On multicurve models for the term structure.

Wolfgang Runggaldier

Dipartimento di Matematica, Università di Padova

WQMIF, Zagreb 2014

Preliminary remarks

- In the wake of the big crisis one has witnessed a significant increase in the spreads between LIBORs of different tenors as well as the spread between a LIBOR and the discount curve (LIBOR-OIS)

$\longrightarrow$ *This has led to the construction of multicurve models where, typically, future cash flows are generated through curves associated to the underlying rates, but are discounted by another curve.*

Preliminary remarks

- The majority of the models that have been considered reflect the usual classical distinction between

  i) short rate models *(bottom-up)*;
  ii) HJM setup;
  iii) BGM or LIBOR market models *(top-down)*.

In addition, methodologies related to foreign exchange.

→ *Concerning i) and ii), short rate models lead more easily to a Markovian structure, while HJM allows for a direct calibration to the initial term structure.*

Preliminary remarks

- Here we concentrate on short rate models. *[Kenyon, Kijima-Tanaka-Wong, Filipovic-Trolle]*
- A major goal with this modeling choice will be to derive an easy relationship between risk-free and “risky” FRAs thereby exhibiting an “adjustment factor” that plays a role analogous to “quanto adjustments” in cross-currency derivatives or to the “multiplicative forward basis” in *[Bianchetti]*.

Preliminary remarks

FRA (*forward rate agreement*) is an OTC derivative that allows the holder to lock in at $t < T$ the interest rate between the inception date T and the maturity S at a fixed value K. At maturity S, a payment based on K is made and one based on the relevant floating rate *(generally the spot Libor rate $L(T;T,S)$)* is received.

$\longrightarrow$ *Considering later on a single tenor, we let the maturity be $S = T + \Delta$ and denote the value of the FRA at $t < T$ by $FRA^T(t, K)$.*

Preliminary remarks

- To present the basic ideas in a simple way, here we consider a two-curve model, namely with a curve for discounting and one for generating future cash flows:
  i) The choice of the discount curve is not unique; we follow the common choice of considering the OIS swap curve.
  ii) For the risky cash flows without collateral we consider a single LIBOR *(i.e. for a given tenor)*.

Preliminary remarks

We describe an approach that we present here for the case of pricing of FRAs (*linear derivatives*).

- *We consider only "clean valuation" formulas, namely without counterparty risk.*
- To account for counterpart risk and funding issues, various value adjustments are generally computed on top of the clean prices.
- *As pointed out in [Crepey, Grbac, Ngor, Skovmand], market quotes typically reflect prices of fully collateralized transactions. The clean price formulas thus turn out to be sufficient also for calibration.*

Preliminary remarks

- Traditionally, interest rates are defined to be coherent with the bond prices $p(t, T)$, which represent the expectation of the market concerning the future value of money.

  → For discrete compounding forward rates this leads to ($t < T < S$)

$$F(t; T, S) = \frac{1}{S - T}\left(\frac{p(t, T)}{p(t, S)} - 1\right)$$

- The formula can also be justified as representing the fair fixed rate at time t of a FRA, where the floating rate received at S is

$$F(T; T, S) = \frac{1}{S - T}\left(\frac{1}{p(T, S)} - 1\right)$$

Preliminary remarks

- In fact, the arbitrage-free price in t of such a FRA is *(using the forward martingale measure Q^S)*

$$FRA^T(t,K) = p(t,S)\, E^{Q^S}\left\{(F(T;T,S) - K) \mid \mathcal{F}_t\right\}$$

  which is zero for

$$\begin{aligned} K &= E^{Q^S}\left\{(F(T;T,S) \mid \mathcal{F}_t\right\} \\ &= E^{Q^S}\left\{\frac{1}{S-T}\left(\frac{p(T,T)}{p(T,S)} - 1\right) \mid \mathcal{F}_t\right\} = \frac{1}{S-T}\left(\frac{p(t,T)}{p(t,S)} - 1\right) \end{aligned}$$

Preliminary remarks

- Since the discount curve is considered to be given by the OIS zero-coupon curve ($p(t,T) = p^{OIS}(t,T)$), one uses also the notation $L^D(t;T,S)$ for $F(t;T,S)$ and calls it OIS forward rate.
- *The pre-crisis (risk-free) forward Libor rate $L(t;T,S)$ was supposed to coincide with the OIS forward rate, namely the following equality was supposed to hold*

$$L(t;T,S) = L^D(t;T,S) = F(t;T,S)$$

Preliminary remarks

- Putting now $S = T + \Delta$ *(tenor Δ)*, recall that the risky LIBOR rates $L(t;T,T+\Delta)$ are determined by the LIBOR panel that takes into account various factors such as credit risk, liquidity, etc. and this implies that in general $L(t;T,S) \neq F(t;T,S)$ thus leading to a LIBOR-OIS spread.
- Following some of the recent literature, in particular *[Crepey- Grbac-Nguyen]* (see also *[Kijima-Tanaka-Wong]*), we keep the formal relationship between discrete compounding forward rates and bond prices also for the LIBORs, but replace the risk-free bond prices $p(t,T)$ by fictitious ones $\bar{p}(t,T)$ that are supposed to be affected by the same factors as the LIBORs.

Preliminary remarks

- Since FRAs are based on the $T-$spot LIBOR $L(T; T, T+\Delta)$, we actually postulate the classical relationship only at the inception time $t = T$. Our starting point is thus

$$L(T; T, S) = \frac{1}{\Delta}\left(\frac{1}{\bar{p}(T, T+\Delta)} - 1\right)$$

→ *Notice that also for our "risky bonds" we have* $\bar{p}(T, T) = 1$.

FRAs

In our two-curve risky setup, the fair price of a FRA in $t < T$ with $S = T + \Delta$, fixed rate K and notional N is then

$$
\begin{aligned}
FRA^T(t,K) &= N\Delta p(t, T+\Delta) E^{T+\Delta}\left[L(T; T, T+\Delta) - K \mid \mathcal{F}_t\right] \\
&= Np(t, T+\Delta) E^{T+\Delta}\left[\frac{1}{\bar{p}(T, T+\Delta)} - (1+\Delta K) \mid \mathcal{F}_t\right]
\end{aligned}
$$

where $E^{T+\Delta}$ denotes expectation under the $(T+\Delta)-$ forward measure.

$\longrightarrow$ *The simultaneous presence of $p(t, T+\Delta)$ and $\bar{p}(t, T+\Delta)$ does not allow for the convenient reduction of the formula to a simpler form as in the one-curve setup.*

FRAs

The crucial quantity to compute in the $FRA^T(t, K)$ expression is

$$\bar{\nu}_{t,T} := E^{T+\Delta}\left[\frac{1}{\bar{p}(T, T+\Delta)} \mid \mathcal{F}_t\right].$$

$\rightarrow$ *The fixed rate to make the FRA a fair contract at time t is then*

$$\bar{K}_t := \frac{1}{\Delta}(\bar{\nu}_{t,T} - 1)$$

FRAs

- In the classical single curve case we have instead

$$\nu_{t,T} := E^{T+\Delta}\left[\frac{1}{p(T,T+\Delta)} \mid \mathcal{F}_t\right] = \frac{p(t,T)}{p(t,T+\Delta)}$$

being $\frac{p(t,T)}{p(t,T+\Delta)}$ an $\mathcal{F}_t-$martingale under the $(T+\Delta)-$forward measure.

The fair fixed rate in the single curve case is then

$$K_t = \frac{1}{\Delta}\left(\nu_{t,T} - 1\right) = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right)$$

$\longrightarrow$ *To compute K_t no interest rate model is needed (contrary to $\bar{K}_t$).*

The model

- To compute the expectation $E^{T+\Delta}$ we need a model for $\bar{p}(t,T)$.
- For this purpose recall first the classical bond price formula *(r_t is the short rate)*

$$p(t,T) = E^Q\left\{\exp\left[-\int_t^T r_u du\right] \mid \mathcal{F}_t\right\}$$

with Q the standard martingale measure,.

The model

We now define the "risky bond prices" as

$$\bar{p}(t,T) = E^Q \left\{ \exp\left[-\int_t^T (r_u + s_u)du \right] \mid \mathcal{F}_t \right\}$$

with s_t representing the short rate spread (*hazard rate in case of only default risk*).

→ The spread is introduced from the outset.

→ $\bar{p}(t,T)$ *is not an actual price*.

The model

- Next we need a dynamical model for r_t and s_t and for this purpose we shall introduce a factor model.
- For various reasons, in particular in view of our main goal to obtain an “adjustment factor”, it is convenient to be able to have the same factor model for FRAs with different maturities. We therefore aim at performing the calculations under a single reference measure, namely Q.

  $\longrightarrow$ *We shall first recall two basic factor models for the short rate*.

The model

Model A. The square-root, exponentially affine model (CIR) model where $r_t = \sum_{i=1}^{I} \gamma_i \Psi_t^i$ with, under Q, (w_t^i independent $Q-$Wiener)

$$d\Psi_t^i = (a^i - b^i \Psi_t^i)dt + \sigma^i \sqrt{c^i \Psi_t^i + d^i}\, dw_t^i$$

It implies

$$\begin{aligned} p(t,T) &= E^Q \left\{ \exp\left[-\int_t^T r_u\, du \right] \mid \mathcal{F}_t \right\} \\ &= \exp\left[A(t,T) - \sum_{i=1}^{I} B^i(t,T) \Psi_t^i \right] \end{aligned}$$

$\longrightarrow$ *For $c^i = 0$ the square-root model becomes a Gaussian mean reverting (Hull-White) model.*

The model

- The above model class includes various specific models that have appeared in the literature such as e.g. the following two-factor Gaussian short rate model from *[Filipovic-Trolle]* (analogous models for the spreads)

$$\begin{cases} dr_t & = & \kappa_r(\gamma_t - r_t)dt + \sigma_r \, dw_t^r \\ d\gamma_t & = & \kappa_\gamma(\theta_\gamma - \gamma_t) + \sigma_\gamma \left(\rho \, dw_t^r + \sqrt{1-\rho^2} \, dw_t^\gamma \right) \end{cases}$$

The model

- It suffices in fact to consider two Gaussian factors

$$d\Psi_t^i = (a^i - b^i \Psi_t^i)dt + \sigma^i\, dw_t^i\,, \quad i = 1, 2$$

and put

$$\begin{cases} r_t &= \lambda^1 \Psi_t^1 + \lambda^2 \Psi_t^2 \\ \gamma_t &= \frac{\lambda^1 a^1 + \lambda^2 a^2}{b^1} + \frac{b^1 - b^2}{b^1} \lambda^2\, \Psi_t^2 \end{cases}$$

- *The given model class can also be easily generalized to affine jump-diffusion models (see e.g.[Bjoerk, Kabanov, R.]); only the notation becomes then more involved.*

The model

Model B. The Gaussian, exponentially quadratic model *[Pelsser, Kijima-Tanaka-Wong] (dual to square-root exponentially affine)*

$$r_t = \sum_{i=1}^{l_1} \gamma_i \Psi_t^i + \sum_{i=l_1+1}^{l_2} \gamma_i (\Psi_t^i)^2$$

$$d\Psi_t^i = -b^i \Psi_t^i dt + \sigma^i dw_t^i$$

It implies

$$\begin{aligned} p(t,T) &= E^Q\left\{\exp\left[-\int_t^T r_u\, du\right] \mid \mathcal{F}_t\right\} \\ &= \exp\left[A(t,T) - \sum_{i=1}^{l_1} B^i(t,T)\Psi_t^i - \sum_{i=l_1+1}^{l_2} C^i(t,T)(\Psi_t^i)^2\right] \end{aligned}$$

$\longrightarrow$ *Advantage of this model in derivative pricing: the distribution of Ψ_t^i remains always Gaussian; in a square-root model it is a χ^2-distribution.*

The model

- In presenting joint models for r_t and s_t we want to allow for non-zero correlation between r_t and s_t.
  - $\longrightarrow$ *It is obtained by considering common factors, the remaining ones being idiosyncratic factors.*
  - $\longrightarrow$ To obtain an adjustment factor, at least one of the common factors has to satisfy a Gaussian model (Vasiček/Hull-White).
  - $\longrightarrow$ *By analogy to the pure short rate case, also here we consider two model classes.*

The model

Model A. (based on Morino-R. 2013)
Given three independent affine factor processes Ψ_t^i, $i = 1, 2, 3$ let

$$\begin{cases} r_t &= \Psi_t^2 - \Psi_t^1 \\ s_t &= \kappa \Psi_t^1 + \Psi_t^3 \end{cases}$$

where the common factor Ψ_t^1 allows for instantaneous correlation between r_t and s_t with correlation intensity κ (negative correlation for $\kappa > 0$). *Other factors may be added to drive s_t.*

The model

- (*Model A. contd.*) Let, under Q,

$$\begin{cases} d\Psi_t^1 & = & (a^1 - b^1)\Psi_t^1 dt + \sigma^1 \, dw_t^1 \\ \\ d\Psi_t^i & = & (a^i - b^i)\Psi_t^i dt + \sigma^i \sqrt{\Psi_t^i} \, dw_t^i, \quad i = 2, 3 \end{cases}$$

where a^i, b^i, σ^i are positive constants with $a^i \geq (\sigma^i)^2/2$ for $i = 2, 3$, and w_t^i independent $Q-$Wiener processes.

$\rightarrow$ Ψ_t^1 may take negative values implying that, not only r_t, but also s_t may become negative (see later).

The model

Model B. (*analogous to above*)
Given again three independent affine factor processes Ψ_t^i, $i = 1, 2, 3$ let

$$\begin{cases} r_t = \Psi_t^1 + (\Psi_t^2)^2 \\ s_t = \kappa\Psi_t^1 + (\Psi_t^3)^2 \end{cases}$$

where the common factor Ψ_t^1 allows again for instantaneous correlation between r_t and s_t with correlation intensity κ . *Other factors may be added to drive s_t.*

- Under Q,

$$d\Psi_t^i = -b^i \Psi_t^i dt + \sigma^i \, dw_t^i, \quad i = 1, 2, 3$$

where w_t^i independent $Q-$Wiener processes.

$\longrightarrow$ *Ψ_t^1 might take negative values so that also here r_t and s_t may become negative.*

Bond price relations

- For case A. we have

$$p(t,T) = \exp\left[A(t,T) - B^1(t,T)\Psi_t^1 - B^2(t,T)\Psi_t^2\right]$$

$$\bar{p}(t,T) = \exp\left[\bar{A}(t,T) - \bar{B}^1(t,T)\Psi_t^1 - \bar{B}^2(t,T)\Psi_t^2 - \bar{B}^3(t,T)\Psi_t^3\right]$$

with $\bar{B}^1(t,T) = (1-\kappa)\, B^1(t,T),\ \ \bar{B}^2(t,T) = B^2(t,T)$.
It follows that

$$\bar{p}(t,T) = p(t,T)\, \exp\left[\tilde{A}(t,T) + \kappa B^1(t,T)\Psi_t^1 - \bar{B}^3(t,T)\Psi_t^3\right]$$

where $\tilde{A}(t,T) := \bar{A}(t,T) - A(t,T)$.

Bond price relations

- Putting for simplicity $\tilde{B}^1 := B^1(T, T+\Delta)$, it follows that

$$\frac{p(T, T+\Delta)}{\bar{p}(T, T+\Delta)} = \exp\left[-\tilde{A}(T, T+\Delta) - \kappa\tilde{B}^1\Psi^1_T + \bar{B}^3(T, T+\Delta)\Psi^3_T\right]$$

and, defining an adjustment factor as

$$Ad_t^{T,\Delta} := E^Q\left\{\frac{p(T, T+\Delta)}{\bar{p}(T, T+\Delta)} \mid \mathcal{F}_t\right\}$$

this factor can be *expressed as*

$$\begin{aligned} Ad_t^{T,\Delta} &:= e^{-\tilde{A}(T,T+\Delta)} E^Q\left\{e^{-\kappa\tilde{B}^1\Psi^1_T + \bar{B}^3(T,T+\Delta)\Psi^3_T} \mid \mathcal{F}_t\right\} \\ &= A(\theta, \kappa, \Psi^1_t, \Psi^3_t) \end{aligned}$$

with $\theta := (a^i, b^i, \sigma^i,\ i = 1, 2, 3)$.

Main result

Proposition: We have

$$\begin{aligned}\bar{\nu}_{t,T} &= \nu_{t,T} \cdot Ad_t^{T,\Delta} \\ &\cdot \exp\left[\kappa \frac{(\sigma^1)^2}{2(b^1)^3}\left(1 - e^{-b^1\Delta}\right)\left(1 - e^{-b^1(T-t)}\right)^2\right]\end{aligned}$$

- The fair value $\bar{K}_t$ of the fixed rate in a “risky” FRA is then related to K_t in a corresponding riskless FRA as follows:

$$\begin{aligned}\bar{K}_t &= \left(K_t + \frac{1}{\Delta}\right) \cdot Ad_t^{T,\Delta} \\ &\cdot \exp\left[\kappa \frac{(\sigma^1)^2}{2(b^1)^3}\left(1 - e^{-b^1\Delta}\right)\left(1 - e^{-b^1(T-t)}\right)^2\right] - \frac{1}{\Delta}\end{aligned}$$

$\longrightarrow$ *The factor given by the exponential is equal to* 1 *for zero correlation* $(\kappa = 0)$.

Comments on the main result: adjustment factors

- An easy intuitive interpretation of the main result can be obtained in the case of $\kappa = 0$ *(independence of r_t and s_t)*. In this case, since $s_t = \Psi_t^3 > 0$, we have $r_t + s_t > r_t$ implying $\bar{p}(T, T+\Delta) < p(T, T+\Delta)$ so that $Ad_t^{T,\Delta} \geq 1$ (the exponential adjustment factor is equal to 1).

  $\longrightarrow$ *As expected we then have*

$$\bar{\nu}_{t,T} \geq \nu_{t,T} \quad , \quad \bar{K}_t \geq K_t$$

Comments on the main result: calibration

The coefficients $a^1, a^2, b^1, b^2, \sigma^1, \sigma^2$ can be calibrated in the usual way on the basis of the observations of default-free bonds $p(t, T)$.

- $\longrightarrow$ To calibrate a^3, b^3, σ^3, notice that, contrary to $p(t, T)$, the "risky" bonds $\bar{p}(t, T)$ are not observable *(there is no unique inverse relationship to determine $\bar{p}(t, T)$ from observations of the LIBORs)*.
- $\longrightarrow$ *One can however observe $K_t = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right)$ as well as the "risky" FRA rate $\bar{K}_t$.*

Comments on the main result: calibration

Recalling then the Corollary, namely

$$\begin{aligned}\bar{K}_t &= \left(K_t + \tfrac{1}{\Delta}\right) \cdot Ad_t^{T,\Delta} \\ &\quad \cdot \exp\left[-\kappa \tfrac{(\sigma^1)^2}{(b^1)^3}\left(e^{-b^1\Delta} - 1\right)\left(1 - e^{-b^1(T-t)}\right)^2\right] - \tfrac{1}{\Delta}\end{aligned}$$

and the fact that $Ad_t^{T,\Delta} = A(\theta, \kappa, \Psi_t^1, \Psi_t^2)$, this allows to calibrate a^3, b^3, σ^3 as well as κ.

Bond price relations

- For case B. we have analogously ((·) *stands for* (t,T))

$$p(t,T) = \exp\left[A(\cdot) - B^1(\cdot)\Psi_t^1 - C^2(\cdot)(\Psi_t^2)^2\right]$$

$$\bar{p}(t,T) = \exp\left[\bar{A}(\cdot) - \bar{B}^1(\cdot)\Psi_t^1 - \bar{C}^2(\cdot)(\Psi_t^2)^2 - \bar{C}^3(\cdot)(\Psi_t^3)^2\right]$$

with $\bar{B}^1(t,T) = (1+\kappa)\, B^1(t,T), \;\; \bar{C}^2(t,T) = C^2(t,T)$.
It follows that

$$\bar{p}(t,T) = p(t,T)\, \exp\left[\tilde{A}(t,T) + \kappa B^1(t,T)\Psi_t^1 - \bar{C}^3(t,T)(\Psi_t^3)^2\right]$$

where, again, $\tilde{A}(t,T) := \bar{A}(t,T) - A(t,T)$.

Bond price relations

- Putting again $\tilde{B}^1 := B^1(T, T+\Delta)$, it follows that

$$\frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} = \exp\left[-\tilde{A}(T,T+\Delta) - \kappa \tilde{B}^1 \Psi_T^1 + C^3(T,T+\Delta)(\Psi_T^3)^2\right]$$

Introducing the same *adjustment factor*

$$Ad_t^{T,\Delta} := E^Q\left\{\frac{p(T,T+\Delta)}{\bar{p}(T,T+\Delta)} \mid \mathcal{F}_t\right\}$$

that can again be *expressed as*

$$\begin{aligned} Ad_t^{T,\Delta} &:= e^{-\tilde{A}(T,T+\Delta)} E^Q\left\{e^{-\kappa \tilde{B}^1 \Psi_T^1 + \bar{B}^3(T,T+\Delta)\Psi_T^3} \mid \mathcal{F}_t\right\} \\ &= A(\theta, \kappa, \Psi_t^1, \Psi_t^3) \end{aligned}$$

where $\theta := (a^i, b^i, \sigma^i,\ i = 1,2,3)$, one obtains completely analogous results as for case A.

Thank you for your attention

Preliminary results for determining $\bar{\nu}_{t,T}$

- Due to the affine dynamics of Ψ_t^i $(i = 1, 2, 3)$ under Q we have for the risk-free bond

$$p(t,T) = E^Q\left\{ \exp\left[-\int_t^T r_u d_u \right] \mid \mathcal{F}_t \right\}$$

$$= E^Q\left\{ \exp\left[\int_t^T (\Psi_u^1 - \Psi_u^2) d_u \right] \mid \mathcal{F}_t \right\}$$

$$= \exp\left[A(t,T) - B^1(t,T)\Psi_t^1 - B^2(t,T)\Psi_t^2 \right]$$

Preliminary results for determining $\bar{\nu}_{t,T}$

- The coefficients satisfy

$$\begin{cases} B^1_t - b^1 B^1 - 1 = 0 & , \quad B^1(T,T) = 0 \\ B^2_t - b^2 B^2 - \frac{(\sigma^2)^2}{2}(B^2)^2 + 1 = 0 & , \quad B^2(T,T) = 0 \\ A_t = a^1 B^1 - \frac{(\sigma^1)^2}{2}(B^1)^2 + a^2 B^2 & , \quad A(T,T) = 0 \end{cases}$$

in particular

$$B^1(t,T) = \frac{1}{b^1}\left(e^{-b^1(T-t)} - 1\right)$$

Preliminary results for determining $\bar{\nu}_{t,T}$

- For the "risky" bond we have instead

$$\bar{p}(t,T) = E^Q\left\{ \exp\left[-\int_t^T (r_u + s_u)du \right] \mid \mathcal{F}_t \right\}$$

$$= E^Q\left\{ \exp\left[-\int_t^T ((\kappa - 1)\Psi_u^1 - \Psi_u^2 - \Psi_u^3)du \right] \mid \mathcal{F}_t \right\}$$

$$= \exp\left[\bar{A}(t,T) - \bar{B}^1(t,T)\Psi_t^1 - \bar{B}^2(t,T)\Psi_t^2 - \bar{B}^3(t,T)\Psi_t^3 \right]$$

Preliminary results for determining $\bar{\nu}_{t,T}$

- The coefficients satisfy

$$\begin{cases} \bar{B}^1_t - b^1 \bar{B}^1 + (\kappa - 1) = 0 & , \quad \bar{B}^1(T,T) = 0 \\ \bar{B}^2_t - b^2 \bar{B}^2 - \frac{(\sigma^2)^2}{2}(\bar{B}^2)^2 + 1 = 0 & , \quad \bar{B}^2(T,T) = 0 \\ \bar{B}^3_t - b^3 \bar{B}^3 - \frac{(\sigma^3)^2}{2}(\bar{B}^3)^2 + 1 = 0 & , \quad \bar{B}^3(T,T) = 0 \\ \bar{A}_t = a^1 \bar{B}^1 - \frac{(\sigma^1)^2}{2}(\bar{B}^1)^2 + a^2 \bar{B}^2 + a^3 \bar{B}^3 & , \quad \bar{A}(T,T) = 0 \end{cases}$$

in particular

$$\bar{B}^1(t,T) = \frac{1-\kappa}{b^1}\left(e^{-b^1(T-t)} - 1\right) = (1-\kappa)\, B^1(t,T)$$

Preliminary results for determining $\bar{\nu}_{t,T}$

>From the 1−st order equations it follows that

$$\begin{aligned}
\bar{B}^1(t,T) &= (1-\kappa)\,B^1(t,T)\\
\bar{B}^2(t,T) &= B^2(t,T)\\
\bar{A}(t,T) &= A(t,T) - a^1\kappa\int_t^T B^1(u,T)du\\
&\quad +\frac{(\sigma^1)^2}{2}\kappa^2\int_t^T (B^1(u,T))^2du + (\sigma^1)^2\kappa\int_t^T B^1(u,T)du\\
&\quad -a^3\int_t^T \bar{B}^3(u,T)du
\end{aligned}$$

- Let

$$\tilde{A}(t,T) := \bar{A}(t,T) - A(t,T)$$

Nonlinear derivatives *CAPs/Caplets*

- We concentrate on the pricing of a single Caplet, with strike K, maturity T on the forward LIBOR for the period $[T, T+\Delta]$. Using the forward measure, its price in $t < T$ is then given by

$$Capl^{T,\Delta}(t) = \Delta p(t, T+\Delta) E^{T+\Delta} \left\{ \left(\bar{L}(T; T, T+\Delta) - K \right)^+ \mid \mathcal{F}_t \right\}$$

$$= p(t, T+\Delta) E^{T+\Delta} \left\{ \left(\frac{1}{\bar{p}(T, T+\Delta)} - \tilde{K} \right)^+ \mid \mathcal{F}_t \right\}$$

with $\tilde{K} := 1 + \Delta K$.

Nonlinear derivatives *CAPs/Caplets*

- We may use the same “ risky” short rate model as for the FRAs that we may consider as already calibrated *(for the standard martingale measure Q)*.
- The aim, pursued in the case of the FRAs, of performing the calculations under the same measure Q leads here to some difficulties and so we stick to forward measures.

  $\longrightarrow$ *Depending on the pricing methodology, one may then need to change the dynamics of the factors to be valid under the various forward measures*.

  $\longrightarrow$ The R.N.-derivative to change from Q to the various forward measures can be expresses in explicit form and it preserves the affine structure.

Nonlinear derivatives *CAPs/Caplets*

- *It may thus suffice to derive just a pricing algorithm that need not also be used for calibration*.
- It remains however desirable to obtain also here an "adjustment factor".

Nonlinear derivatives *CAPs/Caplets*

- For the pricing, *in the forward measure*, we may use Fourier transform methods as in [CGN] and [CGNS] thereby representing the claim as

$$\left(e^X - \tilde{K}\right)^+ \quad \text{with} \quad X := -\log \bar{p}(T, T + \Delta)$$

(possibly also a Gram-Charlier expansion as in [KTW]).

$\longrightarrow$ Need only to compute the moment generating function of X that is a linear combination of the factors *(computation is feasible thanks to the affine structure)* and use the Fourier transform of $f(x) = \left(e^x - \tilde{K}\right)^+$.

Nonlinear derivatives *CAPs/Caplets*

The price in $t = 0$ can then be obtained in the form

$$Capl(0, T, T+\Delta) = \frac{p(0, T+\Delta)}{2\pi} \int \frac{\tilde{K}^{1-iv-R} \bar{M}_X^{T+\Delta}(R+iv)}{(R+iv)(R+iv-1)} \, dv$$

where $\bar{M}_X^{T+\Delta}(\cdot)$ is the moment generating function of X under the $(T+\Delta)-$forward measure.

Nonlinear derivatives *CAPs/Caplets*

- If $M_X^{T+\Delta}(\cdot)$ is the moment generating function of X with $p(T, T+\Delta)$ instead of $\bar{p}(T, T+\Delta)$ then

$$\bar{M}_X^{T+\Delta}(\cdot) = M_X^{T+\Delta}(\cdot)A(\cdot;\theta,\kappa,\Psi_0^1,\Psi_0^2,\Psi_0^3)$$

where, given the affine nature of the factors, $A(\cdot;\theta,\kappa,\Psi_0^1,\Psi_0^2,\Psi_0^3)$ can be explicitly computed.

$\longrightarrow$ *Since, for the above factorization to hold, $A(\cdot;\theta,\kappa,\Psi_0^1,\Psi_0^2,\Psi_0^3)$ contains also $(M_X^{T+\Delta}(\cdot))^{-1}$, this may however not suffice to derive a satisfactory adjustment factor as for FRAs.*