Mixed thinning INAR(1) model

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Integer-valued time series

Fields of usage of counting processes

- meteorology (earthquakes counting),
- insurance theory (counts of accidents),
- communications (transmitted messages),
- medicine (number of patients),
- law and social sciences (crime victimizations) and so on.

History of development of integer-valued time series

- Model based on Markov chains (Cox and Miller (1965))
- MTD models (Raftery (1985a))
- DARMA models (Jacobs and Lewis (1978a,b,c))

First ordered integer-valued autoregressive model (INAR(1))

Defined by the recursion

$$X_t = \sum_{i=1}^{X_{t-1}} B_i(t) + \epsilon_t, \quad t \in Z,$$

with demands:

- $\{B_i(t)\}$ and $\{\epsilon_t\}$ are integer-valued,
- $\{B_i(t)\}$ is i.i.d. sequence independent of X_{t-1} and ϵ_t,
- $\{\epsilon_t\}$ is i.i.d. sequence independent of X_{t-i}, for $i \geq 1$.

Note that $X_{t-1} = 0 \Rightarrow X_t = \epsilon_t$.

Thinning operators

Probabilistic operations that can be applied to random counts.

- Purpose: *shrinking* the observed population
- Method: *randomly deletes* some members of the population

Many different types of thinning operators, refer to a survey of Weiss (2008).

Binomial thinning

Let X be integer-valued r.v. and $\alpha \in [0, 1]$. Define a random variable

$$\alpha \circ X = \sum_{i=1}^{X} Y_i,$$

where $\{Y_i\}$ are i.i.d. Bernoulli indicators with parameter α (called *counting series*), which are independent of X.
We say: $\alpha \circ X$ arises from X by *binomial thinning*, and "$\circ$" is the *binomial thinning operator*.

- $\alpha \circ X | (X = x) : Bin(x, \alpha)$
- $\alpha \circ X \leq X$

$\Rightarrow$ The term is entirely justified.

Interpretation of $\alpha \circ X$

- Observe the population of size X at certain time t.
- At next time point $t + 1$ the population may be shrinked, because some of the elements have left between time points t and $t + 1$.
- Assume that elements under the study leave independently of each other with probability $1 - \alpha$.

$\Rightarrow$ Size of the observed population at time point $t + 1$ is $\alpha \circ X$.

Some properties of binomial thinning

$$\begin{aligned}
\alpha \circ (X + Y) &\overset{d}{=} \alpha \circ X + \alpha \circ Y, \text{ for independent r. v. } X, Y \\
\alpha \circ (\beta \circ X) &\overset{d}{=} (\alpha\beta) \circ X \\
1 \circ X &\overset{wp1}{=} X \\
0 \circ X &\overset{wp1}{=} 0 \\
E[\alpha \circ X] &= \alpha E[X] \\
Var[\alpha \circ X] &= \alpha^2 Var[X] + \alpha(1 - \alpha)E[X]
\end{aligned}$$

INAR(1) model based on binomial thinning

Let $\alpha \in (0, 1)$. The model is defined by the recursion

$$X_t = \alpha \circ X_{t-1} + \epsilon_t, \quad t \in Z,$$

with demands:

- Thinning operations are performed independently of each other and of $\{\epsilon_t\}$
- At each time t thinning operations at that time and ϵ_t are independent of $\{X_s\}_{s<t}$

Special case: geometric marginals (GINAR(1) introduced by Alzaid and Al-Osh (1988))

Interpretations

Basic interpretation

- X_t - size of the population at time t
- $\alpha \circ X_{t-1}$ - survivors of time $t-1$
- ϵ_t - immigration

An alternative interpretation

- X_t - customers at time t
- ϵ_t - new customers arrived between time points $t-1$ and t
- $X_{t-1} - \alpha \circ X_{t-1}$ - customers that have been lost between time points $t-1$ and t

Some generalizations of binomial thinning

Obtained by relaxing conditions specified in the definition of binomial thinning.

- counting variables have full range N_0 - *generalized thinning*
  - special case: *negative binomial thinning*
- negative integers are included - *signed thinning*
- *random coefficient thinning*
- dependent Bernoulli indicators

Model based on negative binomial thinning

Defined by the recursion

$$X_t = \alpha * X_{t-1} + \epsilon_t, \quad \text{where} \quad \alpha * X = \sum_{i=1}^{X} Y_i, \quad \text{for} \quad Y_i : Geom\left(\frac{\alpha}{1+\alpha}\right).$$

- Operator " $*$ " is not actually a "thinning", because $\alpha * X \leqslant X$ is not always true.
- Special case: geometric marginals (NGINAR(1) introduced by Ristić et al. (2009))

$\circ$ VS $*$

The main differences are

- $0 \circ X \overset{wp1}{=} 0$ and $0 * X \overset{wp1}{=} 0$, but
- $1 \circ X \overset{wp1}{=} X$, while $1 * X \overset{d}{=} \begin{cases} 0, & \text{w.p. } \frac{1}{1+\mu} \\ X, & \text{w.p } \frac{\mu^2}{(1+\mu)^2} \\ X+Y, & \text{w.p. } \frac{\mu}{(1+\mu)^2} \end{cases}$ where Y is geometric$\left(\frac{1+\mu}{2+\mu}\right)$ independent of X.
- $\beta \circ (\alpha \circ X) = (\beta\alpha) \circ X$, where counting sequences of "$\alpha \circ$" and "$\beta \circ$" are independent, unfortunately

  $$\beta * (\alpha * X) \overset{d}{=} \begin{cases} 0, & \text{w.p. } \frac{1+\alpha}{1+\alpha+\alpha\mu} \\ (\beta\alpha) * X + Y_1, & \text{w.p. } \frac{\alpha^2\mu^2}{(1+\alpha+\alpha\mu)(1+\alpha\mu)} \\ (\beta\alpha) * X + Y_2, & \text{w.p. } \frac{\alpha\mu}{(1+\alpha+\alpha\mu)(1+\alpha\mu)} \end{cases}$$

  Y_1 and Y_2 are independent and geometrically distributed with parameters $\frac{\beta\alpha}{1+\beta\alpha}$ and $\frac{\beta(1+\alpha+\alpha p)}{1+\beta(1+\alpha+\alpha p)}$, respectively and are independent of X.
- $E(\alpha \circ X)^2 = \alpha^2 E(X^2) + \alpha(1-\alpha)E(X)$, similarly $E(\alpha * X)^2 = \alpha^2 E\left(X^2\right) + \alpha(1+\alpha)E(X)$.

Model based on dependent Bernoulli counting series

1. Generate a sequence of dependent Bernoulli r. v. as

   $U_i = (1 - V_i)W_i + V_i Z$, where

   - $\{W_i\}$ is i.i.d. with $Ber(\alpha)$ distribution,
   - $\{V_i\}$ is i.i.d. with $Ber(\theta)$ distribution,
   - $Z : Ber(\alpha)$.

$$U_1 + U_2 + \ldots + U_n \stackrel{d}{=} \begin{cases} Bin(n, \alpha(1-\theta)), & \text{w.p. } 1-\alpha \\ Bin(n, \alpha+\theta-\alpha\theta), & \text{w.p. } \alpha \end{cases}$$

2. Define thinning operator as $\alpha \circ_\theta X = \sum_{i=1}^{X} U_i$.
3. Obtained model

$$X_t = \alpha \circ_\theta X_{t-1} + \epsilon_t, \quad t \in Z.$$

Introduced by Ristić et al. (2013)

- Models based on binomial thinning
    - elements can enter/survive/leave (contribution to the overall thinning sum 0 or 1)
- Models based on negative binomial thinning
    - elements by replicating themselves contribute to the overall thinning sum more than 1
- By mixing we could deal with elements which are active in some period and passive in another

Applications:

- the number of patients with certain transmitting disease
- the number of crimes in some police district
- the number of bacteria

Construction of the mixed thinning operator

Let $\{W_i\}_{i\in N}$ be i.i.d. sequence, defined as

$$W_i = \begin{cases} B_i, & \text{w.p. } p, \\ G_i, & \text{w.p. } 1-p, \end{cases} \quad p \in [0,1],\ B_i : Ber(\alpha),\ G_i : Geom\left(\frac{\alpha}{1+\alpha}\right).$$

The new thinning operator is

$$\alpha \bullet_p X = \sum_{i=0}^{X} W_i,$$

where $W_0 = 0$, X is nonnegative integer-valued r.v. independent of the counting series $\{W_i\}_{i\in N}$.

Basic properties of the mixed thinning operator

Using p.g.f. of W_i, we obtain

$$\alpha \bullet_p X | \{X = x\} \stackrel{d}{=} \begin{cases} NB\left(x, \frac{\alpha}{1+\alpha}\right) & \text{w.p. } (1-p)^x \\ Bin(i,\alpha) + NB\left(x-i, \frac{\alpha}{1+\alpha}\right) & \text{w.p. } \binom{x}{i} p^i (1-p)^{x-i} \\ Bin(x,\alpha) & \text{w.p. } p^x \end{cases}$$

for $1 \leq i \leq x-1$.

Let $J : Bin(x, p)$. Then

$$\alpha \bullet_p X | \{X = x\} \stackrel{d}{=} Bin(J,\alpha) + NB\left(x-J, \frac{\alpha}{1+\alpha}\right) \stackrel{d}{=} \alpha \circ J + \alpha * (x-J).$$

Also,

$E[\alpha \bullet_p X] = \alpha E[X]$

$Var[\alpha \bullet_p X] = \alpha^2 Var[X] + \alpha(1+\alpha-2\alpha p)E[X]$

Construction of the mixed model with geometric marginals (MixGINAR(1))

MixINAR(1) model is defined by recursion

$$X_t = \alpha \bullet_p X_{t-1} + \varepsilon_t, \quad t \in Z \tag{1}$$

with demands:

- $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of i.i.d. r. v. independent of the counting series $\{W_i\}_{i\in\mathbb{N}}$,
- r. v. X_{t-i} and ε_t are independent for all $i \geqslant 1$.

The mixed model (1) with geometric marginals (MixGINAR(1)) contains two existing models as special cases:

- for $p = 0 \Rightarrow$ MixGINAR(1) $\equiv$ NGINAR(1),
- for $p = 1 \Rightarrow$ MixGINAR(1) $\equiv$ GINAR(1).

Distribution of innovation process for (MixGINAR(1))

Using the p.g.f. of innovation r.v. we obtain next result:

Let $X_t : Geom\left(\frac{\mu}{1+\mu}\right)$ for $t \in Z$ and $\mu \geqslant \alpha(1-\alpha p)/(1-\alpha)$. Then

$$\varepsilon_t \stackrel{d}{=} \begin{cases} 0, & \text{with probability } \alpha p, \\ Geom\left(\frac{\alpha}{1+\alpha}\right), & \text{with probability } \frac{\alpha\mu(1-p)}{\mu-\alpha}, \\ Geom\left(\frac{\mu}{1+\mu}\right), & \text{with probability } \frac{\mu-\alpha(1+\mu-\alpha p)}{\mu-\alpha}. \end{cases}$$

Conditional least squares estimations

The unknown parameters α, μ and p need to be estimated. Since the conditional expectation $E(X_t|X_{t-1}) = \alpha X_{t-1} + (1-\alpha)\mu$ only depends on the first two parameters α and μ, we will use the two-step conditional least squares approach considered by Karlesen and Tjøstheim (1986).

- Step one: estimation of the unknown parameters α and μ,
- Step two: estimation of the unknown parameter p using the conditional least squares estimates of the parameters α and μ obtained in the first step.

Obtained conditional least squares estimates

$$\hat{\alpha}_{cls} = \frac{(n-1)^{-1}\sum_{t=2}^{n-1} X_t X_{t-1} - (n-1)^{-2}\sum_{t=2}^{n} X_t \sum_{t=2}^{n} X_{t-1}}{(n-1)^{-1}\sum_{t=2}^{n} X_{t-1}^2 - (n-1)^{-2}\left(\sum_{t=2}^{n} X_{t-1}\right)^2}$$

$$\hat{\mu}_{cls} = \frac{\sum_{t=2}^{n} X_t - \hat{\alpha}_{cls}\sum_{t=2}^{n} X_{t-1}}{(n-1)(1-\hat{\alpha}_{cls})}$$

$$\hat{p}_{cls} = \frac{\sum_{t=2}^{n} \hat{Z}_t (X_{t-1} - \hat{\mu}_{cls})}{2\hat{\alpha}_{cls}^2 \sum_{t=2}^{n} (X_{t-1} - \hat{\mu}_{cls})^2},$$

where

$\hat{Z}_t = -\hat{Y}_t + \hat{\alpha}_{cls}(1+\hat{\alpha}_{cls})X_{t-1} + \hat{\mu}_{cls}(1-\hat{\alpha}_{cls} - 2\hat{\alpha}_{cls}^2 + \hat{\mu}_{cls} - \hat{\alpha}_{cls}^2\hat{\mu}_{cls})$

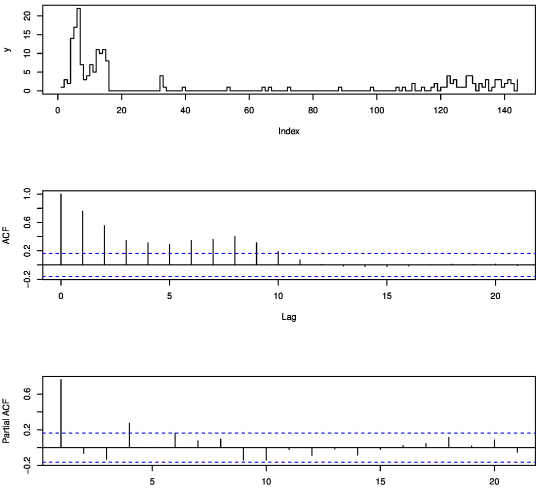
$\hat{Y}_t = X_t - \hat{\alpha}_{cls}X_{t-1} - (1-\hat{\alpha}_{cls})\hat{\mu}_{cls}.$

We will compare GINAR(1), NGINAR(1) and MixGINAR(1).

Consider data series representing a monthly counting of committing a light criminal activity, public drunkenness, in a period from January 1990 to December 2001, constituting sequence of 144 observations.

1. Series PubDrunk-22 is created by the $22nd$ police car beat of Pittsburgh and can be downloaded from a website Forecasting Principles (http://www.forecastingprinciples.com).

The sample mean, variance and autocorrelation of the PubDrunk-22 are respectively, 1.34, 10.1 and 0.761.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0
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15
20
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80
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ACF
−0.2
0.2
0.6
1.0
0
5
10
15
20
Lag
Partial ACF
−0.2
0.2
0.6
5
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15
20

Model	MLE	MLV	RMS
GINAR(1)	$\hat{q} = \mathbf{0.6030}$		
	$\hat{\alpha} = \mathbf{0.4800}$	188.099	2.248
NGINAR(1)	$\hat{\mu} = \mathbf{1.3729}$		
	$\hat{\alpha} = \mathbf{0.5720}$	178.010	2.146
MTGINAR(1)	$\hat{\mu} = \mathbf{1.4544}$		
	$\hat{p} = \mathbf{0.2234}$		
	$\hat{\alpha} = \mathbf{0.6167}$	177.320	2.111

MLE- maximum likelihood parameter estimates
MLV - maximum log-likelihood values
RMS - the root mean squares of differences between the observations and predicted values

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Thank You for Your Attention