

A discrete fracture model for two-phase flow with matrix-fracture interaction

Elyes Ahmed^{1,2}, Jérôme Jaffré¹, Jean E. Roberts¹

1 : INRIA-Rocquencourt, France

2 : ENIT-LAMSIN, Tunisia

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- 1 Introduction
- 2 Two-phase flow with a change of rock type
- 3 Two-phase flow with a fracture

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A problem requiring multi-scale modelling

- Fractures represent heterogeneities in porous media.
 - Usually of much higher permeability than surrounding medium
 - May be of much lower permeability so that they act as a barrier
- Fracture width much smaller than any reasonable parameter of spatial discretization.

The models that we consider

- Discrete fracture models, where the fracture locations are given.
- Models that allow an interchange between the fractures and the matrix rock.
- Reduced models where fracture f is reduced to a 2-D interface γ .

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Saturation equation

$$\phi \frac{\partial s}{\partial t} + \operatorname{div} \mathbf{u}_w = 0, \quad \mathbf{u}_w = \mathbf{r} + \mathbf{f},$$
$$r = \underbrace{-\mathbf{K} \nabla \alpha(s)}_{\text{cap. diffusion}}, \quad \mathbf{f} = \underbrace{b_T(s) \mathbf{u}_T + b_g(s) \mathbf{K} \mathbf{u}_G}_{\text{advection}}.$$

Pressure equation

$$\operatorname{div} \mathbf{u} = 0, \quad \mathbf{u} = -\mathbf{K} k_T(s) (\nabla p - \rho(s) \mathbf{u}_G),$$

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}_{nw}, \text{ total Darcy flow,}$$

$$p = \frac{1}{2}(p_w + p_{nw}) + \beta(s), \text{ global pressure (not physical).}$$

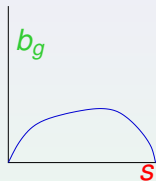
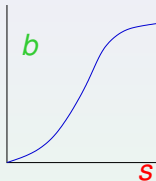
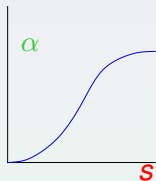
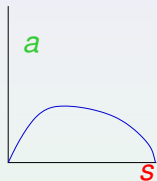
Functions α, β depend on capillary pressure and mobilities

Functions k_T, b_T, b_g, ρ depend on mobilities

$$a(s) = \frac{k_w k_{nw}}{k_w + k_{nw}} \left(-\frac{d\pi}{ds} \right), \quad a(0) = a(1) = 0, \quad \alpha(s) = \int_0^s a(\sigma) d\sigma,$$

$$b(s) = \frac{k_w}{k_w + k_{nw}}, \quad b(0) = 0, \quad b(1) = 1, \quad b_g(s) = \frac{k_w k_{nw}}{k_w + k_{nw}} (\rho_w - \rho_{nw})$$

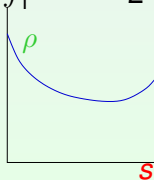
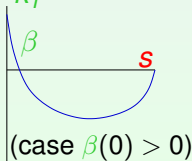
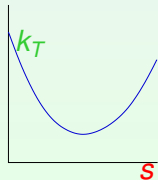
$$b_g(0) \geq 0 \text{ if } \rho_w \geq \rho_{nw}, \quad b_g(0) = b_g(1) = 0.$$



$$k_T(s) = k_w + k_{nw} \geq \underline{k}_T > 0$$

$$\rho(s) = \frac{k_{nw} \rho_{nw} + k_w \rho_w}{k_T}$$

$$\beta(s) = \int_1^s \left(b_T - \frac{1}{2} \right) \left(-\frac{d\pi}{ds} \right)$$



Rock type L		Rock type R
$\phi^L, K^L, k_\ell^L, \pi^L,$	σ	$\phi^R, K^R, k_\ell^R, \pi^R$
$\alpha^L, \beta^L, k_T^L, b_T^L, b_g^L, \rho^L$		$\alpha^R, \beta^R, k_T^R, b_T^R, b_g^R, \rho^R$

Transmission conditions across σ

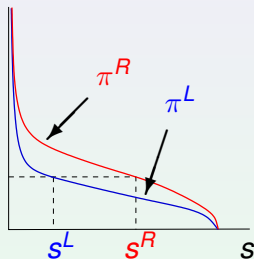
- Conservation of phases

$$\mathbf{u}_\ell^L \cdot \mathbf{n}^L + \mathbf{u}_\ell^R \cdot \mathbf{n}^R = 0, \ell = w, nw \implies \mathbf{u}^L \cdot \mathbf{n}^L + \mathbf{u}^R \cdot \mathbf{n}^R = 0$$

- Continuity of phase pressures

- Discontinuous saturation $s^L \neq s^R$

$$\pi^L(s^L) = \pi^R(s^R)$$



- Discontinuous global pressure $p^L \neq p^R$

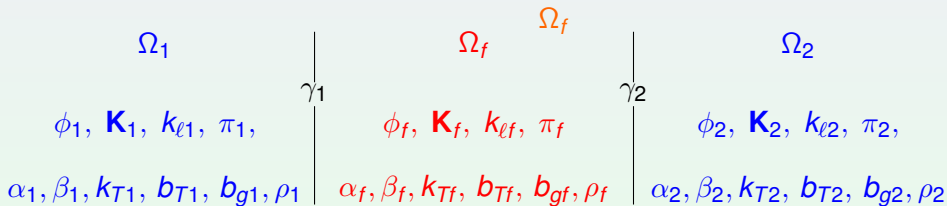
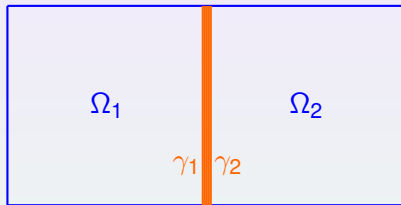
$$p^L - \beta^L(s^L) = p^R - \beta^R(s^R)$$

$$(p = \frac{1}{2}(p_w + p_{nw}) + \beta(s))$$

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Three rock types for $\Omega_1, \Omega_f, \Omega_2$

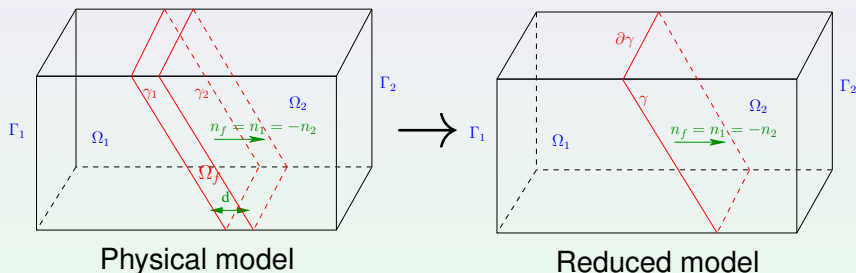
$$\Omega = \Omega_1 \cup \Omega_f \cup \Omega_2 \subset \mathbb{R}^n$$



Natural idea: • Use local grid refinement

Alternative: • Model Ω_f as an interface γ

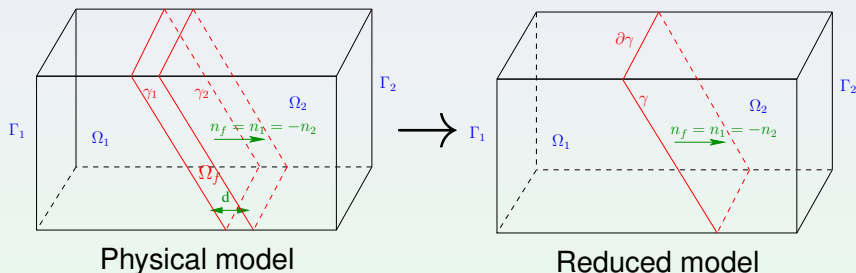
1. Shrink Ω_f to an interface hyperplane γ .



2. On γ introduce tangential and normal components:

$$\mathbf{K}_f = (\mathbf{K}_{f,\tau}, K_{f,n}), \quad \mathbf{u}_f = (\mathbf{u}_{f,\tau}, u_{f,n}).$$

1. Shrink Ω_f to an interface hyperplane γ .



2. On γ introduce tangential and normal components:

$$\mathbf{K}_f = (\mathbf{K}_{f,\tau}, K_{f,n}), \quad \mathbf{u}_f = (\mathbf{u}_{f,\tau}, u_{f,n}).$$

3. Average conservation equation and tangential component of Darcy's law across the fracture:

$$\operatorname{div}_\tau \mathbf{u}_\gamma = Q_\gamma + \mathbf{u}_1 \cdot \mathbf{n}_1 + \mathbf{u}_2 \cdot \mathbf{n}_2, \quad \mathbf{u}_\gamma = -\mathbf{K}_{f,\tau} \nabla_\tau P_\gamma$$

$$\text{with } \mathbf{u}_\gamma = \int_0^d \mathbf{u}_{f,\tau} d\sigma, \quad Q_\gamma = \int_0^d q_f d\sigma, \quad P_\gamma = \frac{1}{d} \int_0^d p_f d\sigma.$$

4. Integrate normal component of Darcy's law across the fracture:

→ Darcy's law across the interface γ :

$$-\frac{1}{\delta(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n} + (p_i - \beta_i(\mathbf{s}_i)) = (P_\gamma - \beta_f(\mathbf{S}_\gamma)), \quad i = 1, 2.$$

- $\delta(\mathbf{s}_\gamma) = \frac{2\mathbf{K}_{f\mathbf{n}} k_f(\mathbf{s}_\gamma)}{d}.$
- Note that when $\delta(\mathbf{s}_\gamma) \rightarrow \infty$, then $(p_i - \beta_i(\mathbf{s}_i)) = (P_\gamma - \beta_f(\mathbf{S}_\gamma)).$

5. Average conservation equation and tangential component of Darcy's law for the wetting phase across the fracture:

$$\phi_\gamma \frac{\partial \mathbf{s}_\gamma}{\partial t} + \operatorname{div}_\tau \mathbf{u}_{w\gamma} = Q_{w\gamma} + \mathbf{u}_{w1} \cdot \mathbf{n}_1 + \mathbf{u}_{w2} \cdot \mathbf{n}_2,$$

$$\mathbf{u}_{w\gamma} = -\mathbf{K}_\gamma \nabla_\tau \alpha_f(\mathbf{s}_\gamma) + \mathbf{f}(\mathbf{s}_\gamma),$$

with $\mathbf{u}_{w\gamma} = \int_0^d \mathbf{u}_{wf,\tau} d\sigma$, $Q_{w\gamma} = \int_0^d q_{wf} d\sigma$, $\mathbf{s}_\gamma = \frac{1}{d} \int_0^d \mathbf{s}_f d\sigma$.

6. Integrate normal component of Darcy's law (phase w) across the fracture:

→ Darcy's law for the wetting phase across the interface γ :

$$\frac{1}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_{wi} \cdot \mathbf{n}_i + \pi_i(\mathbf{s}_i) = \pi_\gamma(\mathbf{s}_\gamma) + \frac{f_\gamma(\mathbf{s}_\gamma)}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n}_i, \quad i = 1, 2.$$

- $\delta_w(\mathbf{s}_\gamma) = \mathbf{K}_f \mathbf{n} k_{wf} \mathbf{n}(s_f)/d$,
- Note that if $\delta_w(\mathbf{s}_\gamma) \rightarrow \infty$, then $\pi_i(\mathbf{s}_i) - \pi_\gamma(\mathbf{s}_\gamma) \rightarrow 0$.

Reduced model

Model in Ω_i

$$\phi_i \frac{\partial \mathbf{s}_i}{\partial t} + \operatorname{div} \mathbf{u}_{wi} = q_{wi}, \quad \mathbf{u}_{wi} = -\mathbf{K}_i \nabla \alpha(\mathbf{s}_i) + \mathbf{f}_i(\mathbf{s}_i),$$
$$\operatorname{div} \mathbf{u}_i = q_i, \quad \mathbf{u}_i = -\mathbf{K}_i k_i(\mathbf{s}_i)(\nabla p_i - \rho_i(\mathbf{s}_i) \mathbf{u}_{Gi}),$$

Model on γ

$$\phi_\gamma \frac{\partial \mathbf{s}_\gamma}{\partial t} + \operatorname{div} \mathbf{u}_{w\gamma} = Q_{w\gamma} + \sum_{i=1}^2 \mathbf{u}_{wi} \cdot \mathbf{n}_i, \quad \mathbf{u}_{w\gamma} = -\mathbf{K}_\gamma \nabla \alpha(\mathbf{s}_\gamma) + \mathbf{f}_\gamma(\mathbf{s}_\gamma),$$
$$\operatorname{div} \mathbf{u}_\gamma = Q_\gamma + \sum_{i=1}^2 \mathbf{u}_i \cdot \mathbf{n}_i, \quad \mathbf{u}_\gamma = -\mathbf{K}_\gamma k_\gamma(\mathbf{s}_\gamma)(\nabla p_\gamma - \rho_\gamma(\mathbf{s}_\gamma) \mathbf{u}_{G\gamma}),$$

Transmission conditions

$$\frac{1}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_{wi} \cdot \mathbf{n}_i + \pi_i(\mathbf{s}_i) = \pi_\gamma(\mathbf{s}_\gamma) + \frac{f_\gamma(\mathbf{s}_\gamma)}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n}_i,$$
$$-\frac{1}{\delta(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n} + p_i = P_\gamma - [\beta(\mathbf{s})]_{\gamma,i}.$$

Robin-to-Neumann operator

$$\mathcal{L}_i^{RtN}(\mathbf{p}_\gamma, \mathbf{u}_G, q_i) = -(\mathbf{u}_i \cdot \mathbf{n}_i)|_\gamma,$$

where $(\mathbf{p}_i, \mathbf{u}_i)$ is the solution of

$$\begin{aligned} \operatorname{div} \mathbf{u}_i &= q_i && \text{in } \Omega_i \times (0, T), \\ \mathbf{u}_i &= -\mathbf{K}_i(\nabla \mathbf{p}_i - \rho_i(\mathbf{s}_i) \mathbf{u}_G) && \text{in } \Omega_i \times (0, T), \\ \mathbf{p}_i &= 0 && \text{on } \partial\Omega_i \cap \partial\Omega \times (0, T), \\ -\frac{1}{\delta(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n}_i + \mathbf{p}_i &= \mathbf{p}_\gamma - [\beta(\mathbf{s})]_{\gamma,i} && \text{on } \gamma \times (0, T). \end{aligned}$$

Interface problem

$$\begin{aligned} \sum_{i=1}^2 \mathcal{L}_i^{RtN}(\mathbf{p}_\gamma, \mathbf{u}_G, q) - \mathbf{u}_\gamma &= Q_\gamma && \text{in } \gamma \times (0, T), \\ \mathbf{u}_\gamma &= -\mathbf{K}_\gamma k_\gamma(\mathbf{s}_\gamma) \nabla \mathbf{p}_\gamma && \text{in } \gamma \times (0, T), \\ \operatorname{div} \mathbf{u}_\gamma &= 0 && \text{in } \gamma \times (0, T), \\ \mathbf{p}_\gamma &= 0 && \text{on } \partial\gamma \times (0, T), \end{aligned}$$

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Interface problem

$$\begin{aligned} \sum_{i=1}^2 \mathcal{L}_i^{RtN}(\mathbf{p}_\gamma, \mathbf{u}_G, q) - \mathbf{u}_\gamma &\equiv Q_\gamma && \text{in } \gamma \times (0, T), \\ \mathbf{u}_\gamma &\equiv -\mathbf{K}_\gamma k_\gamma(\mathbf{s}_\gamma) \nabla \mathbf{p}_\gamma && \text{in } \gamma \times (0, T), \\ \operatorname{div} \mathbf{u}_\gamma &= 0 && \text{in } \gamma \times (0, T), \\ \mathbf{p}_\gamma &= 0 && \text{on } \partial\gamma \times (0, T), \end{aligned}$$

Robin-to-Neumann operator

$$\mathcal{L}_{wi}^{RtN}(\mathbf{s}_\gamma, \mathbf{s}_0, q_{wi}) = -\mathbf{u}_{wi} \cdot \mathbf{n}_i|_\gamma,$$

where $(\mathbf{s}_i, \mathbf{u}_{wi})$ is the solution

$$\begin{aligned} \phi_i \frac{\partial \mathbf{s}_i}{\partial t} + \operatorname{div} \mathbf{u}_{wi} &= q_{wi} && \text{in } \Omega_i \times (0, T), \\ \mathbf{u}_{wi} &= -\mathbf{K}_i \nabla \alpha_i(\mathbf{s}_i) + \mathbf{f}_i(\mathbf{s}_i) && \text{in } \Omega_i \times (0, T), \\ \mathbf{s}_i &= 0 && \text{in } \partial\Omega_i \cap \partial\Omega \times (0, T), \\ \frac{1}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_{wi} \cdot \mathbf{n}_i + \pi_i(\mathbf{s}_i) &= \pi_\gamma(\mathbf{s}_\gamma) + \frac{f_\gamma(\mathbf{s}_\gamma)}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n}_i && \text{on } \gamma \times (0, T). \end{aligned}$$

Interface problem

$$\begin{aligned} \Phi_\gamma \frac{\partial \mathbf{s}_\gamma}{\partial t} + \mathbf{u}_{w\gamma} &= Q_{w\gamma} + \sum_{j=1}^2 \mathcal{L}_w^{RtN}(\mathbf{s}_\gamma, q_w) && \text{in } \gamma \times (0, T), \\ \mathbf{u}_{w\gamma} &= -\mathbf{K}_\gamma \nabla \alpha_\gamma(\mathbf{s}_\gamma) + \mathbf{f}_\gamma(\mathbf{s}_\gamma) && \text{in } \gamma \times (0, T), \\ \mathbf{s}_\gamma &= 0 && \text{in } \partial\gamma \times (0, T), \end{aligned}$$

Robin-to-Neumann operator

$$\mathcal{L}_{wi}^{RtN}(\mathbf{s}_\gamma, \mathbf{s}_0, q_{wi}) = -\mathbf{u}_{wi} \cdot \mathbf{n}_i|_\gamma,$$

where $(\mathbf{s}_i, \mathbf{u}_{wi})$ is the solution

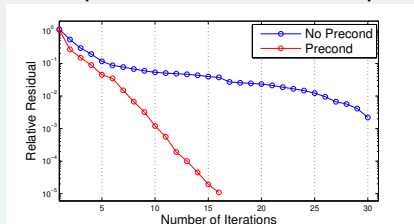
$$\begin{aligned} \phi_i \frac{\partial \mathbf{s}_i}{\partial t} + \operatorname{div} \mathbf{u}_{wi} &= q_{wi} && \text{in } \Omega_i \times (0, T), \\ \mathbf{u}_{wi} &= -\mathbf{K}_i \nabla \alpha_i(\mathbf{s}_i) + \mathbf{f}_i(\mathbf{s}_i) && \text{in } \Omega_i \times (0, T), \\ \mathbf{s}_i &= 0 && \text{in } \partial\Omega_i \cap \partial\Omega \times (0, T), \\ \frac{1}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_{wi} \cdot \mathbf{n}_i + \pi_i(\mathbf{s}_i) &= \pi_\gamma(\mathbf{s}_\gamma) + \frac{f_\gamma(\mathbf{s}_\gamma)}{\delta_w(\mathbf{s}_\gamma)} \mathbf{u}_i \cdot \mathbf{n}_i && \text{on } \gamma \times (0, T). \end{aligned}$$

Interface problem

$$\begin{aligned} \phi_\gamma \frac{\partial \mathbf{s}_\gamma}{\partial t} + \mathbf{u}_{w\gamma} &= Q_{w\gamma} + \sum_{j=1}^2 \mathcal{L}_w^{RtN}(\mathbf{s}_\gamma, q_w) && \text{in } \gamma \times (0, T), \\ \mathbf{u}_{w\gamma} &= -\mathbf{K}_\gamma \nabla \alpha_\gamma(\mathbf{s}_\gamma) + \mathbf{f}_\gamma(\mathbf{s}_\gamma) && \text{in } \gamma \times (0, T), \\ \mathbf{s}_\gamma &= 0 && \text{in } \partial\gamma \times (0, T), \end{aligned}$$

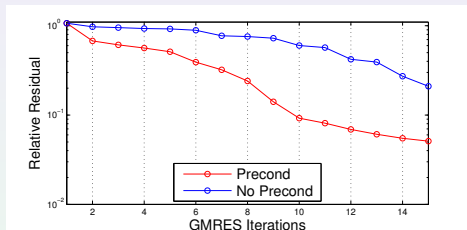
- Splitting diffusion and advection
- Mixed finite elements for 2nd order terms
- Cell-centered finite volumes and Godunov's method for advection
- Implicit Euler for diffusion, explicit Euler for advection
- Different time steps for diffusion and advection
- Domain decomposition solution

- Local preconditioner for the pressure: $(\text{div}_\tau)^{-1}$



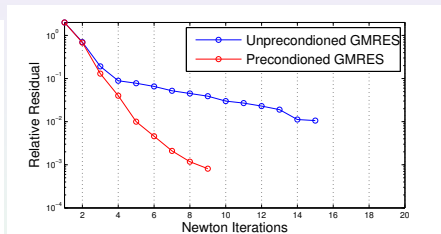
Efficiency of the CG
preconditioner

Interface GMRES-Newton convergence



GMRES- Relatif residual

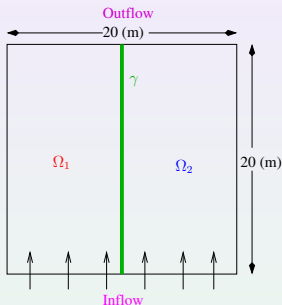
- Unpreconditioned GMRES-Newton
- Preconditioned GMRES-Newton



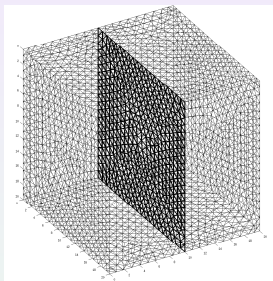
Inexact Newton- Relatif Residual

Effect of preconditioner on interface Newton convergence,
15 GMRES iterations

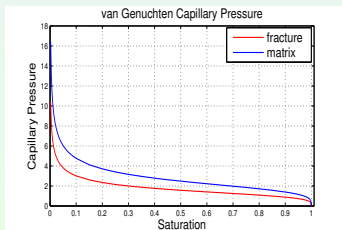
Numerical experiment with a high transmissivity fracture



Domain of calculation

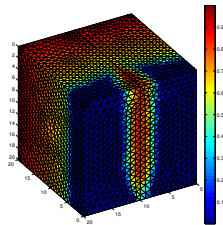
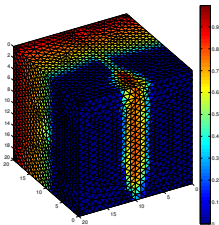


A conforming mesh with
72088 tetrahedra

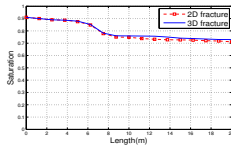
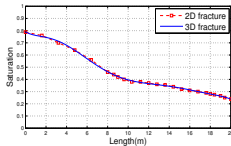


Capillary pressure curves

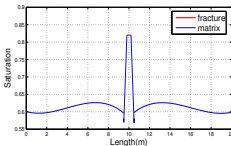
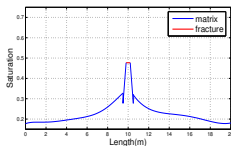
Numerical experiment with a high transmissivity fracture



Saturation at two different times

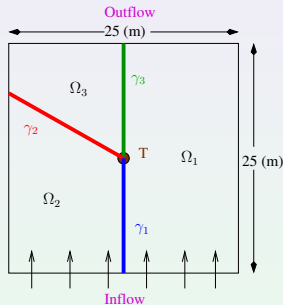


Along the fracture



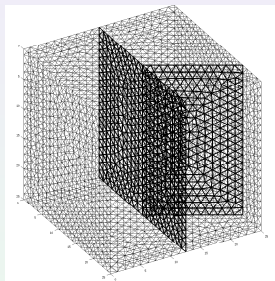
Along a line orthogonal to the fracture

Numerical experiment with intersecting fractures



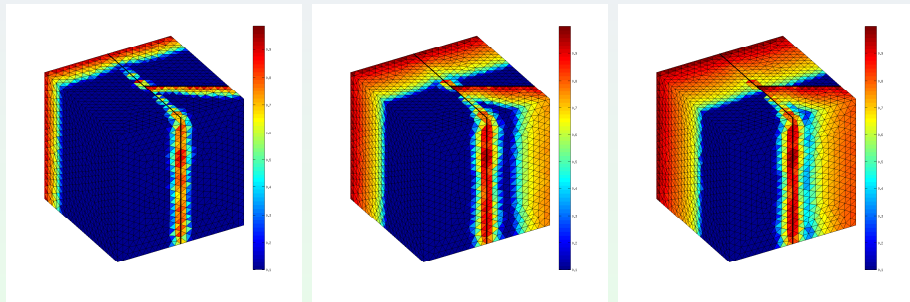
Domain of calculation

Domain Ω_3 has a larger permeability.

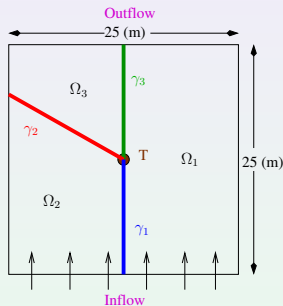


A conforming mesh with
65104 tetrahedra

Saturation at three different times

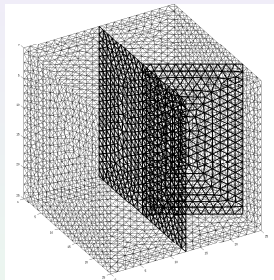


Numerical experiment with a fracture intersecting a barrier



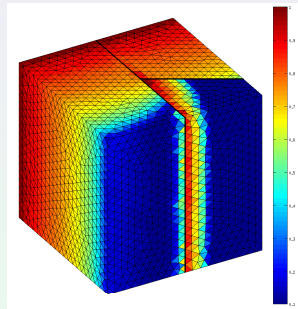
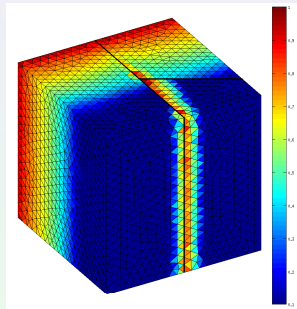
Domain of calculation

γ_2 is now a barrier.



A conforming mesh with
65104 tetrahedra

Saturation at two different times:



1 Mesh generation

3-D and surface meshing by [P. Laug](#), Inria, Gamma3.

- **BLSURF**: mesh generator for composite parametric surfaces
- **GHS3D**: 3-D mesh generator for tetrahedral elements

2 Modeling software

MATLAB Reservoir Simulation Toolbox (**MRST**) developed by SINTEF Applied Mathematics, [K.-A. Lie et al.](#)

3 Articles

- [J. Jaffré](#), [M. Mnejja](#), and [J. E. Roberts](#), A discrete fracture model for two-phase flow with matrix-fracture interaction (2011).
- [V. Reichenberger](#), [H. Jakobs](#), [P. Bastian](#), and [R. Helmig](#), A mixed-dimensional finite volume method for two-phase flow in fractured porous media (2006).
- [J. E. P Monteagudo](#) and [A. Firoozabadi](#), Control-volume model for simulation of water injection in fractured media: incorporating matrix heterogeneity and reservoir wettability effects (2006).

- Simulate actual 3-D examples
- Consider fractures which do not extend up to the boundary
- Use nonconforming meshes
- Use different time steps in fractures and in matrix rock