Regular variations for functions, random variables and stationary processes

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Outline

Solvency II and the SCR

Definition (Solvency Capital Requirement)

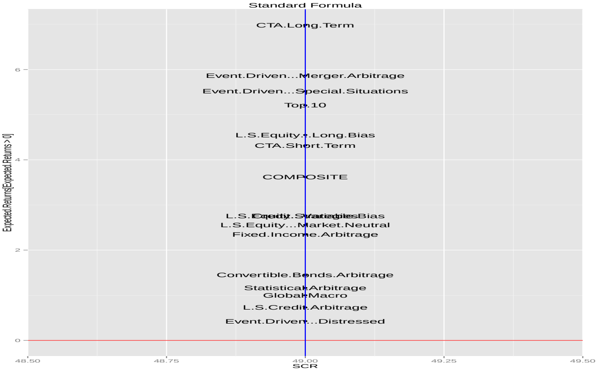
"The SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability of at least 99.5%." (Wikipedia)

Extrapolation of the magnitude of an event that occurs once per 200 years, i.e. that is not observed!

Motivation: insurance companies as Hedge Fund investors

Standard formula for "other equities"

The capital requirement is 48% of the investment whatever is the HF strategy.



Annualized returns Lyxor indices

SCR as a VaR of the log ratios

Definition

Log ratios $r_t = \log(P_{t+1}/P_t)$ where (P_t) are weakly prices of HF indices.

As $\log(x) \sim x - 1$ and $\log(xy) = \log(x) + \log(y)$, we have

$$\mathbb{P}\left(\frac{P_{T+1\text{ an}} - P_T}{P_T} \leqslant -SCR\right) = 0.005 \quad <\approx> \quad \mathbb{P}\left(\log\left(\frac{P_{T+1\text{ an}}}{P_T}\right) \leqslant -SCR\right) = 0.005$$

$$<\approx> \quad \mathbb{P}\left(\sum_{t=T}^{T+52} r_t \leqslant -SCR\right) = 0.005.$$

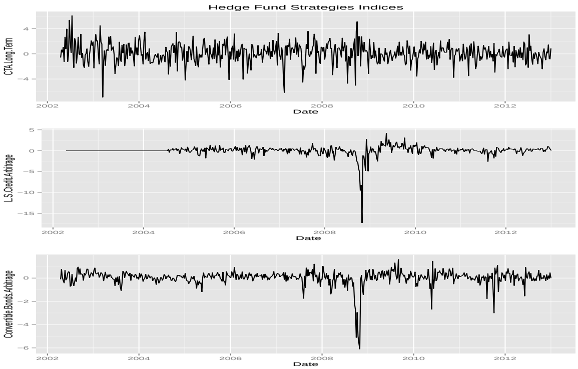
Definition

The Value at Risk (VaR, quantile) at the confidence level α of the r.v. X is

$$VaR_\alpha = \inf\{x\,;\,\mathbb{P}(X \leqslant x) \geqslant \alpha\}$$

$SCR \approx -VaR_{0.005}$ with $X = \sum_{t=T}^{T+52} r_t$

The need for Quantitative Risk Management



3 different strategies, same SCRs under standard formula

Solvency II: a challenge for the mathematician

Use a standard formula or find a more realistic calculation of the SCR (quantiles, VaR) using an internal model.

Outline

Modern portfolio theory

Definition (Standard gaussian distribution)

A r.v. X is standard gaussian $\mathcal{N}(0,1)$ if

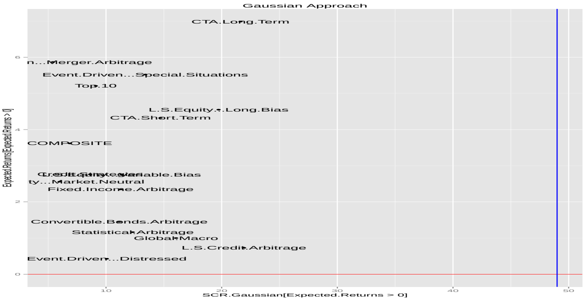
$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), \qquad x \in \mathbb{R}.$$

Properties

1. $Y = \mu + \sigma X$ then $Y \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = \mathbb{E}[Y]$ and $\sigma^2 = \mathrm{Var}(Y) = \mathbb{E}[(Y-\mu)^2]$,
2. If $X_i \sim \mathcal{N}(0,1)$ iid and $X = (X_1, \ldots, X_d)'$ then $Y = A + \Sigma X \sim \mathcal{N}_d(A, \Sigma^2)$ with $\Sigma^2 = \Sigma\Sigma'$.

$$\mathrm{qnorm(0.005)\ [1]\ \text{-}2.575829} \Longrightarrow SCR \approx -2.6 \times \mathrm{var}\left(\sum_{t=T}^{T+52} r_t\right).$$

Modern portfolio theory



Mean-VaR Portfolio Optimization: Markowitz (1952)

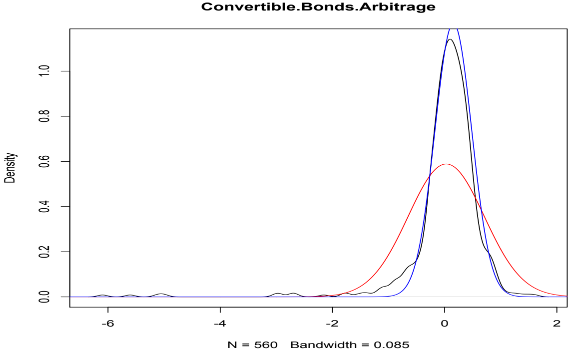
Remark

Work for other elliptical distribution where $SCR \approx -\beta \times \mathrm{var}(\sum_{t=T}^{T+52} r_t)$, β being the risk aversion coefficient, see McNeil et al. (2006).

Advantages

1. Easy calculations: VaR reduces to Var,
2. Any point of the convex hull corresponds to a feasible strategy: stability of the gaussian law.

Gaussian modeling works on average

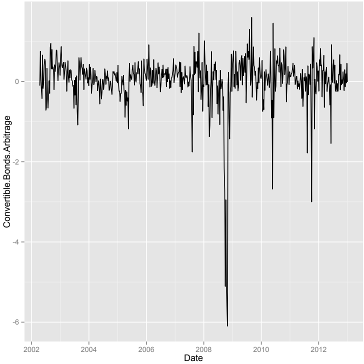


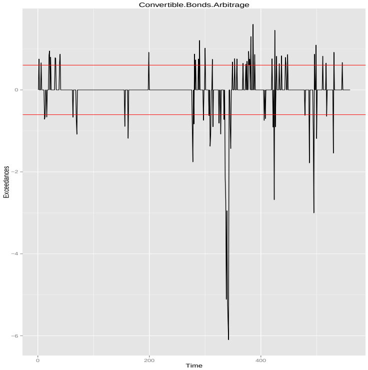
Dark: marginal density of the log-ratio of an index
Red: Gaussian model
Blue: Gaussian model excluding the 40 worst days

Outline

Exceedances

"Let the tails speak by themselves"! (Embrechts et al., 1997)





Definition (Exceedance)

For a threshold u, exceedance E is the excess of the r.v. X above u:

$$E_u = X - u \qquad \text{conditionally to} \qquad X \geqslant u.$$

Feasible calculations when $u \longrightarrow \infty$

The threshold u is large and varies. Let U be a monotone function on $(0, \infty)$ (for instance $U(u) = \mathbb{E}[|E_u|^k]$).

Lemma (Feller, 1971)

It exists a function Ψ such that for any $x > 0$

$$\frac{U(xu)}{U(u)} \to \Psi(x) \qquad u \to \infty$$

iff $\Psi(x) = x^\rho$, $\rho \in \mathbb{R}$,
iff $U(u) = u^\rho L(u)$ for some slowly varying function L satisfying

$$\frac{L(xu)}{L(u)} \to 1, \qquad u \to \infty.$$

Proof: for any $x, y > 0$ then $\frac{U(xyu)}{U(u)} = \frac{U(xyu)}{U(yu)}\frac{U(yu)}{U(u)}$ and then $\Psi(xy) = \Psi(x)\Psi(y)$.

Regularly varying functions

Definition (Karamata, 1930)

The function U on $(0, \infty)$ is regularly varying with index ρ iff $U(u) = u^\rho L(u)$.

Theorem (Karamata, 1930, Feller, 1971)

If $U \in RV_\rho$ and if $U_p(x) = \int_x^\infty u^p U(u) du$ exists then

$$\frac{x^{p+1} U(x)}{U_p(x)} \to \lambda = -(p + \gamma + 1) \geqslant 0.$$

Conversely, if $\lambda > 0$, then $U \in RV_\gamma$ with $\gamma = -\lambda - p - 1$ and $U_p \in RV_{-\lambda}$.

Remark

The regularly varying functions U and U_p have slowly varying functions that are equivalent up to a constant at $+\infty$.

Outline

Regularly varying nonnegative r.v.

Definition (Feller, 1971)

The r.v. $X \geqslant 0$ is regularly varying of index $\alpha > 0$, $X \in RV_\alpha$ iff

$$1 - \mathbb{P}(X \leqslant u) = \mathbb{P}(X > u) = u^{-\alpha} L(u), \qquad u > 0.$$

Definition (Pareto distribution)

The r.v. X follows a Pareto distribution (α, λ) if

$$\mathbb{P}(X > x) = \left(\frac{\lambda}{x}\right)^{\alpha}, \qquad x > \lambda.$$

Maxima of random regularly varying variables

Let X_i iid and $M_n = \max\{X_i;\ 1 \leqslant i \leqslant n\}$.

Theorem (Fisher, 1927)

There exists a non-decreasing sequence (a_n) such that M_n/a_n converges to a non degenerate limit iff X_is are regularly varying of index $\alpha > 0$. For some $c > 0$,

$$\mathbb{P}(M_n \leqslant xa_n) \to \exp(-cx^{-\alpha}), \qquad n \to \infty.$$

Proof: Denote F and G the X_i's and asymptotic distribution. Then

$$F^n(xa_n) \to G(x) \Leftrightarrow n(1 - F(xa_n)) \to -\log(G(x))$$

We conclude as $1 - F$ is non-increasing and for $a_n \leqslant t \leqslant a_{n+1}$

$$\frac{1 - F(xa_{n+1})}{1 - F(a_n)} \leqslant \frac{1 - F(xt)}{1 - F(t)} \leqslant \frac{1 - F(xa_n)}{1 - F(a_{n+1})}.$$

Exceedances

Theorem (Pickands-Balkema-de Haan, 1975-1974)

Let (X_t) iid regularly varying r.v. with index $\alpha > 0$. Denote F_u the distribution of the exceedances E_u over $u > 0$:

$$F_u(x) = \mathbb{P}(X - u \leqslant x \mid X > u), \qquad x \geqslant 0, \quad X \sim F.$$

Then

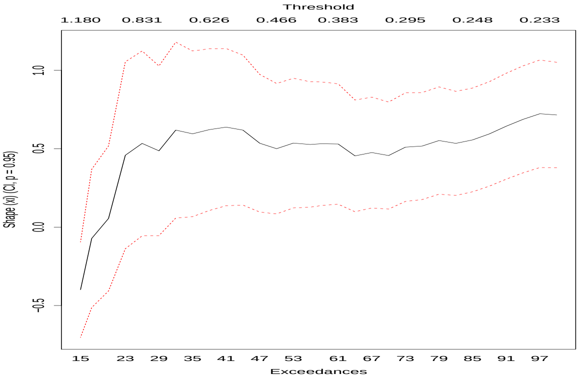
$$\lim_{u \to \infty} \sup_{0 < x < \infty} |F_u(x) - G_{\xi,\beta}(x)| = 0$$

where $G_{\xi,\beta}$ is the Generalized Pareto Distribution

$$G_{\xi,\beta}(x) = 1 - (1 - \xi(x-u)/\beta)^{-1/\xi}, \qquad \xi = \frac{1}{\alpha}, \beta > 0.$$

Peak Over Threshold approach, Embrechts et al. (1997)

POT approach: fit by MLE a GPD on the exceedances E_u for many $u \Longrightarrow \hat{\xi}_u$:



Sums of random regularly varying variables

Let X_i iid and $M_n = \max\{X_i;\, 1 \leqslant i \leqslant n\}$.

Theorem (Feller, 1971)

If X_1 and X_2 are independent regularly varying r.v. with index $\alpha > 0$ and with respective slowly varying function L_1 and L_2 then

$$\mathbb{P}(X_1 + X_2 > x) \sim x^{-\alpha}(L_1(x) + L_2(x))$$

Proof: We show that

$$\mathbb{P}(X_1 + X_2 > x) \sim \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x).$$

Regularly varying distribution

Definition (Feller, 1971)

A r.v. $X \in RV_\alpha$ iff it exist $p, q \geqslant 0$ with $p + q = 1$ and a slowly varying function L such that

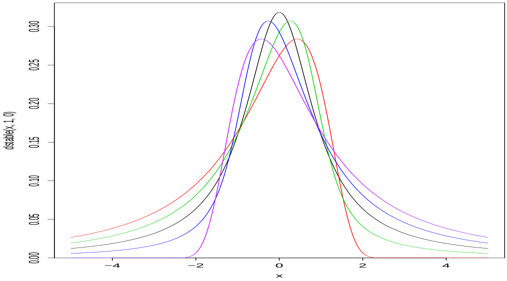
$$\mathbb{P}(X > x) \sim p\,\frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(X \leqslant -x) \sim q\,\frac{L(x)}{x^\alpha}\,, \quad x \to \infty.$$

Example: strictly stable r.v.

Definition

A r.v. Y is strictly α-stable distributed iff $\exists\ a > 0$, Y_1 and Y_2 independent, distributed as Y such that $Y_1 + Y_2 = aY$ in distribution.
Then Y is strictly α-stable with $0 < \alpha \leqslant 2$ and c.f. $\exp(-\sigma^\alpha |x|^\alpha \chi_\alpha(x,p,q))$,

$$\chi_\alpha(x,p,q) = \frac{\Gamma(2-\alpha)}{1-\alpha}(\cos(\pi\alpha/2) - i\operatorname{sgn}(x)(p-q)\sin(\pi\,\alpha/2)).$$



Domain of attraction

Let X_i iid and $S_n = \sum_{i=1}^n X_i$.

Theorem (Central Limit Theorem with heavy tails, Feller 1971)

Assume that $\mathbb{E}[X^2] = +\infty$ and X is centered if $\mathbb{E}[X] < \infty$.
The sequence $(a_n^{-1} S_n)$ has a non degenerate limit for some (a_n) iff $X \in RV_\alpha$.
The limit is distributed according to a (strictly) α-stable law (for $\alpha \neq 1$).

Precise large deviations

Theorem (A.V. Nagaev, 1969)

(X_i) iid random variables with $\alpha > 0$ regularly varying (centered if $\alpha > 1$) distribution then $S_n = \sum_{i=1}^n X_i$ satisfies the precise large deviations relation

$$\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n > x)}{n\,\mathbb{P}(|X| > x)} - p \right| = 0 \text{ and } \lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n \leqslant -x)}{n\,\mathbb{P}(|X| > x)} - q \right| = 0$$

with $b_n = n^{\delta + 1/(\alpha \wedge 2)}$ for any $\delta > 0$.

Remark

$$q\, n\,\mathbb{P}(|X| > x) \sim n\,\mathbb{P}(X \leqslant x)$$

If (r_t) iid $\Longrightarrow \mathbb{P}\Big(\sum_{j=T}^{T+52} r_t \leqslant -SCR\Big) = 0.005 <\approx> \mathbb{P}\Big(r_t \leqslant -SCR\Big) = 0.0001.$

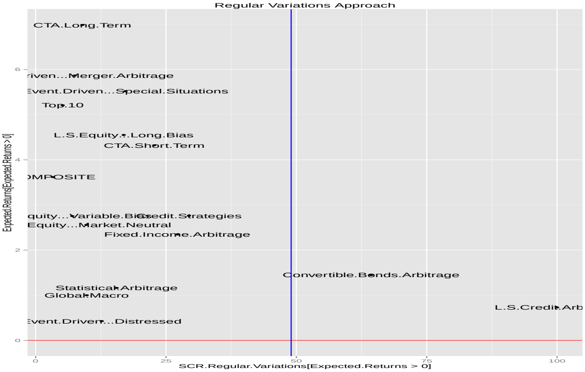
$$SCR \approx -VaR_{0.005} \text{ with } X = r_t.$$

SCR calculation in the iid case

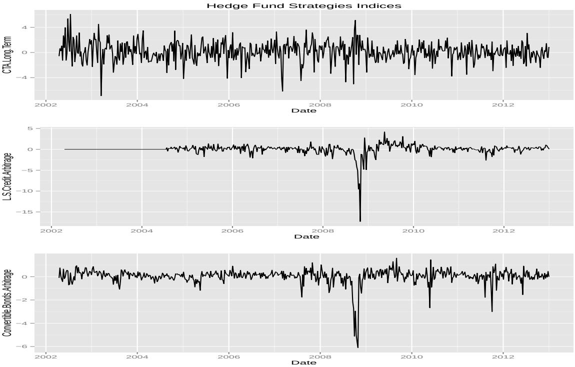
SCR calculation with POT approach, Smith (197)

$$\widehat{SCR} = \max_{15 \leqslant m \leqslant 40} u_m + \frac{\hat{\beta}}{\hat{\xi}} \left[\left(\frac{n * 0.0001}{m} \right)^{-\hat{\xi}} - 1 \right]$$

where m is the number of exceedances.



SCR extrapolation when extremes cluster



What is happening for dependent sequences for whom extremes cluster?

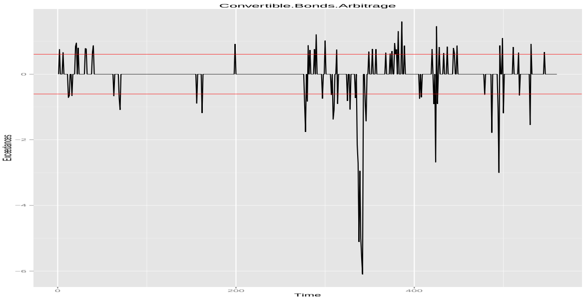
Outline

Regular variations revisited

Remark

A r.v. X is regularly varying with index $\alpha > 0$ iff there are r.v. $\Theta \in \{-1, +1\}$ and $Y \sim \mathit{Pareto}(\alpha, 1)$ independent satisfying

$$\mathbb{P}(|X|^{-1}X \in \cdot, |X_0| > xu \mid |X| > u) \to \mathbb{P}(\Theta \in \cdot), \qquad u \to \infty,$$
$$\mathbb{P}(|X| \leqslant xu \mid |X| > u) \to \mathbb{P}(Y \leqslant x), \qquad u \to \infty.$$



Breiman's Lemma

Lemma (Breiman (1965))

If $X_1 \geqslant 0$ *and* $X_2 \geqslant 0$ *are independent,* $X_1 \in RV_\alpha$ *and* $\mathbb{E}[X_2^{\alpha+\varepsilon}] < \infty$*, then*

$$\mathbb{P}(X_1 X_2 > u) \sim \mathbb{E}[X_2^\alpha]\mathbb{P}(X_1 > u).$$

Proof: (Jessen and Mikosch, 2006)

$$\frac{\mathbb{P}(X_1 X_2 > u)}{\mathbb{P}(X_1 > u)} \approx \int \frac{\mathbb{P}(X_1 > u/y)}{\mathbb{P}(X_1 > u)} d\mathbb{P}(X_2 > y) \approx \int y^\alpha d\mathbb{P}(X_2 > y).$$

Remark

Let $X \in RV_\alpha$ associated with Θ. We also have

$$\mathbb{P}(X > u) \approx \mathbb{P}(\Theta_+ |X| > u) \approx \mathbb{E}[\Theta_+^\alpha]\mathbb{P}(|X| > u)$$
$$= \mathbb{P}(\Theta = 1)\mathbb{P}(|X| > u) = p\mathbb{P}(|X| > u).$$

Regularly varying multivariate distributions

Regular variations, Basrak et al. (2002)

A random vector $X = (X_1, \dots, X_d)$ is regularly varying if a non-null Radon measure μ_d is such that

$$n\,\mathbb{P}(a_n^{-1}(X_1, \dots, X_d) \in \cdot) \xrightarrow{v} \mu_d(\cdot)\,,$$

where (a_n) satisfies $n\,\mathbb{P}(|X| > a_n) \to 1$ and $\mu_d(tA) = t^{-\alpha}\mu_d(A)$, $t > 0$.

Definition (Resnick, 1987)

It is equivalent to the existence of the spectral tail vector $\Theta = (\Theta_1, \dots, \Theta_d)$ satisfying

$$\mathbb{P}(|X|^{-1}(X_1, \dots, X_d) \in \cdot \mid |X| > u) \to \mathbb{P}((\Theta_1, \dots, \Theta_d) \in \cdot)\,, \quad u \to \infty\,,$$
$$\mathbb{P}(|X| \leqslant xu \mid |X| > u) \to \mathbb{P}(Y \leqslant x)\,, \qquad u \to \infty\,.$$

Where is the trick?

Choose $d = 2$, $X = (X_1, X_2)$ with $X_1 \in RV_\alpha$ independent of $X_2 \in RV_\alpha$ and $|X| = X_1 + X_2$. If $L_2(u) = o(L_1(u))$

$$\mathbb{P}(X_1 > u) \sim \mathbb{P}(|X| > u) \sim u^{-\alpha}(L_1(u) + L_2(u)),$$

$\mathbb{P}(X_1/|X| = 1 \mid |X| > u) \to 1$, $\mathbb{P}(X_2/|X| = 1 \mid |X| > u) \to 1$ and $\Theta = (1, 0)$.

To avoid degenerate Θ, we assume that X_i are identically distributed.

Remark

When $L_1(u) \sim L_2(u)$ then $\Theta = (1, 0)$ or $= (0, 1)$ w.p. $1/2$.

If $X_i \sim F_i$ then we standardize via the transform $1/(1 - F_i(X_i)) \sim Pareto(1, 1)$.

Identically distributed margins

Definition (Basrak and Segers, 2009)

Assume that X_i are identically regularly varying distributed with index $\alpha > 0$ iff it exists the spectral tail vector $\Theta = (\Theta_1, \ldots, \Theta_d)$ satisfying

$$\mathbb{P}(|X_1|^{-1}(X_1, \ldots, X_d) \in \cdot \mid |X_1| > u) \to \mathbb{P}((\Theta_1, \ldots, \Theta_d) \in \cdot)\,, \quad u \to \infty\,.$$

Example

If the X_i are independent then $\Theta = (1, 0, \ldots, 0)$.

Outline

Regularly varying processes

Definition (Basrak and Segers, 2009)

A stationary sequence (X_t) is regularly varying of order $\alpha > 0$ iff $X_0 \in RV_\alpha$ and it exists the spectral tail process (Θ_t) defined for any $k \geqslant 0$, any $u > 0$ by the relation

$$\mathbb{P}(|X_0|^{-1}(X_0, \dots, X_k) \in \cdot \mid |X_0| > u) \to \mathbb{P}((\Theta_0, \dots, \Theta_k) \in \cdot), \qquad u \to \infty.$$

Example

If the (X_t) are iiid then $\Theta_t = 0$, $|t| \neq 0$.

The AR(1) model

Definition (AR(1) model)

The AR(1) model is the solution of $X_t = \phi X_{t-1} + Z_t$, $|\phi| < 1$ with (Z_t) is an iid regularly varying sequence if order $\alpha > 0$.

Proposition

We have $X_0 \in RV_\alpha$ and $\Theta_t = \phi^t$, $t > 0$.

The GARCH(1,1) model

Definition (Bollerslev, 1986)

The GARCH(1,1) model (X_t) is the solution of $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$ with (Z_t) is an iid mean zero and unit variance sequence of random variables and (σ_t^2) satisfies the stochastic recurrence equation

$$\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2, \qquad t \in \mathbb{Z}\,.$$

Proposition

If $X_0 \in RV_\alpha$ then we have

$$\mathbb{P}(|X_0|^{-1}(X_0,\ldots,X_t) \in \cdot \mid |X_0| > x) \to \frac{1}{\mathbb{E}|Z_0|^\alpha}\mathbb{E}\Big[|Z_0|^\alpha 1_{(Z_0, Z_1\Pi_i^{0.5},\ldots,Z_t\Pi_t^{0.5}) \in |Z_0|\,\cdot}\Big]\,,$$

where $\Pi_t = A_1 \cdots A_t$ with $A_t = \alpha_1 Z_{t-1}^2 + \beta_1$.

Alternative measures of dependance

Definition (The extremal index)

The stationary sequence (X_t) has extremal index θ if for any $x > 0$ we have

$$\mathbb{P}(\max\{|X_1|, |X_2|, \dots, |X_n|\} \geqslant a_n x) \sim \mathbb{P}(\max\{|X_1'|, |X_2'|, \dots, |X_n'|\} \geqslant a_n x)^\theta$$

where X_t' are iid copies.

Proposition (Basrak and Segers, 2009)

If $(X_t) \in RV_\alpha$ then

$$\theta = \mathbb{E}[\max\{|\Theta_0|, |\Theta_1|, |\Theta_2|, \dots\}^\alpha - \max\{|\Theta_1|, |\Theta_2|, |\Theta_3|, \dots\}^\alpha].$$

Alternative measures of dependance

Definition (Upper tail dependence coefficient)

For any vector (X_0, X_h) the upper tail dependence coefficient $\rho(h)$ satisfies

$$\rho(h) = \lim_{u \to \infty} \mathbb{P}(X_h > u \mid X_0 > u).$$

Proposition (Davis et al., 2013)

If $(X_t) \in RV_\alpha$ then

$$\rho(h) = \frac{\mathbb{E}[\min\{\Theta_0, \Theta_t\}_+^\alpha]}{\mathbb{E}[(\Theta_0)_+^\alpha]}.$$

Proof: Applying Breiman's Lemma, we have

$$\mathbb{P}(\min\{X_h, X_0\} > u) \approx \mathbb{P}(\min\{\Theta_h, \Theta_0\}|X_0| > u) \approx \mathbb{E}[\min\{\Theta_h, \Theta_0\}_+^\alpha]\mathbb{P}(|X_0| > u)$$

Examples in the *m*-dependent case

Assume $(X_t,\ t \leqslant 0)$ is independent of $\sigma(X_t,\ t \geqslant m+1)$ then $\Theta_t = 0$ for $|t| \geqslant m$.

Definition (Conditional spectral tail process)

Define for *m*-dependent RV(α) processes $(\Theta'_0, \dots, \Theta'_m) = (\Theta_0, \dots, \Theta_m)$ conditionally to $\Theta_{-j} = 0, 0 < j \leqslant m$.

Example

1. $X_t = \max(Z_{t-1}, Z_t)$ then $(\Theta'_0, \Theta'_1) = (1, 1)$,
2. $X_t = Z_t + \frac{1}{2} Z_{t-1}$ then $(\Theta'_0, \Theta'_1) = (1, \frac{1}{2})$.

Large deviations in the m-dependent case

Theorem (Mikosch and W., 2012)

Assume (X_t) is $\alpha > 0$ regularly varying (centered if $\alpha > 1$) distribution then

$$\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n > x)}{n\,\mathbb{P}(|X| > x)} - b_+ \right| = 0 \text{ and } \lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(S_n \leqslant -x)}{n\,\mathbb{P}(|X| > x)} - b_- \right| = 0\,,$$

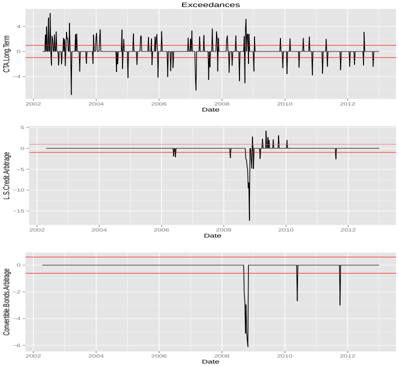
with $b_n = n^{\delta+1/(\alpha\wedge 2)}$ for any $\delta > 0$ and cluster indices

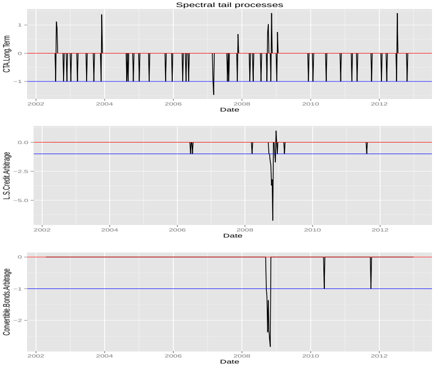
$$b_\pm = \mathbb{E}\Big[\Big(\sum_{t=0}^{m} \Theta_t\Big)_\pm^\alpha - \Big(\sum_{t=1}^{m} \Theta_t\Big)_\pm^\alpha\Big] = \mathbb{P}(\Theta_{-j} = 0, 1 < j \leqslant m)\,\mathbb{E}\Big[\Big(\sum_{t=0}^{m} \Theta'_t\Big)_\pm^\alpha\Big].$$

Application to risk management

Definition (Empirical conditional spectral tail process)

Define $(\hat{\Theta}'_j, \ldots, \hat{\Theta}'_{j+k}) = (r_j/|r_j|, \ldots, r_{j+k}/|r_j|)$ if $|r_t| > u$, $t \in I = \{j, \ldots, j+k\}$, $|r_{j-1}| < \varepsilon u$ and $|r_{j+k+1}| < u$ are clusters of exceedances starting with a jump.





Approximation of the cluster index

$$\mathbb{P}\Big(\sum_{j=T}^{T+52} r_t \leqslant -SCR\Big) = 0.005 <\approx> \mathbb{P}\Big(|r_t| > SCR\Big) = 0.0001/b_-,$$

with $b_- = \gamma\,\mathbb{E}[(\sum_{t=0}^{m}\Theta'_t)_-^\alpha]$ where $\gamma = \mathbb{P}(\Theta_{-j} = 0, 1 < j \leqslant m)$ can be interpreted as the inverse of the average length of the clusters of exceedances starting with a jump.

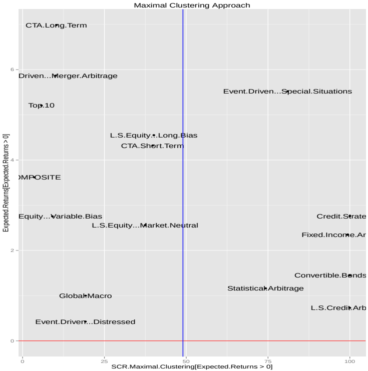
Definition (Empirical cluster index)

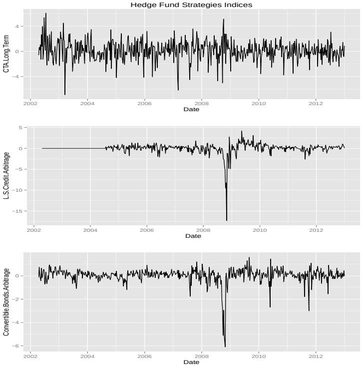
$$\hat{b}_- = \frac{1}{\ell}\sum_{i=1}^{\ell}\frac{1}{k_i}\Big(\sum_{t=0}^{k_i}\hat{\Theta}'_{j_i+t}\Big)_-^{\hat{\alpha}}.$$

Calculation of the SCR when extremes cluster

Definition

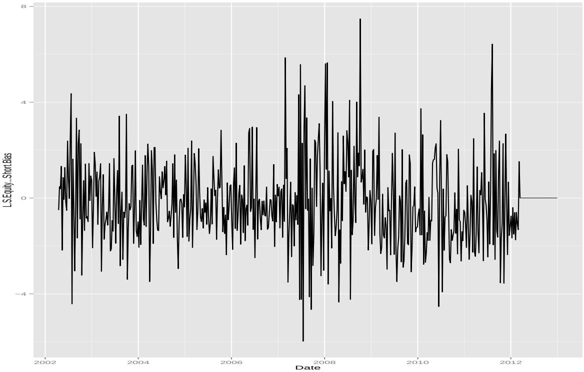
$$\widehat{SCR} = \max_{15 \leqslant m \leqslant 40} \max_{\ell} u_m + \frac{\hat{\gamma}}{\hat{\beta}} \left[\left(\frac{n * 0.0001}{m * \hat{b}_{-}} \right)^{-\hat{\xi}} - 1 \right].$$





Conclusion

- An alternative approach of risk management based on exceedances and not on variances,
- Regular variations ensure stability and feasible computations,
- The statistical inference remains to be done.



Conclusion

Thank you for your attention!