The Modeling and Detection of Financial Bubbles

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Famous bubbles of history

- Tulipomania; Amsterdam, 17th century (circa 1630s)
- John Law and the Banque Royale (Paris, 1716 – 1720)
- The South Sea Company (London, 1711 – 1720)
- In the United States:
  - After the War of 1812, real estate speculation, fostered by the Second Bank of the United States, created in 1816;
  - Runaway speculation tied to advances in infrastructure through the building of canals and turnpikes, ending in the crash of 1837;
  - Speculation due to the creation of the railroads led to the panic of 1873.
  - The Wall Street panic of 1907; (Banking crisis due to speculation; stock market fell 50%, led to development of Federal Reserve in 1913 [Glass-Owen bill]); role of J.P. Morgan.
  - Florida land speculation in the first half of the 1920s, followed by stock market speculation in the second half of the 1920s created in part by margin loans, led to the Great Crash of 1929, leading to many bank failures and the worldwide depression of the 1930s.

US Stock Prices 1929 (Donaldson & Kamstra [1996])

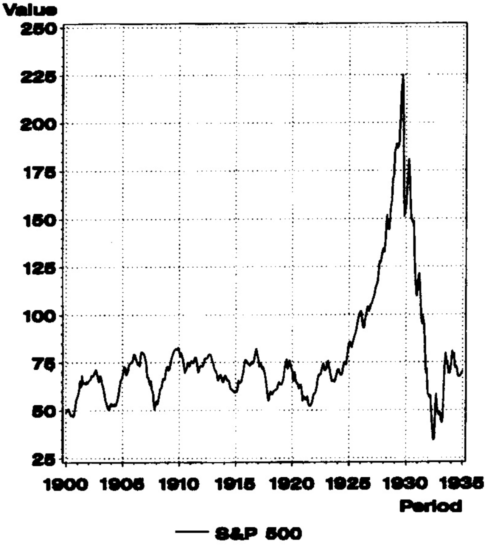


Figure 1
S&P 500 stock price index
This figure plots the S&P 500 stock price index. See Appendix 1 for data sources.

More recent bubbles

- Minor crashes in the 1960s and 1980s
- Junk bond financing led to the major crash of 1987
- Japanese housing bubble circa 1970 to 1989
- The "dot com" crash, from March 11th, 2000 to October 9th, 2002. Led by speculation due to the promise of the internet; The Nasdaq Composite lost 78% of its value as it fell from 5046.86 to 1114.11.
- Current US housing bubble and subprime mortgages

NASDAQ Index 1998-2000 (Brunnermeir & Nagel)

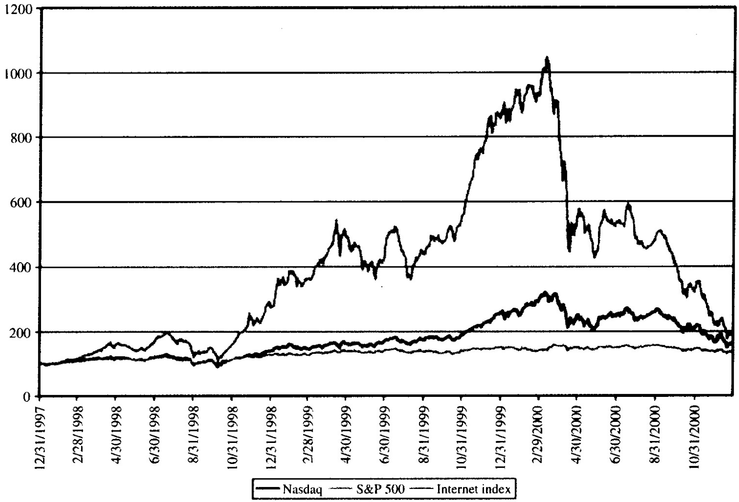
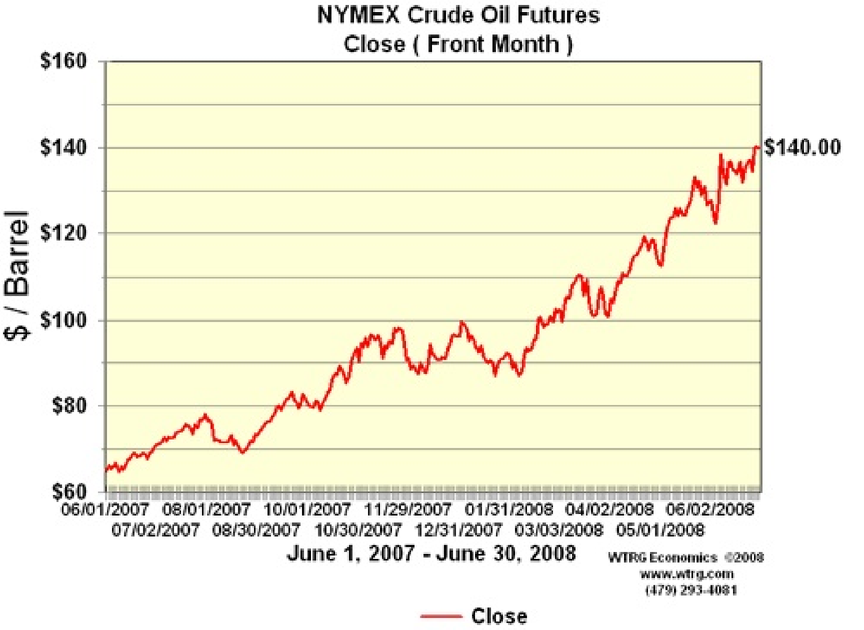


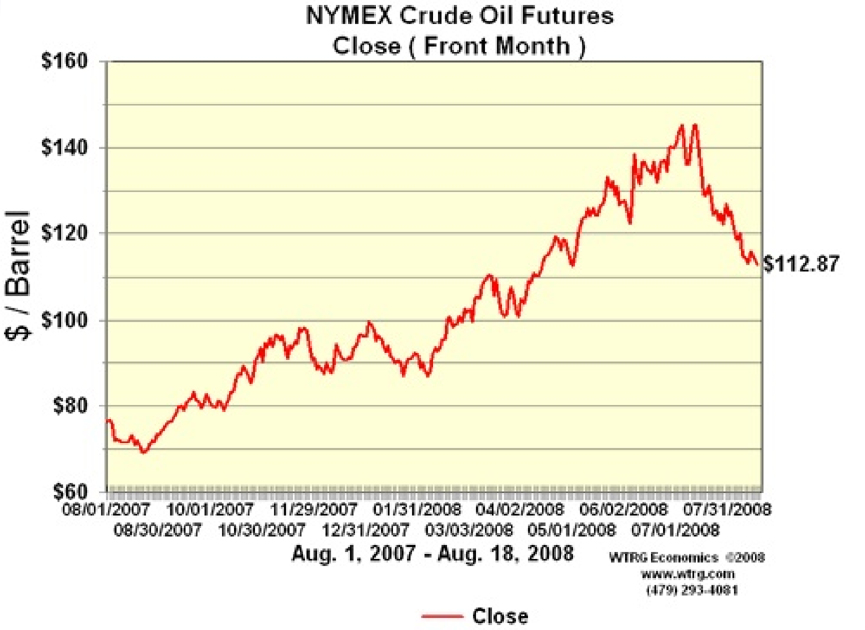
Figure 1. Returns on equally weighted Internet index, S&P 500 and Nasdaq composite. Comparison of index levels of the equally weighted Internet index, the S&P 500 index, and the Nasdaq composite index for the period 1/1/1998–12/31/2000. All three indexes are scaled to be 100 on 12/31/1997.

Current US Housing Price Trend (Center for Responsible Lending)

Oil Futures (WTRG Economics)



Oil Futures (WTRG Economics)



The Basic Framework

- We assume the **No Free Lunch With Vanishing Risk** framework of F.Delbaen and W. Schachermayer. In words, there are no arbitrage opportunities and there are no trading strategies which approximate arbitrarily closely arbitrage opportunities.
- A risky asset with maturity τ and a money market account with constant value 1 are traded.
- $D = (D_t)_{0 \leq t \leq \tau} \geq 0$ is the **cumulative dividend process**.
- $X_\tau \geq 0$ is the payoff at time τ;
- The **market price** is $S = (S_t)_{0 \leq t \leq \tau} \geq 0$.
- The **wealth process** is

$$W_t = S_t + \int_0^{t \wedge \tau} dD_u + X_\tau 1_{\{t \geq \tau\}}.$$

- A **trading strategy** is a pair of adapted processes (π, η) representing the number of units of the risky asset and money market account held at time t.
- The **wealth process** V of the trading strategy (π, η) is given by

$$V_t^{\pi,\eta} = \pi_t S_t + \eta_t. \tag{1}$$

- A **self-financing trading strategy** is a trading strategy (π, η) with π predictable and η optional such that $V_0^{\pi} = 0$ and

$$V_t^{\pi,\eta} = \int_0^t \pi_u dW_u = (\pi \cdot W)_t \tag{2}$$

- We say that the trading strategy π is $a-$**admissible** if it is self-financing and $V_t^\pi \geq -a$ for all $t \geq 0$ almost surely.
- We say a trading strategy is **admissible** if it is self-financing and there exists an $a \in \mathbb{R}_+$ such that $V_t^\pi \geq -a$ for all t almost surely.
  - Admissibility needed to exclude doubling strategies.
  - **Admissibility is the reason for the existence of bubbles**.
  - Admissibility is an implicit restriction on shorting the risky asset.

- Theorem (D & S, 1998; First Fundamental Theorem)

  A process S has No Free Lunch with Vanishing Risk (NFLVR) if and only if there exists an equivalent probability measure Q such that S is a sigma martingale under Q.

  Definition

  A market is **complete** if every bounded contingent claim can be perfectly hedged.

  Theorem (Second Fundamental Theorem)

  A market is complete if and only if there is only one and only one risk neutral measure (sigma martingale measure)

  - Since $W \geq 0$ always, we can replace sigma martingale with local martingale.

A market is said to said to satisfy **No Dominance** if, given any two assets with their associated payoff structures (dividends + terminal payoff) and market prices, neither asset's payoff structure is always (weakly) greater than the other's, **and also has a strictly lower market price**

Lemma

No Dominance implies NFLVR; however the converse is false.

From now on, we assume No Dominance holds.

The Fundamental Price

In complete markets with a finite horizon T, we use the risk neutral measure Q, and for $t < T$ the **fundamental price** of the risky asset is defined to be:

$$S_t^\star = E_Q\{\int_t^T dD_u + X_T|\mathcal{F}_t\}$$

Definition (Bubble)

A bubble in a static market for an asset with price process S is defined to be:

$$\beta = S - S^\star$$

Static Markets

Theorem (Three types of bubbles)

1. *β is a local martingale (which could be a uniformly martingale) if $P(\tau = \infty) > 0$;*
2. *β is a local martingale but not a uniformly integrable martingale, if it is unbounded, but with $P(\tau < \infty) = 1$;*
3. *β is a strict Q local martingale, if τ is a bounded stopping time.*

- Type 1 is akin to fiat money
- Type 2 is tested in the empirical literature
- Type 3 is essentially "new." Type 3 are the most interesting!

Theorem (Bubble Decomposition)

The risky asset price admits a unique decomposition

$$S = S^{\star} + (\beta^1 + \beta^2 + \beta^3)$$

where

1. β^1 *is a càdlàg nonnegative uniformly integrable martingale with* $\lim_{t\to\infty} \beta^1_t = X_\infty$ *a.s.*
2. β^2 *is a càdlàg nonnegative NON uniformly integrable martingale with* $\lim_{t\to\infty} \beta^2_t = 0$ *a.s.*
3. β^3 *is a càdlàg non-negative supermartingale (and strict local martingale) such that* $\lim_{t\to\infty} E\{\beta^3_t\} = 0$ *and* $\lim_{t\to\infty} \beta^3_t = 0$ *a.s.*

Black-Scholes Model (Static Market, Finite Horizon)

- Fix T and let S be the price process of a stock without dividends following

$$S_t = \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma B_t\}, \quad 0 \leq t \leq T,$$

  where $\mu, \sigma \in \mathbb{R}_+$, and B is a standard Brownian motion
- The finite horion $\Rightarrow$ only a type 3 bubble can exist
- Since S is a Q martingale, no type 3 bubbles are possible
- This holds more generally for complete markets, under NFLVR, and without needing No Dominance

Black-Scholes Model (Static Market, Infinite Horizon)

- If we extend S to times in $[0, \infty)$ then the situation changes
- The fundamental value of the stock is $S_t^\star = 0$. (There are no dividends.)
- The definition of the bubble β is

$$\beta = S_t - S_t^\star = S_t,$$

  and the entire stock is a bubble!
- Under No Dominance, if the asset does not have a bubble, S must be the zero process, since there are no dividends and the terminal payout is zero
- Therefore the model is a bubble, and only the finite horizon Black-Scholes model is reasonable

Incomplete markets

- There are an infinite number of risk neutral measures
- We need to choose one to define the concept of **fundamental value**.
- We assume that enough derivative securities trade so that a risk neutral measure is uniquely determined **by the market**. To do this we could use the ideas of Jacod and Protter.
- We allow **regime/structural shifts** in the economy to generate changes in the market selected risk neutral measure across time (this might be compared to Ising models for phase change)
  - If there are no regime shifts, we say the market is **static**
  - If there is at least one regime shift possible, we say that the market is **dynamic**.

Regime Change

- This idea of regime change is new; previously a risk neutral measure in an incomplete market was chosen in some manner (often *ad hoc*) and fixed for all $t \geq 0$
- The new approach is that the market has chosen one of the infinitely many risk neutral measures with which to price derivatives; in theory, one can determine this choice if (for example) there are enough put options traded, and they are priced consistently with each other and with the price process (Jacod and Protter, 2010)
- Then, it seems possible that over time the risk neutral measure chosen by the market can change, from one to another member of the infinite collection
- This idea is roughly analogous to the Ising model (and related models) of phase changes in physics

The Fundamental Price

- In complete markets with a finite horizon T, we use the risk neutral measure Q, and for $t < T$ the **fundamental price** of the risky asset is defined to be:

$$S_t^\star = E_Q\{\int_t^T dD_u + X_T | \mathcal{F}_t\}$$

- In incomplete markets, if one Q is chosen by the market for all time (ie, a **static market**), the definition is analogous.
- If an incomplete market is **dynamic** with an infinite horizon, then the **fundamental price** of the risky asset is defined to be, with end time τ for the asset, $t < \tau$, and supposing we are in regime i at time t:

$$S_t^\star = E_{Q^i}\{\int_t^\tau dD_u + X_\tau 1_{\{\tau < \infty\}} | \mathcal{F}_t\}$$

  where Q^i is the risk neutral measure chosen by the market.
- Note that $X_\tau 1_{\{\tau = \infty\}}$ is not included.

We can piece all of these measures Q^i together to get one measure $Q^\star$, but $Q^\star$ **need not be risk neutral measure**; we call $Q^\star$ the **evaluation measure**, and write it $Q^{t\star}$ to denote that it changes with the time t.

Written this way, the previous equation becomes:

$$S_t^\star = E_{Q^{t\star}}\{\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\}$$

Recall the definition of a bubble:

Definition (Bubble)

A bubble in a static market for an asset with price process S is defined to be:

$$\beta = S - S^{\star}$$

A bubble in a dynamic market for $t < \tau$ in regime i is:

$$\beta = S - E_{Q^{t\star}}\{\int_t^{\tau} dD_u + X_{\tau}1_{\{\tau<\infty\}}|\mathcal{F}_t\}$$

Since we are in regime i, we have in this case $Q^{t\star} = Q^i$.

If there are no bubbles, a change to a new risk neutral measure can create a bubble; we call this **bubble birth**

Derivative Securities

- Assume S pays no dividends
- A derivative security is written on the market price of S
- Let H be such a contingent claim, and denote its market price by Λ_t^H
- Suppose we are in regime i at time t; the **fundamental price** of H is $E_{Q^{t\star}}\{H|\mathcal{F}_t\}$
- The derivative security's price **bubble** is defined as

$$\delta_t = \Lambda_t^H - E_{Q^{t\star}}\{H|\mathcal{F}_t\}.$$

European Call and Put Options

We have a risky asset with market price $S = (S_t)_{t \geq 0}$. We consider contingent claims with a maturity date T and a strike price K

- A **forward contract** has payoff $S_T - K$. Its market price at time t is denoted $V_t^f(K)$.
- A **European call option** has payoff $(S_T - K)_+$. Its market price at time t is denoted $C_t(K)$.
- A **European put option** has payoff $(K - S_T)_+$. Its market price at time t is denoted $P_t(K)$.
- We let $V_t^{f^\star}(K)$, $C_t^\star(K)$ and $P_t^\star(K)$ be the **fundamental prices** of the forward, call, and put, respectively

Theorem (Put-Call parity for Fundamental Prices)

$$C_t^{\star}(K) - P_t^{\star}(K) = V_t^{f\star}(K).$$

Theorem (Put-Call Parity for Market Prices)

$$C_t(K) - P_t(K) = V_t^f(K) = S_t - K$$

- The Fundamental Price Theorem follows by properties of expectations
- The Market Price Theorem follows by No Dominance using the argument of Merton (1973)

Theorem (Equality of European Put Prices)

For all $K \geq 0$

$$P_t(K) = P_t^\star(K)$$

European puts have no bubbles, due to the payoff being bounded.

Theorem (European Call Prices)

For all $K \geq 0$

$$C_t(K) - C_t^\star(K) = S_t - E_{Q^{t\star}}\{S_T|\mathcal{F}_t\} = \beta_t^3 - E_{Q^{t\star}}\{\beta_T^3|\mathcal{F}_t\}$$

- Only type 3 bubbles are reflected in call prices
- Risk neutral valuation need not hold in an NFLVR and No Dominance market

American Call Options (Static Market)

- We introduce a risk free savings account D given by

$$D_t = \exp\left(\int_0^t r_s ds\right)$$

where r is a non-negative, adapted process representing the default free spot rate of interest

- The **fundamental value** of an American Call option with strike price K and maturity T is

$$C_t^{A\star}(K) = \sup_{\eta \in [t,T]} E_Q\{(S_\eta - \frac{K}{D_\eta})_+ | \mathcal{F}_t\}$$

where η is a stopping time and Q is the risk neutral measure.

- We let $C^A(K)_t$ denote the **market price** at time t of this same option

Theorem

Assume that the jumps of the asset price S satisfy some mild regularity conditions. Then for all K,

$$C_t^E(K) = C_t^A(K) = C_t^{A\star}(K)$$

- This is an extension of Merton's famous "no early exercise" theorem (1973)
- American call options do not exhibit bubbles
- $C_t^A(K) - C_t^{E\star}(K) = \beta_t^3 = S_t - E_{Q^{t\star}}[S_T|\mathcal{F}_t]$
- While the market prices of European and American options agree, the fundamental prices need not agree.

How do we detect bubbles in real time?

- In a relatively simple context of an SDE

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt \tag{3}$$

it basically comes down to deciding if X is a local martingale or not under the risk neutral measure

- Thus equation (3) becomes

$$dX_t = \sigma(X_t)d\tilde{B}_t \tag{4}$$

- X in (4) is a strict local martingale if and only if

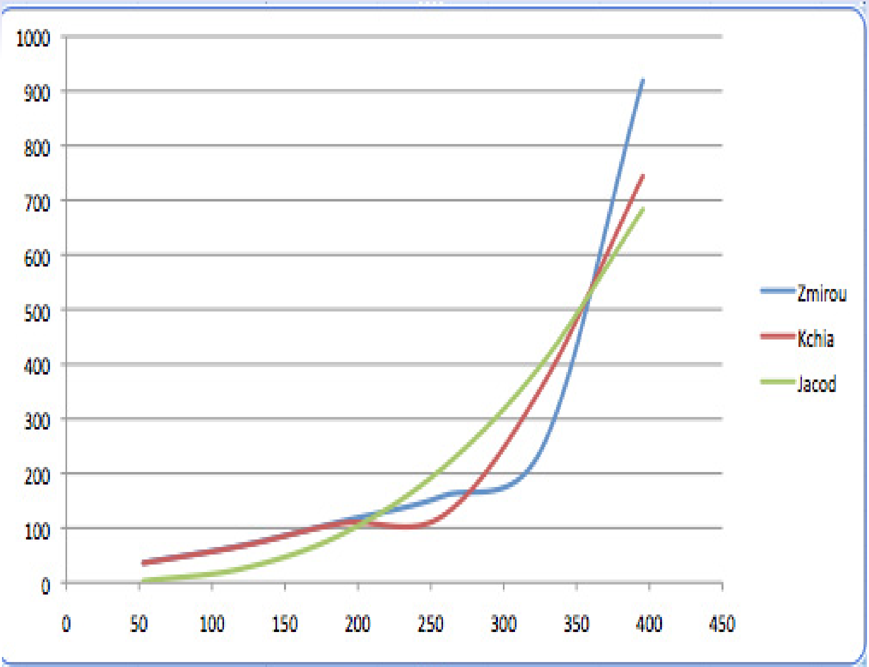
$$\int_{\epsilon}^{\infty} \frac{x}{\sigma(x)^2} dx < \infty \tag{5}$$

- To decide whether or not this integral is finite from data is *a priori* impossible
- The problem becomes one of determining what the asset's volatility is, or perhaps more appropriately σ^2
- We are limited to a finite interval since data is finite
- With some luck, the behavior of σ^2 will be sufficiently regular as to justify the assumption that its behavior continues into its tail

- We try to determine the form of σ^2 from market data: both high/low daily, and tick data
- Luckily, estimators of σ^2 have been developed in the literature by D. Florens-Zmirou, V. Genon-Catalot, J. Jacod, M. Hoffmann, and their collaborators.
- The techniques are both parametric and non parametric
- We next show the graph of Lastminute.com daily open/close date, 2000-2002


600
500
400
300
200
100
0
0
200
400
600
800
1000
1200
1400
1600

- We use open/close daily data, and three estimators of σ^2: one by Jacod, one by Florens-Zmirou, and one developed by us, and labeled Kchia
- We suspect there is a bubble in the years 2000-2002 (the dot come bubble period)


1000
900
800
700
600
500
400
300
200
100
0
0
50
100
150
200
250
300
350
400
450
Zmirou
Kchia
Jacod

- We see from these results that all three estimators indicate the presence of a bubble, using our test.
- Other dot com stocks give first indications of being inconclusive; however we are gradually finding that by using tick data, we can decide easily in favor of the there being a bubble
- We are also trying a reproducing kernel Hilbert space approach
- We are also broaching the multidimensional case, and the case of incomplete markets

Thank You for Your Attention

Why doesn't "no arbitrage" exclude bubbles in an NFLVR economy?

- The obvious candidate strategy: short the risky asset during the bubble, and cover the short after the bubble crashes
  - For type 1 and type 2 bubbles, the trading strategy fails to be an arbitrage because all trading strategies must terminate in finite time, and the bubble may outlast this trading strategy with positive probability
  - For type 3 bubbles this trading strategy fails because of the admissibility requirement. With positive probability a type 3 bubble can increase such that the short position's losses violate the admissibility condition
- In a complete market, No Dominance excludes these bubbles because there are two ways to create the asset's payoff (synthetic versus buy and hold)
- In an incomplete market, synthetic replication need not be possible. Hence, bubbles can exist!

A static market with NFLVR only

Corollary

Any asset price bubble has the following properties:

Bubbles are non-negative

For assets with possibly unbounded but finite lifetimes, bubbles may burst at the asset's maturity

Bubbles cannot be "born" after time 0

Implications

- As a local martingale, a typical pattern (a price increase, then a decrease) may not occur.
- A bubble is a supermartingale (a local martingale which is bounded below)
- Bubbles may be more common (and exist in individual assets as well as in sectors) than is widely believed