Limit theorems for regularly varying functions of Markov chains

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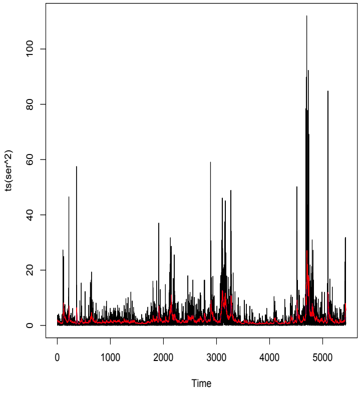
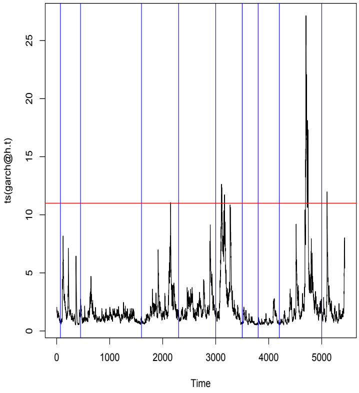
Illustrations





Motivation: characterization of limit theorems

Iid case

The α-stable limit in CLT, the large deviations of the partial sums, the ruin probabilities... are characterized by the regular variations of the margins.

Clusters of extremes

For regularly varying processes, there are clusters of extremes.
How do the clusters modify the limit characteristics?

Outline

1. Markov chains
   - Regular variation, splitting scheme and drift condition
   - Regular variation of cycles
2. Limit theorems for functions of Markov chains
   - Central Limit Theorem
   - Large deviations and ruin probabilities
3. Markov chains with extremal linear behavior

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Regularly varying sequences

Regularly varying condition of order $\alpha > 0$

A stationary sequence (X_t) is regularly varying if a non-null Radon measure μ_d is such that

$$(RV_\alpha) \qquad n\,\mathbb{P}(a_n^{-1}(X_1,\ldots,X_d) \in \cdot) \xrightarrow{v} \mu_d(\cdot)\,,$$

where (a_n) satisfies $n\,\mathbb{P}(|X| > a_n) \to 1$ and $\mu_d(tA) = t^{-\alpha}\mu_d(A)$, $t > 0$.

Definition (Basrak & Segers, 2009)

It is equivalent to the existence of the spectral tail process (Θ_t) defined for $k \geqslant 0$,

$$\mathbb{P}(|X_0|^{-1}(X_0,\ldots,X_k) \in \cdot \mid |X_0| > x) \xrightarrow{w} \mathbb{P}((\Theta_0,\ldots,\Theta_k) \in \cdot)\,, \quad x \to \infty\,.$$

Regeneration of Markov chains with an accessible atom (Doeblin, 1939)

Definition

(Φ_t) is a Markov chain of kernel P on $\mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

- A is an atom if $\exists$ a measure ν on $\mathcal{B}(\mathbb{R}^d)$ st $P(x, B) = \nu(B)$ for all $x \in A$.
- A is accessible, i.e. $\sum_k P^k(x, A) > 0$ for all $x \in \mathbb{R}^d$,

Let $(\tau_A(j))_{j \geqslant 1}$ visiting times to the set A, i.e.
$\tau_A(1) = \tau_A = \min\{k > 0 : X_k \in A\}$ and $\tau_A(j+1) = \min\{k > \tau_A(j) : X_k \in A\}$.

Regeneration cycles

1. $N_A(t) = \#\{j \geqslant 1 : \tau_A(j) \leqslant t\}$, $t \geqslant 0$, is a renewal process,
2. The cycles $(\Phi_{\tau_A(t)+1}, \ldots, \Phi_{\tau_A(t+1)})$ are iid.

Irreducible Markov chain and Nummelin scheme

Definition (Minorization condition, Meyn and Tweedie, 1993)

$\exists\ \delta > 0,\ C \in \mathcal{B}(\mathbb{R}^d)$ and a distribution ν on C such that

$$(MC_k) \qquad P^k(x, B) \geqslant \delta\nu(B), \qquad x \in C, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

(MC_1) is called the strongly aperiodic case.

If P is an irreducible aperiodic Markov chain then it satisfies (MC_k) for some $k \in \mathbb{N}$.

Nummelin splitting scheme

Under (MC_1) an enlargement of (Φ_t) on $\mathbb{R}^d \times \{0, 1\} \subset \mathbb{R}^{d+1}$ possesses an accessible atom $A = C \times \{1\} \Longrightarrow$ the enlarged Markov chain regenerates.

Main assumptions

Assume that (Φ_t) (possibly enlarged) possesses an accessible atom A, the existence of its invariant measure π and $\Phi_0 \sim \pi$.

Assume the existence of f such that:

1. There exist constants $\beta \in (0,1)$, $b > 0$ such that for any y,

$$(DC_p) \qquad \mathbb{E}(|f(\Phi_1)|^p \mid \Phi_0 = y) \leqslant \beta\,|f(y)|^p + b\,1_A(y).$$

2. $(X_t = f(\Phi_t))$ satisfies (RV_α) with index $\alpha > 0$ and spectral tail process (Θ_t).

Remarks

1. it is absolutely $(\beta-)$mixing with exponential rate,
2. $\sup_{x\in A} \mathbb{E}_x(\kappa^{\tau_A})$ for some $\kappa > 1$.
3. $(\mathrm{DC}_p) \Longrightarrow (\mathrm{DC}_{p'})$ for $0 < p' \leqslant p$.

The cluster index

Under (RV_α) denote $b_k(\pm) = \lim_{n\to\infty} n\,\mathbb{P}(\pm S_k > a_n)\,, \quad k \geqslant 1.$

Theorem (4)

Assume $(\mathbf{RV}_\alpha)$ *for some* $\alpha > 0$ *and* $(\mathbf{DC_p})$ *for positive* $p \in (\alpha - 1, \alpha)$. *Then the limits (called cluster indices)*

$$
\begin{aligned}
b_\pm : &= \lim_{k\to\infty} (b_{k+1}(\pm) - b_k(\pm)) \\
&= \lim_{k\to\infty} \mathbb{E}\Big[\Big(\sum_{t=0}^{k} \Theta_t\Big)_\pm^\alpha - \Big(\sum_{t=1}^{k} \Theta_t\Big)_\pm^\alpha\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{t=0}^{\infty} \Theta_t\Big)_\pm^\alpha - \Big(\sum_{t=1}^{\infty} \Theta_t\Big)_\pm^\alpha\Big]
\end{aligned}
$$

exist and are finite. Here (Θ_t) *is the spectral tail process of* (X_t).

Other indices

The extremal index $0 < \theta = \mathbb{E}\Big[\Big(\sup_{t\geqslant 0}\Theta_t\Big)_+^\alpha - \Big(\sup_{t\geqslant 1}\Theta_t\Big)_+^\alpha\Big] \leqslant \mathbb{E}[(\Theta_0)_+^\alpha]$.
Under (RV$_\alpha$) denote $\tilde{b}_k = \lim_{n\to\infty} n\,\mathbb{P}(\sup_{t\geqslant k} S_t > a_n)\,, \quad k \geqslant 1$.

Theorem (Under the hypothesis of the Theorem 4)

The limit (called cluster index)

$$\begin{aligned} \tilde{b} : &= \lim_{k\to\infty}(\tilde{b}_{k+1} - \tilde{b}_k) \\ &= \mathbb{E}\Big[\Big(\sup_{k\geqslant 0}\sum_{t=0}^{k}\Theta_t\Big)_\pm^\alpha - \Big(\sup_{k\geqslant 1}\sum_{t=1}^{k}\Theta_t\Big)_\pm^\alpha\Big] \end{aligned}$$

exist and is finite. Here (Θ_t) is the spectral tail process of (X_t).

Regular variation of cycles

Theorem (Under the hypothesis of the Theorem 4)

Assume (RV_α) with $\alpha > 0$ and (DC_p) with $(\alpha - 1)_+ < p < \alpha$ and $b\pm \neq 0$ then

$$\mathbb{P}_A\Big(\sup_{1\leqslant i\leqslant \tau_A} f(\Phi_i) > x\Big) \sim_{x\to\infty} \theta\, \mathbb{E}_A(\tau_A)\, \mathbb{P}(|X| > x),$$

$$\mathbb{P}_A\Big(\pm\sum_{i=1}^{\tau_A} f(\Phi_i) > x\Big) \sim_{x\to\infty} b_\pm\, \mathbb{E}_A(\tau_A)\, \mathbb{P}(|X| > x),$$

$$\mathbb{P}_A\Big(\sup_{1\leqslant i\leqslant \tau_A} \sum_{i=1}^{i} f(\Phi_i) > x\Big) \sim_{x\to\infty} \tilde{b}\, \mathbb{E}_A(\tau_A)\, \mathbb{P}(|X| > x),$$

Remarks

1. We always have $\mathbb{E}_A(\tau_A)\,\mathbb{P}(X > x) = \mathbb{E}_A[\sum_{i=1}^{\tau_A} 1_{f(\Phi_i)>x}]$,
2. If τ_A was independent of (X_t) then $\mathbb{P}_A(S_A(1) > x) \sim_{x\to\infty} \mathbb{E}_A(\tau_A)\,\mathbb{P}(X > x)$.

Outline

Stable Central Limit Theorem

Theorem (Under the hypothesis of the Theorem 4)

If $0 < \alpha < 2$, $\alpha \neq 1$ *and* X *is centered if* $1 < \alpha < 2$ *then* $a_n^{-1} S_n \xrightarrow{d} \xi_\alpha$, *with the characteristic function* ξ_α *given by* $\exp(-|x|^\alpha \chi_\alpha(x, b_+, b_-))$, *where*

$$\chi_\alpha(x, b_+, b_-) = \frac{\Gamma(2-\alpha)}{1-\alpha}\left((b_+ + b_-)\cos\left(\frac{\pi\alpha}{2}\right) - isgn(x)(b_+ - b_-)\sin\left(\frac{\pi\alpha}{2}\right)\right).$$

Precise large deviations and ruin probabilities

Theorem (Under the hypothesis of the Theorem 4)

If $0 < \alpha < 1$ *then* $\lim_{n\to\infty} \sup_{x \geqslant b_n} \left| \frac{\mathbb{P}(\pm S_n > x)}{n\,\mathbb{P}(|X| > x)} - b_\pm \right| = 0$*. If* $\alpha > 1$ *and* X *is centered then* $\lim_{n\to\infty} \sup_{b_n \leqslant x \leqslant c_n} \left| \frac{\mathbb{P}(\pm S_n > x)}{n\,\mathbb{P}(|X| > x)} - b_\pm \right| = 0$ *with* $\sqrt{n} = o(b_n)$ *if* $\alpha > 2$, $n^{1/\alpha} L(n) = o(b_n)$ *otherwise and* $\mathbb{P}(\tau_A > n) = o(n\mathbb{P}(|X| > c_n))$.

Theorem (Under the hypothesis of the Theorem 4)

Assume that (X_t) *is regularly varying with index* $\alpha > 1$ *and centered. Then we have for any* $\rho > 0$,

$$\mathbb{P}\Big(\sup_{t \geqslant 0} (S_t - \rho t) > x \Big) \sim \frac{\tilde{b}\, x\, \mathbb{P}(|X| > x)}{(\alpha - 1)\rho}, \quad x \to \infty.$$

Consequence: if $b_+ \neq \tilde{b}$ the functional CLT cannot hold.

Outline

The AR(1) model

Definition (AR(1) model)

The AR(1) model is the solution of $X_t = \phi X_{t-1} + Z_t$, $|\phi| < 1$ with (Z_t) is an iid regularly varying sequence if order $\alpha > 0$.

Proposition

We have $(X_t) \in RV_\alpha$ and the conclusions of the theorems hold with $\Theta_t = \phi^t$, $t > 0$.

Markov chains with extremal linear behavior (Kesten, 1974, Goldie, 1991, Segers, 2007, Mirek, 2011)

Assume (A, B) is absolutely continuous on $\mathbb{R}^+ \times \mathbb{R}$ with $\mathbb{E}A^\alpha = 1$, $X_t = \Psi_t(X_{t-1})$ with iid iterated Lipschitz functions Ψ_t with negative top Lyapunov exponent and

$$A_t X_{t-1} - |B_t| \leqslant X_t \leqslant A_t X_{t-1} + |B_t|.$$

Proposition

We have $(X_t) \in RV_\alpha$ and the conclusions of the theorems hold with

$$\Theta_t = \prod_{i=1}^{t} A_i, \qquad t \geqslant 0.$$

The GARCH(1,1) model

Definition (Bollerslev, 1986)

The GARCH(1,1) model (X_t) is the solution of $X_t = \sigma_t Z_t$, $t \in \mathbb{Z}$ with (Z_t) is an iid mean zero and unit variance sequence of random variables and (σ_t^2) satisfies the stochastic recurrence equation

$$\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2, \qquad t \in \mathbb{Z}\,.$$

Proposition

If $X_0 \in RV_\alpha$ then we have

$$\mathbb{P}(|X_0|^{-1}(X_0, \ldots, X_t) \in \cdot \mid |X_0| > x) \to \frac{1}{\mathbb{E}|Z_0|^\alpha}\mathbb{E}\Big[|Z_0|^\alpha 1_{(Z_0, Z_1\Pi_i^{0.5}, \ldots, Z_t\Pi_t^{0.5}) \in |Z_0|\,\cdot}\Big]\,,$$

where $\Pi_t = A_1 \cdots A_t$ with $A_t = \alpha_1 Z_{t-1}^2 + \beta_1$.

Conclusion

- Cluster indices $b_\pm$, $\tilde{b}$ together with θ determine the limit theorems of dependent and regularly varying variables,
- Under the hypothesis of the theorems

$$\mathbb{P}(S_n > x) \sim_{n\to\infty} b_+ n\mathbb{P}(X > x) \quad \text{for} \quad b_n \leqslant x \leqslant c_n$$

  with $b_+/\theta >> 1$. Consequences in risk management....
- Inference of the cluster indices $b_\pm$, $\tilde{b}$...

Conclusion

Thank you for your attention!