Dividend problem for a general Lévy insurance risk process

Zbigniew Palmowski

Joint work with F. Avram, I. Czarna, A. Kyprianou, M. Pistorius

Croatian Quants Day, Zagreb
The word *dividend* comes from the Latin word *dividendum* meaning *the thing which is to be divided* and has got sense of *portion of interest on a loan, stock, etc.*

Dividends are usually defined as the distribution of earnings in real assets among the shareholders of the firm (in proportion to their ownership).

Dividends are paid from the firm’s after-tax income. For the recipient, dividends are considered regular income and are therefore fully taxable.

There are two sides of dividends policies in the modern corporate firms. The first are managers of the firm (insiders), the second are shareholders (outsiders). The interest of management and shareholders may not coincide. This has important consequences for dividend policy. There is a suggestion that former typically prefer a low payout in order to pursue growth maximizing strategies or consume additional benefits, while letters generally wish for a high payout since this will force the management to incur the inspection of the capital markets for each new project undertaken.

We focus in this talk on the maximizing the cumulant dividend payments (we look at it only from the point of view of beneficiaries).
The reserve of an insurance company can be described by a Cramér-Lundberg process (Filip Lundberg (1903) and Harald Cramér (1933)):

\[ X_t = x + ct - \sum_{k=1}^{N_t} C_k \]

where

- \( C_k \) - sequence of independent, identically distributed random variables with distribution function \( F \)
- \( N_t \) - independent Poisson process with intensity \( \lambda \)
- \( c \) - the premium income per unit time
$X_t$ - spectrally negative Lévy process, which is not subordinator, that is $X_t$ - process with stationary and independent increments having only negative jumps

Process $X_t$ models the risk-process of an insurance company before dividends are deducted.

Lévy-Khitchine formula:

$$Ee^{i\theta X_t} = e^{-\Psi(\theta)t},$$

where

$$\Psi(\theta) = -ic\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,-1)} (1 - e^{i\theta x}) \Pi(dx) + \int_{(-1,0)} (1 - e^{i\theta x} + i\theta x) \Pi(dx)$$

(1)

where we assume that $\int_{(-\infty,0)} (1 \wedge x^2) \Pi(dx) < \infty$
De Finetti problem

We assume that \( X_t \to \infty \) a.s. that is \( c - \int_{(-\infty,-1]} |x| \Pi(dx) > 0 \).

"That is why de Finetti (1957) proposed another, economically motivated, criterion to the actuarial world. Instead of focusing on the safety aspect (measured by the probability of ruin) he proposed to measure the performance of an insurance portfolio by the maximal dividend payout that can be achieved over the lifetime of the portfolio. In particular, he proposed to look for the expected discounted sum of dividend payments until the time of ruin, where the discounting is with respect to some constant discount rate \( q > 0 \). Whereas de Finetti himself solved the problem to identify the optimal such dividend strategy in a very simple discrete random walk model, since then many research groups have tried to address this optimality question under more general and more realistic model assumptions and until nowadays this turns out to be a rich and challenging field of research that needs the combination of tools from analysis, probability and stochastic control."

(Albrecher and Thonhauser 2009)
$X_t$ - spectrally negative Lévy process

$\pi$ - a dividend strategy consisting of a non-decreasing, left-continuous $\mathbb{F}$-adapted process $\pi = \{L_t^\pi, t \geq 0\}$ with $L_0^\pi = 0$, where $L_t^\pi$ represents the cumulative dividends paid out by the company up till time $t$

The risk process controlled by a dividend policy $\pi$ is then given by

$$U_t^\pi = X_t - L_t^\pi$$

Ruin time:

$$\sigma^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}$$

Discounted value of paid dividend:

$$I_q^\pi = \int_0^{\sigma^\pi} e^{-qt} dL_t^\pi$$

$$v_*(x) = \sup_{\pi} \mathbb{E}_x [I_q^\pi]$$ - the value function
Barrier strategy $\pi^a$
Barrier strategy

\[ X_t \]

\[ L_t^\pi \]

\[ U_t^\pi \]

\[ a \]

\[ t \]

\[ \sigma^\pi \]
'If the barrier is too high, then we will wait too long for the risk process to hit the barrier and if we put the barrier too low then we derive the ruin too quickly.' We can then expect the existence of the 'optimal barrier'.
For the barrier strategy with the barrier \( a \):

\[
L_t = a \lor \overline{X}_t - a, \quad \text{where } \overline{X}_t = \sup_{s \leq t} X_s
\]

and

\[
\sigma_a = \inf\{t > 0 : Y_t > a\},
\]

where

\[
Y_t = (a \lor \overline{X}_t) - X_t
\]
Controlled ruin process once again
Discounted local time
Laplace exponent: \( \psi(\theta) \):

\[
\mathbb{E}[e^{\theta X_t}] = e^{t\psi(\theta)}
\]

\( \Phi(q) \) - greatest root of equation \( \psi(\theta) = q \)

First scaling function: \( W^{(q)} : [0, \infty) \rightarrow [0, \infty) \):

\[
\int_0^\infty e^{-\theta x} W^{(q)}(y) dy = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q)
\]

\( W^{(q)} \) is differentiable (not necessary continuously) and \( W(x) = W^{(0)}(x) \)

Second scaling function:

\[
Z^{(q)}(y) = 1 + q \int_0^y W^{(q)}(z) \, dz
\]
For the barrier strategy the upper index $\pi$ will be skipped. Under this strategy the value function

$$v_a(x) = \mathbb{E}_x I_q = \begin{cases} \frac{W(q)(x)}{W(q)'(a)} & 0 \leq x \leq a \\ x - a + \frac{W(q)(a)}{W(q)'(a)} & x > a \end{cases}$$

Hence optimal barrier is:

$$a^* = \inf\{c > 0 : W(q)'(c) \leq W(q)'(x) \text{ for all } x\}$$

where $\inf \emptyset = \infty$

If $W(q) \in C^2(0, \infty)$, then

$$W^{(q)''}(a^*) = 0$$
Optimality of the barrier strategy

Hamilton-Jacobi-Bellman’s (HJB) system of equations:

\[ \max \{ \Gamma f(x) - q f(x), 1 - f'(x) \} = 0, \quad x > 0, \]

where \( \Gamma \) denotes the extended generator of \( X \)

**Theorem 1.** (Avram, Palmowski and Pistorius AAP 17 2007) Assume that \( \sigma > 0 \) or that \( X \) has bounded variation or, otherwise, suppose that \( v_{a^*} \in C^2(0, \infty) \). In classical dividend setting \( a^* < \infty \) and the following hold true:

(i) \( \pi_{a^*} \) is the optimal strategy in the set \( \Pi_{\leq a^*} \) of all bounded by \( a \) strategies and \( v_{a^*} = \sup_{\pi \in \Pi_{\leq a^*}} v_{\pi^*} \).

(ii) If \( (\Gamma v_{a^*} - q v_{a^*})(x) \leq 0 \) for \( x > a^* \), the value function and optimal strategy are given by \( v_* = v_{a^*} \) and \( \pi_* = \pi_{a^*} \), respectively.
Brownian motion with drift

\[ X_t = \sigma B_t + \mu t \]

\[ W^{(q)}(x) = \frac{1}{\sigma^2 \delta} [e^{-(\omega+\delta)x} - e^{-(\omega+\delta)x}] \]

\[ Z^{(q)}(y) = y + \frac{2q}{\sigma^2} + \frac{q}{\sigma^2 \delta} \left[ \frac{1}{\omega + \delta} e^{-(\omega+\delta)y} - \frac{1}{\delta - \omega} e^{-(\omega+\delta)y} \right] \]

where

\[ \delta = \sigma^{-2} \sqrt{\mu^2 + 2q\sigma^2} \]

and

\[ \omega = \mu / \sigma^2 \]

Hence:

\[ a^* = \log \left| \frac{\delta + \omega}{\delta - \omega} \right|^{1/\delta} \]

Jeanblanc and Shiryaev 1995, Gerber and Shiu 2004
\[(\Gamma v_{a^*} - qv_{a^*})(x) \leq 0 \quad \text{for} \quad x > a^*\]

where

\[
\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + cf'(x) \\
+ \int_{-\infty}^{0} \left[ f(x + y) - f(x) - f'(x)y 1_{\{|y|<1\}} \right] \Pi(dy)
\]

for \( f \in C^2(0, \infty) \) and

\( \Pi \) is a Lévy measure of process \( X \)

\( \sigma^2 \) is a Gaussian coefficient
Cramér-Lundberg model with Gamma distributed claims:

\[ F(dx) = xe^{-x} dx, \]

the discount rate \( q = 0.1 \), the intensity \( \lambda = 10 \) of arrival Poisson process \( N_t \),
the premium rate \( c = 2(1 + 0.07)\lambda \).

Then

\[
v^*_v(x) = \begin{cases} 
  x + 2.119 & x \in [0, 1.803) \\
  0.0944e^{-1.4882x} - 9.431e^{-0.07953x} + 11.257e^{-0.03957x} & x \in [1.803, 10.22) \\
  x + 2.456 & x \geq 10.22 
\end{cases}
\]
Band strategy
Band strategy
Impulse control

$$\pi = \{(J_k, T_k), k \geq 0\}$$

where \(0 \leq T_1 \leq T_2 \leq \ldots\) is an increasing sequence of \(\mathcal{F}\)-stopping times representing the times at which a dividend payment is made and \(J_i\) be a sequence of positive \(\mathcal{F}_{T_i}\)-measurable random variables representing the sizes of the dividend payments

\(K\) - a fixed cost

The controlled risk process

$$\bar{U}_t^\pi = X_t - L_t^\pi - KN_t^\pi,$$

where

$$N_t^\pi = \#\{k : T_k \leq t\} \quad L_t^\pi = \sum_{k=1}^{N_t^\pi} J_k$$
The value function:

\[ v_\pi(x) = \mathbb{E}_x \left[ \int_0^\sigma \! \! e^{-qt} dL_t^\pi - K \int_0^\sigma \! \! e^{-qt} dN_t^\pi \right] \]

A band strategy with \( a^- < a^+ \):
1. Reducing the risk process \( U \) to level \( a^- \) if \( x > a^+ \)

2. Each time when \( U \) hits the upper level \( a^+ \) make a payment of size \( a_+ - a_- \)

\[ a^*(d) = \inf\{a \geq 0 : W^{(q)}(a + d) - W^{(q)}(a) \leq W^{(q)}(x + d) - W^{(q)}(x) \ \forall x \geq 0\}. \]

\[ d^* = \inf\{d \geq 0 : W^{(q)}(a^*(d) + d) - W^{(q)}(a^*(d)) - (d - K)W^{(q)'}(a^*(d) + d) = 0\} \]

Optimal levels:

\[ a_-^* = a^*(d^*) \quad a_+^* = a^*(d^*) + d^* \]
\[ v^\pi(x) = B^\pi(x) + H^\pi_w(x), \]

where

\[ B^\pi(x) = \mathbb{E}_x \left[ \int_0^{\sigma^\pi} e^{-qt} dL^\pi_t \right] \]

and \( H^\pi_w \) denotes the Gerber-Shiu penalty function

\[ H^\pi_w(x) = \mathbb{E}_x \left[ e^{-q\sigma^\pi} w(U_{\sigma^\pi}) \right] \]

associated to a penalty \( w : \mathbb{R} \rightarrow \mathbb{R}_- \cup \{0\} \) \((w(x) = 0 \text{ for } x \geq 0)\). Furthermore, we assume \( w \) is an increasing function on \( \mathbb{R}_- \), left-differentiable at 0. We want to find

\[ v^*_*(x) = \sup_{\pi} v^\pi(x) \]
**Definition 1.** According to the band strategy $\pi_{b,a}$ a lump-sum payment $U_{t}^{b,a} - a_i$ is made if $U_{t}^{b,a}$ is in $(a_i, b_i)$, while no dividends are paid while $U_{t}^{b,a}$ is in $[b_{i-1}, a_i)$ and the “overflow” of $U_{t}^{b,a}$ over $a_i$ are paid out as dividends.

**Theorem 2.** (Avram, Palmowski and Pistorius 2009) For $i \geq 1$, it holds that

\[
v_{b,a}(x) = \begin{cases} 
  v_{b,a}(b_{i-1}-) + \frac{W(q)(x-b_{i-1})}{W(q)'(a_{i-1}b_{i-1})} [1 - D_{i-1}(a_i - b_{i-1})] \\
  + D_{i-1}(x - b_{i-1}) & \text{if } x \in [b_{i-1}, a_i) \\
  v_{b,a}(a_i-) + x - a_i & \text{if } x \in [a_i, b_i),
\end{cases}
\]

where $v_{b,a}(x) = w(x)$ for $x < 0$ and $D_{i}(y) = H_{i}(y) + F_{i}(y)$ ($i \geq 1$) with

\[
H_{i}(y) = Z^{(q)}(y) - [\psi'(0) - qv_{b,a}(b_i-)]W^{(q)}(y),
\]

\[
F_{i}(y) = - \int_{0}^{y} W^{(q)'(y-z)}K_{i}(z)dz - W^{(q)}(0)K_{i}(y),
\]

\[
K_{i}(y) = \int_{y}^{\infty} (v_{b,a}(b_{i}+y-z) - [v_{b,a}(b_{i}+) + y-z]) \nu(dz),
\]

and $K_{0}(y) = \int_{y}^{\infty} (w(y-z) - w(0-) - w'(0)(y-z)) \nu(dz)$, $H_{0}(y) = w'(0-)Z^{(q)}(y) - [w'(0-\psi'(0) - w(0-)q]W^{(q)}(y)$, $\Pi(-\infty, -x) = \nu(x, \infty)$. 
Band strategies

Level $a_i$ is determined by the smooth fit condition of singular control:

$$0 = \lim_{x \downarrow a_i} v''_{a,b}(x) = \lim_{x \uparrow a_i} v''_{a,b}(x), \quad (2)$$

and similarly the level $b_i > 0$ is determined by the smooth fit condition

$$1 = \lim_{x \uparrow b_i} v'_{a,b}(x) = \lim_{x \downarrow b_i} v'_{a,b}(x), \quad (3)$$

if $X$ has unbounded variation (or, equivalently, if $X$ when starting at 0 immediately enters the positive half-axis almost surely), and determined by the continuous fit condition

$$v_{a,b}(b_i-) = \lim_{x \uparrow b_i} v_{a,b}(x) = \lim_{x \downarrow b_i} v_{a,b}(x), \quad (4)$$

if $X$ has bounded variation (or, equivalently, if it takes a strictly positive time for $X$ to enter the positive half-axis almost surely).
A numerical example for Cramér-Lundberg model
Taking $w(x) = 0.2x$ (penalty function), $\lambda = 10$ (intensity of claims arrivals), $\mu = 1$, $c = 21.4$ (premium rate), $q = 0.1$ (discounting rate) the optimal strategy is a 2-band strategy:

$$v_*(x) = \begin{cases} 0.2x & \text{for } x < 0 \\ x + 1.72277 & \text{for } 0 \leq x \leq 1.211 \\ 11.1287e^{0.039567x} - 9.6499e^{-0.079355x} + 0.149139e^{-1.48825x} & \text{for } 1.211 < x \leq 10.5051 \\ x + 2.16631 & \text{for } x > 10.5051 \end{cases}$$
Refraction

Process $U$ solves equation:

$$U_t = X_t - \delta \int_0^t 1_{\{U_s > a\}} \, ds$$

Discounted cumulant dividends (Kyprianou and Loeffen 2010 and Gerber and Shiu 2006):

$$E_x \int_0^{\sigma_a} e^{-qt} 1_{\{U_s > a\}} \, ds = - \int_0^{0 \wedge (x-a)} W^{(q)}(z) \, dz$$

$$+ \frac{W^{(q)}(x) + \delta 1_{\{x > a\}} \int_x^a W^{(q)}(x - y) W^{(q),'}(y) \, dy}{\phi(q) \int_0^\infty e^{-\phi(q)y} W^{(q),'}(y + a) \, dy}$$

where

$$\phi(q) = \sup \{ \psi(\theta) - \delta \theta = q \}$$

and $W^{(q)}$ and $Z^{(q)}$ are the scale function associated with process $X_t - \delta t$. 
The process $U^\gamma$ models the surplus process of an insurance company that pays out taxes according to a loss-carried-forward tax scheme, using a surplus-dependent rate $\gamma(\cdot)$. In other words, tax are collected when the company has recovered from its previous losses, i.e., is in a so-called profitable situation. Finally, note that when $\gamma(\cdot) = \gamma \in [0, 1]$, this model amounts to the situation studied in Albrecher et al. 2008 where the tax rate is constant, and when $\gamma = 1$, we retrieve the model where the company pays out as dividends any capital above its initial value $U^\gamma = x$ as in a risk model with a horizontal barrier strategy at level $u$ (see e.g. Renaud and Zhou 2007).

$$E_x \int_0^{\sigma^a} e^{-qt} \gamma(X_s) dX_s = \int_x^\infty \exp \left\{ - \int_x^t \frac{W(q,\gamma(s)) ds}{W(q)(\gamma(s))} \right\} \gamma(t) dt$$

where $\overline{\gamma}(y) = y - \int_x^y \gamma(s) ds$. 

Taxes
Consider now a particular two-dimensional risk model in which two companies split the amount they pay out of each claim in fixed proportions (for simplicity we assume that they are equal), and receive premiums at rates $c_1$ and $c_2$, respectively (so-called proportional reinsurance). That is,

$$X_t = (X_1(t), X_2(t)) = \left( u_1 + c_1 t - \beta_1 \sum_{i=1}^{N_t} C_i, u_2 + c_2 t - \beta_2 \sum_{i=1}^{N_t} C_i \right).$$

Without loss of generality we will assume that $\beta_1 = \beta_2 = 1$ and $c_1 > c_2$. 

Two-dimensional risk process
Two-dimensional risk process

\[ a = (-1, a) \]

\[ Y(t) \]
Controlled risk process:

\[ U_t = (U_1(t), U_2(t)) = X_t - L_t \]

where

\[ L(t) = \left( \delta_1 \int_0^t 1_{\{Y(t) \in B\}}, \delta_2 \int_0^t 1_{\{Y(t) \in B\}} \right) \]

describes the two-dimensional linear drift at rate \( \underline{\delta} = (\delta_1, \delta_2) > (0, 0) \) which is subtracted from the increments of the risk process whenever it enter the fixed set:

\[ B = \{(x, y) : x, y \geq 0 \text{ and } y \geq b - ax\}, \quad a, b > 0. \]

The case \( \underline{\delta} = c - a \) for \( c = (c_1, c_2) \) and \( a = (-1, a) \) corresponds to the reflecting the risk process at the line \( y = b - ax \). Let

\[ v_n(u_1, u_2) = v_n(u) = \mathbb{E}_u \left[ (1, 1) \cdot \int_0^\sigma e^{-qt} dL(t) \right]^n \]

where \( \sigma = \inf\{t \geq 0 : U_1(t)U_2(t) < 0\} \).
Two-dimensional risk process

Theorem 3. (Czarna and Palmowski 2009)

\[ c \cdot \frac{\partial v_n}{\partial u}(u) - (\lambda + nq)v_n(u) + \lambda \int_0^{\min(u_1,u_2)} v_n(u - (1,1)v) \, dF(v) = 0 \]

with the boundary conditions:

\[ n\delta_0 \ V_{n-1}(u) = \delta \cdot \frac{\partial v_n}{\partial u} \bigg|_{u \in \mathcal{B}}, \quad u \in \mathcal{B} \]

\[ \lim_{b \to \infty} v_n(u) = 0, \quad u \in \mathcal{B}^c \]

\[ v_n(0,b) = 0. \]
Assume that we have $\text{Exp}(\mu)$ claims with $\mu = 2$ and that $c_1 = 4$, $c_2 = 3$, $\lambda = 1$, $q = 0.1$.

Note that always there exists optimal choice of linear barrier (choice of its upper left end $(0, b)$ and it slope $a$). This choice depends on the initial reserves $(u_1, u_2)$. For $(u_1, u_2) = (1, 2)$ the optimal barrier is determined by $b = 14$ and $a = 0.1$ and for $(u_1, u_2) = (2, 3)$ the optimal barrier is determined by $b = 15$ and $a = 0.1$. This is contrast to the one-dimensional case where the choice of the barrier is given only via the premium rate and the distribution of the arriving claims.

<table>
<thead>
<tr>
<th>$a$</th>
<th>6</th>
<th>8</th>
<th>14</th>
<th>15</th>
<th>20</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>19.85</td>
<td>27.20</td>
<td>34.95</td>
<td>34.93</td>
<td>32.48</td>
<td>25.89</td>
</tr>
<tr>
<td>0.2</td>
<td>16.33</td>
<td>24.31</td>
<td>33.82</td>
<td>34.19</td>
<td>33.32</td>
<td>28.03</td>
</tr>
<tr>
<td>0.5</td>
<td>11.76</td>
<td>17.74</td>
<td>28.98</td>
<td>30.01</td>
<td>32.54</td>
<td>31.21</td>
</tr>
<tr>
<td>1</td>
<td>7.22</td>
<td>11.40</td>
<td>21.35</td>
<td>22.59</td>
<td>27.17</td>
<td>30.07</td>
</tr>
</tbody>
</table>

Expected value of dividend payments depending on $a$ and $b$ for fixed $(u_1, u_2) = (1, 2)$. 
THANK YOU

● for the Invitation!
● for Your Attention!


