widehat{\Omega}_d \\ \psi_d &= p_d - p_f \quad \text{on } \Gamma_2 \end{aligned}$$

- pseudo-Euler-Lagrange equations:

$$(\mathbf{u}_f - \mathbf{u}_d) + \boldsymbol{\psi}_d = \mathbf{0} \text{ on } \Gamma_1 \qquad (p_d - p_f) + q_f = 0 \text{ on } \Gamma_2.$$

Numerical results (i)



BiCGStab iterations:

- Dependence on K

K	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-14}
iter	30	11	5	3	2	2	1
inf J	9.6e-21	6.8e-22	2.9e-25	1.3e-25	7.0e-27	8.2e-31	1.5e-24
$\|\mathbf{u}_f - \mathbf{u}_d\|_{L^2(\Omega_{fd})}$	3.9e-6	1.7e-6	1.9e-6	1.9e-6	1.9e-6	1.9e-6	1.9e-6
$\|p_f - p_d\|_{L^2(\Omega_{fd})}$	4.5e-10	2.7e-9	4.7e-9	5.2e-9	5.3e-9	5.3e-9	5.3e-9

- Dependence on δ

δ	$7.5 \cdot 10^{-4}$	$7.5 \cdot 10^{-5}$	$7.5 \cdot 10^{-6}$
iter ($K = 10^{-8}$)	4	10	14
iter ($K = 10^{-12}$)	1	2	2

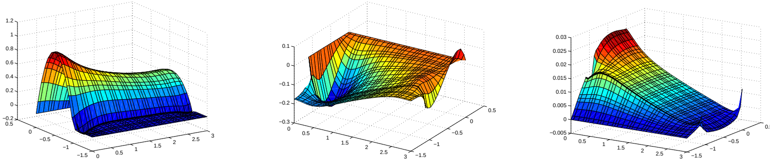
Physical considerations lead to choose $\delta \approx \sqrt{|K|}$.

- Dependence on h

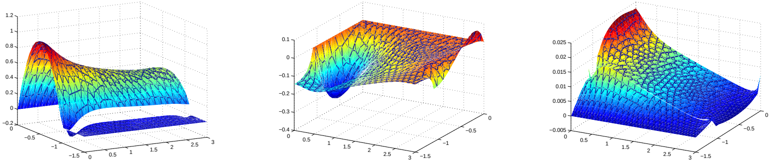
h	0.1	0.05	0.025
iter ($K = 10^{-8}$)	16	30	41
iter ($K = 10^{-12}$)	2	2	2

Comparison with the “sharp-interface” solution ($K = 10^{-7}\ m^2$)

ICDD:



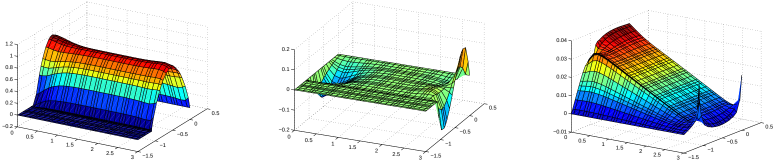
Sharp interface (Beavers-Joseph-Saffman):



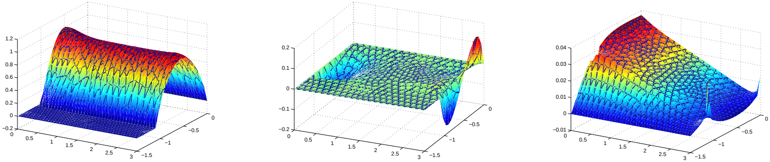
Horizontal velocity | Vertical velocity | Hydrodynamic pressure

Comparison with the “sharp-interface” solution ($K = 10^{-12}$ m^2)

ICDD:



Sharp interface (Beavers-Joseph-Saffman):

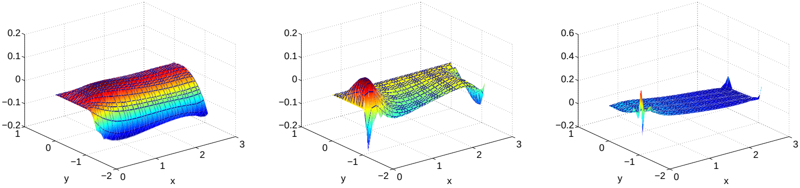


Horizontal velocity | Vertical velocity | Hydrodynamic pressure

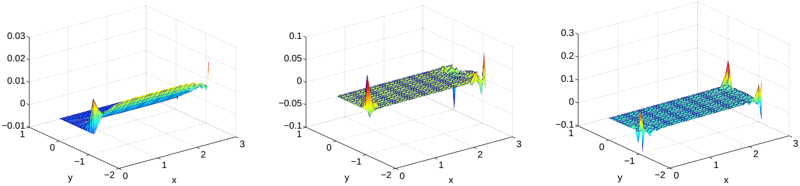
Comparison with the “sharp-interface” solution

Stokes domain

$K = 10^{-7}$ m^2



$K = 10^{-12}$ m^2



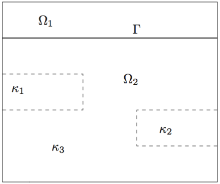
$$\frac{u_{ICDD} - u_{BJS}}{\|u_{BJS}\|_\infty} \qquad \frac{v_{ICDD} - v_{BJS}}{\|v_{BJS}\|_\infty} \qquad \frac{p_{ICDD} - p_{BJS}}{\|p_{BJS}\|_\infty}$$

Numerical results (ii)

$K_1 = 10^{-10}$, $K_2 = 10^{-14}$, $K_3 = 10^{-6}$ m^2

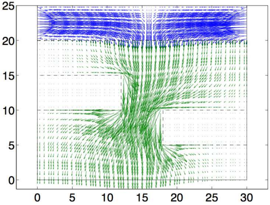
$\delta = 0.05\%$, convergence in 4 iterations

($tol = 10^{-9}$)

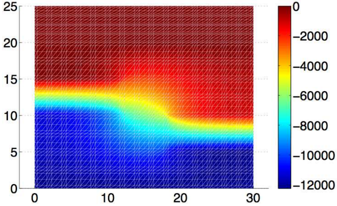


ICDD:

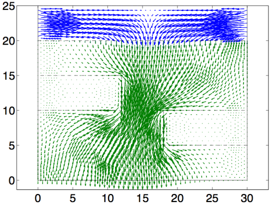
Velocity field

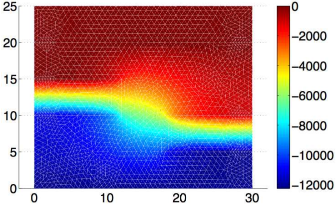


Hydrodynamic pressure



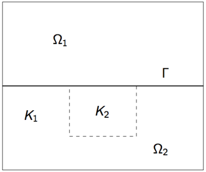
Sharp interface (Beavers-Joseph-Saffman):



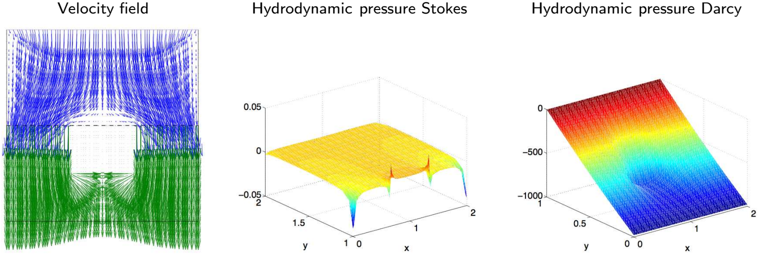


Numerical results (iii)

$K_1 = 10^{-7}$, $K_2 = 10^{-10}$ m^2, $Re = 100$.
$\delta = 0.05\%$, convergence in 2 iterations
$(tol = 10^{-9})$



ICDD:



This case is not meaningful using Beavers-Joseph-Saffman:
ICDD can treat more general fluid regimes.

Conclusions

- “Sharp-interface” approach:
  An interface problem and ad-hoc preconditioners for the Schur complement system (Steklov-Poincaré equation).

- “Transition-region” approach:
  A minimization problem which takes into account the physical properties of the problem without relying on any specific interface conditions.

Optimized Schwarz (Robin-Robin type) preconditioner (i)

How to deal with intermediate values of Da ?

Considering the Schur complement of the linear system associated to the Stokes-Darcy problem with respect to both $(\mathbf{u}_f)_n$ and $p_{d|\Gamma}$, we obtain the following *augmented interface system*:

$$\begin{pmatrix} \boldsymbol{\Sigma}_s & \mathrm{C}_\Gamma^T \\ -\mathrm{C}_\Gamma & \boldsymbol{\Sigma}_p \end{pmatrix} \begin{pmatrix} (\mathbf{u}_f)_n \\ p_{d|\Gamma} \end{pmatrix} = \begin{pmatrix} \chi_s \\ \chi_p \end{pmatrix}$$

where $\boldsymbol{\Sigma}_p \approx \boldsymbol{\Sigma}_d^{-1}$.

[MD (2013)]

Optimized Schwarz (Robin-Robin type) preconditioner (ii)

Consider the Robin-Robin method:

- Solve the Stokes problem in Ω_f with interface conditions on Γ:

$$
\begin{aligned}
-\boldsymbol{\tau} \cdot \boldsymbol{\sigma}(\mathbf{u}_f^{(k)}, p_f^{(k)}) \cdot \mathbf{n} &= \tfrac{\alpha_d \mu}{\sqrt{\boldsymbol{\tau} \cdot \mathbf{K} \cdot \boldsymbol{\tau}}} \mathbf{u}_f^{(k)} \cdot \boldsymbol{\tau} \\
-\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}_f^{(k)}, p_f^{(k)}) \cdot \mathbf{n} - \alpha_f \mathbf{u}_f^{(k)} \cdot \mathbf{n} &= p_d^{(k-1)} - \alpha_f\, \mathbf{u}_d^{(k-1)} \cdot \mathbf{n}
\end{aligned}
$$

- Solve the Darcy problem in Ω_d with interface condition on Γ:

$$
p_d^{(k)} + \alpha_p \mathbf{u}_d^{(k)} \cdot \mathbf{n} = -\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{u}_f^{(k)}, p_f^{(k)}) \cdot \mathbf{n} + \alpha_p \mathbf{u}_f^{(k)} \cdot \mathbf{n}
$$

where the parameters α_f and α_p must be chosen in a suitable way to ensure (optimal) convergence.

Optimized Schwarz (Robin-Robin type) preconditioner (iii)

The Robin-Robin method can be proved to be equivalent to a Gauss-Seidel scheme for the augmented interface Schur complement system multiplied by the preconditioner:

$$P_{OS}^{-1} = \begin{pmatrix} I & -\alpha_f I \\ \alpha_p^{-1} I & I \end{pmatrix}$$

which can be used in the framework of Krylov methods like GMRES or BiCGStab.

The theory of Optimized Schwarz methods allows to characterize α_f and α_p to maximize the convergence rate of the method.

[MD, Gerardo-Giorda (in preparation)]

Numerical results (i)

Domain $\Omega_f = (0,1) \times (1,2)$ and $\Omega_d = (0,1) \times (0,1)$. Solution

$$\mathbf{u}_f = ((y-1)^2 + (y-1) + \sqrt{\mathrm{Da}}, x(x-1))$$
$$p_f = \frac{2}{\mathrm{Re}\,\mathrm{Eu}}(x+y-1)$$
$$p_d = \frac{1}{\mathrm{Re}\,\mathrm{Eu}}\left(\frac{1}{\mathrm{Da}}(x(1-x)(y-1) + (y-1)^3/3) + 2x\right)$$

We set Re $= 10^3$ and Da $= 10^{-6}$, 10^{-8}, 10^{-10}, 10^{-12} m^2.
Using optimized Schwarz methods we can estimate

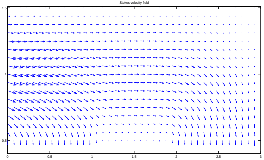
$$\alpha_f = 5.7873,\ \alpha_p = 0.0528.$$

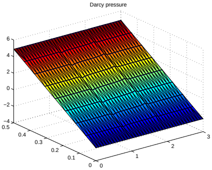
Da :	10^{-6}	10^{-8}	10^{-10}	10^{-12}
$h = 1/8$	18	18	18	18
$h = 1/16$	20	20	18	18
$h = 1/32$	17	20	20	20

BiCGStab iterations with $tol = 10^{-10}$

Numerical results (iii)

Setting as in [Hanspal et al, 2009] but with discontinuous permeability (reduced by one order of magnitude) and inflow from the left (no-slip conditions on the remaining fluid boundary):





Here Re $= 500$ and Da between $1.34 \cdot 10^{-5}$ and $1.34 \cdot 10^{-6}$.
Optimal coefficients: $\alpha_f = 13.24$ and $\alpha_p = 0.024$.

h:	$3/10$	$3/20$	$3/40$
elements	120	480	1920
iterations	19	20	20

BiCGStab iterations with $tol = 10^{-10}$

The ICDD method

ICDD is inspired to both the Virtual Control Method by J.L. Lions, O. Pironneau (1998) and the Least Squares Method by R. Glowinski, Q. Dinh, J. Periaux (1983).

VCM/LSM	ICDD
$J(\boldsymbol{\lambda}) = \frac{1}{2}\|u_1 - u_2\|_{L^2(\Omega_{12})}$	$J(\boldsymbol{\lambda}) = \frac{1}{2}\|u_1 - u_2\|_{L^2(\Gamma_1 \cup \Gamma_2)}$
distributed observation	interface observation
classical Optimality System	non-standard OS
(integration by parts)	

Interface observation makes ICDD more suitable for heterogeneous problems and it leads to more efficient algorithms than distributed observation.