Credit Risk in Lévy Libor Modeling: Rating Based Approach

Zorana Grbac

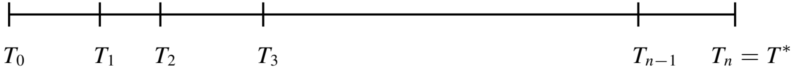
Department of Math. Stochastics, University of Freiburg

Joint work with Ernst Eberlein

Croatian Quants Day
University of Zagreb, 9th April 2010

Libor market models – introduction

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$



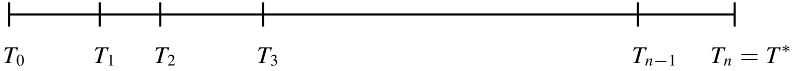
Libor market models – introduction

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$

Default-free zero coupon bonds: $B(\cdot, T_1), \ldots, B(\cdot, T_n)$

Libor market models – introduction

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$



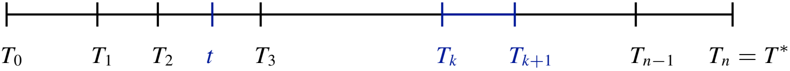
Default-free zero coupon bonds: $B(\cdot, T_1), \ldots, B(\cdot, T_n)$

Forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L(t, T_k) = \frac{1}{\delta_k}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right)$$

Libor market models – introduction

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$



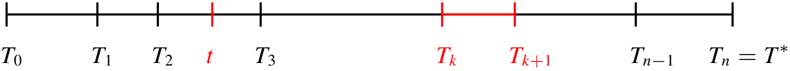
Default-free zero coupon bonds: $B(\cdot, T_1), \ldots, B(\cdot, T_n)$

Forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L(t, T_k) = \frac{1}{\delta_k}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right)$$

Libor market models – introduction

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$



Defaultable zero coupon bonds with credit ratings: $B_C(\cdot, T_1), \ldots, B_C(\cdot, T_n)$

Defaultable forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L_C(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_C(t, T_k)}{B_C(t, T_{k+1})} - 1\right)$$

Libor modeling

- modeling under forward martingale measures, i.e. risk-neutral measures that use zero-coupon bonds as numeraires
- on a given stochastic basis, construct a family of Libor rates $L(\cdot, T_k)$ and a collection of mutually equivalent probability measures $\mathbb{P}_{T_k}$ such that

$$\left(\frac{B(t,T_j)}{B(t,T_k)}\right)_{0\le t\le T_k\wedge T_j}$$

are $\mathbb{P}_{T_k}$-local martingales

Libor modeling

- modeling under forward martingale measures, i.e. risk-neutral measures that use zero-coupon bonds as numeraires
- on a given stochastic basis, construct a family of Libor rates $L(\cdot, T_k)$ and a collection of mutually equivalent probability measures $\mathbb{P}_{T_k}$ such that

$$\left(\frac{B(t,T_j)}{B(t,T_k)}\right)_{0\le t\le T_k\wedge T_j}$$

are $\mathbb{P}_{T_k}$-local martingales

- model additionally defaultable Libor rates $L_C(\cdot, T_k)$ such that

$$\left(\frac{B_C(t,T_j)}{B(t,T_k)}\right)_{0\le t\le T_k\wedge T_j}$$

are $\mathbb{P}_{T_k}$-local martingales

Credit risk with ratings

- Credit risk: risk associated to any kind of credit-linked events (default, changes in the credit quality etc.)

- Credit rating: measure of the credit quality (i.e. tendency to default) of a company

Credit risk with ratings

- Credit risk: risk associated to any kind of credit-linked events (default, changes in the credit quality etc.)
- Credit rating: measure of the credit quality (i.e. tendency to default) of a company
- Credit ratings identified with elements of a finite set $\mathcal{K} = \{1, 2, \ldots, K\}$, where 1 is the best possible rating and K is the default event
- Credit migration is modeled by a conditional Markov chain C with state space $\mathcal{K}$, where K is the absorbing state
- Default time τ: the first time when C reaches state K, i.e.

$$\tau = \inf\{t > 0 : C_t = K\}$$

Defaultable bonds with ratings

- Consider defaultable bonds with credit migration process C and fractional recovery of Treasury value $q = (q_1, \ldots, q_{K-1})$ upon default

- Payoff of such a bond at maturity equals

$$\begin{aligned} B_C(T_k, T_k) &= \mathbf{1}_{\{\tau > T_k\}} + q_{C_{\tau-}} \mathbf{1}_{\{\tau \leq T_k\}} \\ &= \sum_{i=1}^{K-1} \mathbf{1}_{\{C_{T_k} = i\}} + q_{C_{\tau-}} \mathbf{1}_{\{C_{T_k} = K\}}, \end{aligned}$$

  where $C_{\tau-}$ denotes the pre-default rating.

Defaultable bonds with ratings

- Consider defaultable bonds with credit migration process C and fractional recovery of Treasury value $q = (q_1, \ldots, q_{K-1})$ upon default

- Payoff of such a bond at maturity equals

$$\begin{aligned} B_C(T_k, T_k) &= \mathbf{1}_{\{\tau > T_k\}} + q_{C_{\tau-}} \mathbf{1}_{\{\tau \le T_k\}} \\ &= \sum_{i=1}^{K-1} \mathbf{1}_{\{C_{T_k}=i\}} + q_{C_{\tau-}} \mathbf{1}_{\{C_{T_k}=K\}}, \end{aligned}$$

  where $C_{\tau-}$ denotes the pre-default rating.

- time-t price of such a defaultable bond can be expressed as

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau-}} B(t, T_k) \mathbf{1}_{\{C_t=K\}},$$

  where $B_i(t, T_k)$ represents the bond price at time t provided that the bond has rating i during the time interval $[0, t]$.
  We have $B_i(T_k, T_k) = 1$, for all i.

Canonical construction of C

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a given complete stochastic basis.

- Let $\Lambda = (\Lambda_t)_{0 \le t \le T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

which is the stochastic infinitesimal generator of C.

Canonical construction of C

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T^*}, \mathbb{P}_{T^*})$ be a given complete stochastic basis.

- Let $\Lambda = (\Lambda_t)_{0\leq t\leq T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

  which is the stochastic infinitesimal generator of C.
- Enlarge probability space

$$(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}) \rightarrow (\tilde{\Omega}, \mathcal{G}_{T^*}, \mathbb{Q}_{T^*})$$

  and use canonical construction to construct C (Bielecki and Rutkowski, 2002)

Canonical construction of C

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a given complete stochastic basis.

- Let $\Lambda = (\Lambda_t)_{0 \le t \le T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

  which is the stochastic infinitesimal generator of C.

- Enlarge probability space

$$(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}) \to (\tilde{\Omega}, \mathcal{G}_{T^*}, \mathbb{Q}_{T^*})$$

  and use canonical construction to construct C (Bielecki and Rutkowski, 2002)

The process C is a *conditional Markov chain* relative to $\mathbb{F}$ if for every $0 \le t \le s$ and any function $h : \mathcal{K} \to \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s) | \mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s) | \mathcal{F}_t \vee \sigma(C_t)],$$

where $\mathbb{F}^C = (\mathcal{F}_t^C)$ denotes the filtration generated by C.

(skip details)

Canonical construction - details

- Let $\Lambda = (\Lambda_t)_{0 \le t \le T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process on $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*})$

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

  where λ_{ij} are nonnegative processes, integrable on every $[0, t]$ and $\lambda_{ii}(t) = -\sum_{j \in \mathcal{K} \setminus \{i\}} \lambda_{ij}(t)$.
- Let $\mu = (\delta_{ij}, j \in \mathcal{K})$ be a probability distribution on $\overline{\Omega} = \mathcal{K}$.
- Define

$$(\tilde{\Omega}, \mathcal{G}_{T^*}, \mathbb{Q}_{T^*}) = (\Omega \times \Omega^U \times \overline{\Omega}, \mathcal{F}_{T^*} \otimes \mathcal{F}^U \otimes 2^{\overline{\Omega}}, \mathbb{P}_{T^*} \otimes \mathbb{P}^U \otimes \mu),$$

- On $(\Omega^U, \mathcal{F}^U, \mathbb{P}^U)$ a sequence $(U_{i,j})$, $i, j \in \mathbb{N}$, of mutually independent random variables, uniformly distributed on $[0, 1]$.

- The jump times τ_k are constructed recursively as

$$\tau_k := \tau_{k-1} + \inf\left\{ t \geq 0 : \exp\left(\int_{\tau_{k-1}}^{\tau_{k-1}+t} \lambda_{\overline{C}_{k-1}\overline{C}_{k-1}}(u) \mathrm{d}u \right) \leq U_{1,k} \right\},$$

  with $\tau_0 := 0$.

- The new state at the jump time τ_k is defined as

$$\overline{C}_k := \mathbb{C}(U_{2,k}, \overline{C}_{k-1}, \tau_k),$$

  with $\overline{C}_0(\omega, \omega^U, \overline{\omega}) = \overline{\omega}$ and where $\mathbb{C} : [0,1] \times \mathcal{K} \times \mathbb{R}_+ \times \Omega \rightarrow \mathcal{K}$ is any mapping such that for any $i, j \in \mathcal{K}$, $i \neq j$, it holds

$$\text{Leb}\left(\{ u \in [0,1] : \mathbb{C}(u, i, t) = j \} \right) = -\frac{\lambda_{ij}(t)}{\lambda_{ii}(t)},$$

  if $\lambda_{ii}(t) < 0$ and 0, if $\lambda_{ii}(t) = 0$.

- Finally, for every $t \geq 0$

$$C_t := \overline{C}_{k-1}, \quad \text{for } \; t \in [\tau_{k-1}, \tau_k), \; k \geq 1.$$

Definition

The process C is a conditional Markov chain relative to $\mathbb{F}$, i.e. for every $0 \le t \le s$ and any function $h : \mathcal{K} \to \mathbb{R}$ it holds

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)],$$

where $\mathbb{F}^C = (\mathcal{F}_t^C)$ denotes the filtration generated by C.

Proposition

The conditional expectations with respect to enlarged σ-algebras can be expressed in terms of $\mathcal{F}_t$-conditional expectations. It holds

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t \vee \sigma(C_t)] = \sum_{i=1}^{K} \mathbf{1}_{\{C_t=i\}} \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[Y\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]},$$

for any $\mathcal{G}$-measurable random variable Y.

Properties of C

(a) for every $t \leq s \leq u$ and any function $h : \mathcal{K} \to \mathbb{R}$ a stronger version of conditional Markov property holds:

$$\mathbb{E}_{\mathbb{Q}_{T*}}[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T*}}[h(C_s)|\mathcal{F}_u \vee \sigma(C_t)]$$

(b) for every $t \leq s$ and $B \in \mathcal{F}_t^C$:

$$\mathbb{E}_{\mathbb{Q}_{T*}}[\mathbf{1}_B|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T*}}[\mathbf{1}_B|\mathcal{F}_t]$$

(c) $\mathbb{F}$-*conditional* Chapman-Kolmogorov equation

$$P(t,s) = P(t,u)P(u,s),$$

where $P(t,s) = [p_{ij}(t,s)]_{i,j \in \mathcal{K}}$ and

$$p_{ij}(t,s) := \frac{\mathbb{Q}_{T*}(C_s = j, C_t = i|\mathcal{F}_s)}{\mathbb{Q}_{T*}(C_t = i|\mathcal{F}_s)}$$

(d) $\mathbb{F}$-*conditional* forward Kolmogorov equation

$$\frac{\mathrm{d}P(t,s)}{\mathrm{d}s} = P(t,s)\Lambda(s)$$

The progressive enlargement of filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T^*}$, where

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C,$$

satisfies the $(\mathcal{H})$-hypothesis:

$(\mathcal{H})$ Every local $\mathbb{F}$-martingale is a local $\mathbb{G}$-martingale.

The progressive enlargement of filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T^*}$, where

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C,$$

satisfies the $(\mathcal{H})$-hypothesis:

$(\mathcal{H})$ Every local $\mathbb{F}$-martingale is a local $\mathbb{G}$-martingale.

It is well-known that $(\mathcal{H})$ is equivalent to

$(\mathcal{H}1)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t]$,

for any bounded, $\mathcal{F}_t^C$-measurable random variable Y.

The progressive enlargement of filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$, where

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C,$$

satisfies the $(\mathcal{H})$-hypothesis:

$(\mathcal{H})$ Every local $\mathbb{F}$-martingale is a local $\mathbb{G}$-martingale.

It is well-known that $(\mathcal{H})$ is equivalent to

$(\mathcal{H}1)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t]$,

for any bounded, $\mathcal{F}_t^C$-measurable random variable Y.

But this follows easily from property

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_B|\mathcal{F}_t], \qquad t \leq s, B \in \mathcal{F}_t^C,$$

which is proved as a consequence of the canonical construction.

Risk-free Lévy Libor model

(Eberlein and Özkan, 2005)

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis.

- as driving process take a time-inhomogeneous Lévy process $X = (X^1, \ldots, X^d)$ whose Lévy measure satisfies certain integrability conditions, i.e.

Risk-free Lévy Libor model

(Eberlein and Özkan, 2005)

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis.

- as driving process take a time-inhomogeneous Lévy process $X = (X^1, \ldots, X^d)$ whose Lévy measure satisfies certain integrability conditions, i.e. an adapted, cádlág process with $X_0 = 0$ and such that
  - (1) X has independent increments
  - (2) the law of X_t is given by its characteristic function

$$\mathbb{E}[\exp(\mathrm{i}\langle u, X_t \rangle)] = \exp\left(\int_0^t \theta_s(\mathrm{i}u) \mathrm{d}s \right) \quad \text{with}$$

$$\theta_s(\mathrm{i}u) = \mathrm{i}\langle u, b_s \rangle - \frac{1}{2}\langle u, c_s u \rangle + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\langle u, x \rangle} - 1 - \mathrm{i}\langle u, x \rangle F_s^{T^*}(\mathrm{d}x) \right).$$

- X is a special semimartingale with canonical decomposition

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

where W^{T^*} denotes a $\mathbb{P}_{T^*}$-standard Brownian motion and μ is the random measure of jumps of X with $\mathbb{P}_{T^*}$-compensator ν^{T^*}.
We assume that $b = 0$.

Construction of Libor rates

Begin by specifying the dynamics of the most distant Libor rate under $\mathbb{P}_{T^*}$ (regarded as the forward measure associated with date T^*)

$$L(t, T_{n-1}) = L(0, T_{n-1}) \exp\left(\int_0^t b^L(s, T_{n-1}) \mathrm{d}s + \int_0^t \sigma(s, T_{n-1}) \mathrm{d}X_s \right),$$

where the drift is chosen in such a way that $L(\cdot, T_{n-1})$ becomes a $\mathbb{P}_{T^*}$-martingale:

$$\begin{aligned} b^L(s, T_{n-1}) &= -\frac{1}{2} \langle \sigma(s, T_{n-1}), c_s \sigma(s, T_{n-1}) \rangle \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\langle \sigma(s, T_{n-1}), x \rangle} - 1 - \langle \sigma(s, T_{n-1}), x \rangle \right) F_s^{T^*}(\mathrm{d}x). \end{aligned}$$

Construction of Libor rates

Begin by specifying the dynamics of the most distant Libor rate under $\mathbb{P}_{T^*}$ (regarded as the forward measure associated with date T^*)

$$L(t, T_{n-1}) = L(0, T_{n-1}) \exp\left(\int_0^t b^L(s, T_{n-1})\mathrm{d}s + \int_0^t \sigma(s, T_{n-1})\mathrm{d}X_s\right),$$

where the drift is chosen in such a way that $L(\cdot, T_{n-1})$ becomes a $\mathbb{P}_{T^*}$-martingale:

$$\begin{aligned} b^L(s, T_{n-1}) &= -\frac{1}{2}\langle \sigma(s, T_{n-1}), c_s\sigma(s, T_{n-1})\rangle \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\langle \sigma(s,T_{n-1}),x\rangle} - 1 - \langle \sigma(s, T_{n-1}), x\rangle\right) F_s^{T^*}(\mathrm{d}x). \end{aligned}$$

Next, define the forward measure $\mathbb{P}_{T_{n-1}}$ associated with date T_{n-1} via

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{n-1}}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \frac{1 + \delta_{n-1}L(t, T_{n-1})}{1 + \delta_{n-1}L(0, T_{n-1})}$$

and proceed with modeling of $L(\cdot, T_{n-2})$...

General step: for each T_k

(i) define the forward measure $\mathbb{P}_{T_{k+1}}$ via

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(t,T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T^*)}{B(0,T_{k+1})} \frac{B(t,T_{k+1})}{B(t,T^*)}.$$

(ii) the dynamics of the Libor rate $L(\cdot, T_k)$ under this measure

$$L(t,T_k) = L(0,T_k) \exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \sigma(s,T_k)\mathrm{d}X_s^{T_{k+1}} \right), \tag{1}$$

where

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x)$$

General step: for each T_k

(i) define the forward measure $\mathbb{P}_{T_{k+1}}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\bigg|_{\mathcal{F}_t} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(t,T_l)}{1+\delta_l L(0,T_l)} = \frac{B(0,T^*)}{B(0,T_{k+1})} \frac{B(t,T_{k+1})}{B(t,T^*)}.$$

(ii) the dynamics of the Libor rate $L(\cdot,T_k)$ under this measure

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \sigma(s,T_k)\mathrm{d}X_s^{T_{k+1}}\right), \tag{1}$$

where

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}})(\mathrm{d}s,\mathrm{d}x)$$

with $\mathbb{P}_{T_{k+1}}$-Brownian motion $W^{T_{k+1}}$ and

$$\nu^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=k+1}^{n-1}\left(\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}(e^{\langle\sigma(s,T_l),x\rangle}-1)+1\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

The drift term $b^L(s,T_k)$ is chosen such that $L(\cdot,T_k)$ becomes a $\mathbb{P}_{T_{k+1}}$-martingale.

- This construction guarantees that the forward bond price processes

$$\left(\frac{B(t,T_j)}{B(t,T_k)}\right)_{0\le t\le T_j\wedge T_k}$$

are martingales for all $j=1,\dots,n$ under the forward measure $\mathbb{P}_{T_k}$ associated with the date T_k ($k=1,\dots,n$).

- The arbitrage-free price at time t of a contingent claim with payoff X at maturity T_k is given by

$$\pi_t^X = B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[X|\mathcal{F}_t].$$

How to include credit risk with ratings in the Lévy Libor model?

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

(2) Adopt the backward construction of Eberlein and Özkan (2005) to model default-free Libor rates

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

(2) Adopt the backward construction of Eberlein and Özkan (2005) to model default-free Libor rates

(3) Define and model the pre-default term structure of rating-dependent Libor rates

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

(2) Adopt the backward construction of Eberlein and Özkan (2005) to model default-free Libor rates

(3) Define and model the pre-default term structure of rating-dependent Libor rates

To include credit migration between different rating classes:

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

(2) Adopt the backward construction of Eberlein and Özkan (2005) to model default-free Libor rates

(3) Define and model the pre-default term structure of rating-dependent Libor rates

To include credit migration between different rating classes:

(4) Enlarge probability space: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{T^*}) \longrightarrow (\widetilde{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$ and construct the migration process C

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

(2) Adopt the backward construction of Eberlein and Özkan (2005) to model default-free Libor rates

(3) Define and model the pre-default term structure of rating-dependent Libor rates

To include credit migration between different rating classes:

(4) Enlarge probability space: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{T^*}) \rightarrow (\widetilde{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$ and construct the migration process C

(5) The $(\mathcal{H})$-hypothesis $\Rightarrow X$ *remains* a time-inhomogeneous Lévy process with respect to $\mathbb{Q}_{T^*}$ and $\mathbb{G}$ with the *same* characteristics

How to include credit risk with ratings in the Lévy Libor model?

(1) Use defaultable bonds with ratings to introduce a concept of defaultable Libor rates

(2) Adopt the backward construction of Eberlein and Özkan (2005) to model default-free Libor rates

(3) Define and model the pre-default term structure of rating-dependent Libor rates

To include credit migration between different rating classes:

(4) Enlarge probability space: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{T^*}) \to (\widetilde{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$ and construct the migration process C

(5) The $(\mathcal{H})$-hypothesis $\Rightarrow$ X *remains* a time-inhomogeneous Lévy process with respect to $\mathbb{Q}_{T^*}$ and $\mathbb{G}$ with the *same* characteristics

(6) Define on this space the forward measures $\mathbb{Q}_{T_k}$ by:

for each tenor date T_k $\mathbb{Q}_{T_k}$ is obtained from $\mathbb{Q}_{T^*}$ in the same way as $\mathbb{P}_{T_k}$ from $\mathbb{P}_{T^*}$ $(k = 1, \ldots, n-1)$

Conditional Markov chain C under forward measures

Note that

$$\frac{d\mathbb{Q}_{T_k}}{d\mathbb{Q}_{T^*}} = \psi^k,$$

where ψ^k is an $\mathcal{F}_{T_k}$-measurable random variable with expectation 1.

Conditional Markov chain C under forward measures

Note that

$$\frac{d\mathbb{Q}_{T_k}}{d\mathbb{Q}_{T^*}} = \psi^k,$$

where ψ^k is an $\mathcal{F}_{T_k}$-measurable random variable with expectation 1.

Theorem

Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$. Then C is a conditional Markov chain with respect to every forward measure $\mathbb{Q}_{T_k}$ and

$$p_{ij}^{\mathbb{Q}_{T_k}}(t,s) = p_{ij}^{\mathbb{Q}_{T^*}}(t,s)$$

i.e. the matrices of transition probabilities under $\mathbb{Q}_{T^*}$ and $\mathbb{Q}_{T_k}$ are the same.

Conditional Markov chain C under forward measures

Note that

$$\frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T^*}} = \psi^k,$$

where ψ^k is an $\mathcal{F}_{T_k}$-measurable random variable with expectation 1.

Theorem

Let C be a canonically constructed conditional Markov chain with respect to $\mathbb{Q}_{T^*}$. Then C is a conditional Markov chain with respect to every forward measure $\mathbb{Q}_{T_k}$ and

$$p_{ij}^{\mathbb{Q}_{T_k}}(t,s) = p_{ij}^{\mathbb{Q}_{T^*}}(t,s)$$

i.e. the matrices of transition probabilities under $\mathbb{Q}_{T^*}$ and $\mathbb{Q}_{T_k}$ are the same.

Theorem

The $(\mathcal{H})$-hypothesis holds under all $\mathbb{Q}_{T_k}$, i.e. every $(\mathbb{F}, \mathbb{Q}_{T_k})$-local martingale is a $(\mathbb{G}, \mathbb{Q}_{T_k})$-local martingale.

Rating-dependent Libor rates

- The forward Libor rate for credit rating class i

$$L_i(t,T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t,T_k)}{B_i(t,T_{k+1})} - 1\right), \quad i = 1,2,\ldots,K-1$$

We put $L_0(t,T_k) := L(t,T_k)$ (default-free Libor rates).

Rating-dependent Libor rates

- The forward Libor rate for credit rating class i

$$L_i(t,T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t,T_k)}{B_i(t,T_{k+1})} - 1\right), \quad i = 1,2,\dots,K-1$$

We put $L_0(t,T_k) := L(t,T_k)$ (default-free Libor rates).

- The corresponding discrete-tenor forward inter-rating spreads

$$H_i(t,T_k) := \frac{L_i(t,T_k) - L_{i-1}(t,T_k)}{1 + \delta_k L_{i-1}(t,T_k)}$$

Observe that the Libor rate for the rating i can be expressed as

$$\begin{aligned} 1 + \delta_k L_i(t, T_k) &= (1 + \delta_k L_{i-1}(t, T_k))(1 + \delta_k H_i(t, T_k)) \\ &= \underbrace{(1 + \delta_k L(t, T_k))}_{\text{default-free Libor}} \prod_{j=1}^{i} \underbrace{(1 + \delta_k H_j(t, T_k))}_{\text{spread } j-1 \to j} \end{aligned}$$

Observe that the Libor rate for the rating i can be expressed as

$$\begin{aligned} 1+\delta_k L_i(t,T_k) &= (1+\delta_k L_{i-1}(t,T_k))(1+\delta_k H_i(t,T_k)) \\ &= \underbrace{(1+\delta_k L(t,T_k))}_{\text{default-free Libor}} \prod_{j=1}^{i} \underbrace{(1+\delta_k H_j(t,T_k))}_{\text{spread } j-1\to j} \end{aligned}$$

Idea: model $H_j(\cdot, T_k)$ as exponential semimartingales and thus ensure automatically the *monotonicity* of Libor rates w.r.t. the credit rating:

$$L(t,T_k) \le L_1(t,T_k) \le \cdots \le L_{K-1}(t,T_k)$$

$\Longrightarrow$ worse credit rating, higher interest rate

Pre-default term structure of rating-dependent Libor rates

For each rating i and tenor date T_k we model $H_i(\cdot, T_k)$ as

$$H_i(t, T_k) = H_i(0, T_k) \exp\left(\int_0^t b^{H_i}(s, T_k) \mathrm{d}s + \int_0^t \gamma_i(s, T_k) \mathrm{d}X_s^{T_{k+1}} \right) \quad (2)$$

with initial condition

$$H_i(0, T_k) = \frac{1}{\delta_k} \left(\frac{B_i(0, T_k) B_{i-1}(0, T_{k+1})}{B_{i-1}(0, T_k) B_i(0, T_{k+1})} - 1 \right).$$

$X^{T_{k+1}}$ is defined as earlier and $b^{H_i}(s, T_k)$ is the drift term (we assume $b^{H_i}(s, T_k) = 0$, for $s > T_k \Rightarrow H_i(t, T_k) = H_i(T_k, T_k)$, for $t \geq T_k$).

Pre-default term structure of rating-dependent Libor rates

For each rating i and tenor date T_k we model $H_i(\cdot, T_k)$ as

$$H_i(t, T_k) = H_i(0, T_k) \exp\left(\int_0^t b^{H_i}(s, T_k) \mathrm{d}s + \int_0^t \gamma_i(s, T_k) \mathrm{d}X_s^{T_{k+1}} \right) \tag{2}$$

with initial condition

$$H_i(0, T_k) = \frac{1}{\delta_k} \left(\frac{B_i(0, T_k) B_{i-1}(0, T_{k+1})}{B_{i-1}(0, T_k) B_i(0, T_{k+1})} - 1 \right).$$

$X^{T_{k+1}}$ is defined as earlier and $b^{H_i}(s, T_k)$ is the drift term (we assume $b^{H_i}(s, T_k) = 0$, for $s > T_k \Rightarrow H_i(t, T_k) = H_i(T_k, T_k)$, for $t \geq T_k$).

$\Rightarrow$ the forward Libor rate $L_i(\cdot, T_k)$ is obtained from relation

$$1 + \delta_k L_i(t, T_k) = (1 + \delta_k L(t, T_k)) \prod_{j=1}^{i} (1 + \delta_k H_j(t, T_k)).$$

Theorem

Assume that $L(\cdot, T_k)$ and $H_i(\cdot, T_k)$ are given by (1) and (2). Then:

(a) *The rating-dependent forward Libor rates satisfy for every T_k and $t \leq T_k$*

$$L(t, T_k) \leq L_1(t, T_k) \leq \cdots \leq L_{K-1}(t, T_k),$$

i.e. Libor rates are monotone with respect to credit ratings.

(b) *The dynamics of the Libor rate $L_i(\cdot, T_k)$ under $\mathbb{P}_{T_{k+1}}$ is given by*

$$L_i(t, T_k) = L_i(0, T_k) \exp\left(\int_0^t b^{L_i}(s, T_k) \mathrm{d}s + \int_0^t \sqrt{c_s}\sigma_i(s, T_k) \mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} S_i(s, x, T_k)(\mu - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x) \right),$$

where

$$\begin{aligned}\sigma_i(s,T_k) &:= \ell_i(s-,T_k)^{-1}\Big(\ell_{i-1}(s-,T_k)\sigma_{i-1}(s,T_k) + h_i(s-,T_k)\gamma_i(s,T_k)\Big)\\ &= \ell_i(s-,T_k)^{-1}\Big[\ell(s-,T_k)\sigma(s,T_k) + \sum_{j=1}^{i} h_j(s-,T_k)\gamma_j(s,T_k)\Big]\end{aligned}$$

represents the volatility of the Brownian part and

$$S_i(s,x,T_k) := \ln\Big(1 + \ell_i(s-,T_k)^{-1}(\beta_i(s,x,T_k) - 1)\Big)$$

controls the jump size. Here we set

$$h_i(s,T_k) := \frac{\delta_k H_i(s,T_k)}{1+\delta_k H_i(s,T_k)},$$

$$\ell_i(s,T_k) := \frac{\delta_k L_i(s,T_k)}{1+\delta_k L_i(s,T_k)},$$

and

$$\begin{aligned}\beta_i(s,x,T_k) &:= \beta_{i-1}(s,x,T_k)\Big(1 + h_i(s-,T_k)(e^{\langle\gamma_i(s,T_k),x\rangle} - 1)\Big)\\ &= \Big(1 + \ell(s-,T_k)(e^{\langle\sigma(s,T_k),x\rangle} - 1)\Big)\\ &\quad\times\prod_{j=1}^{i}\Big(1 + h_j(s-,T_k)(e^{\langle\gamma_j(s,T_k),x\rangle} - 1)\Big).\end{aligned}$$

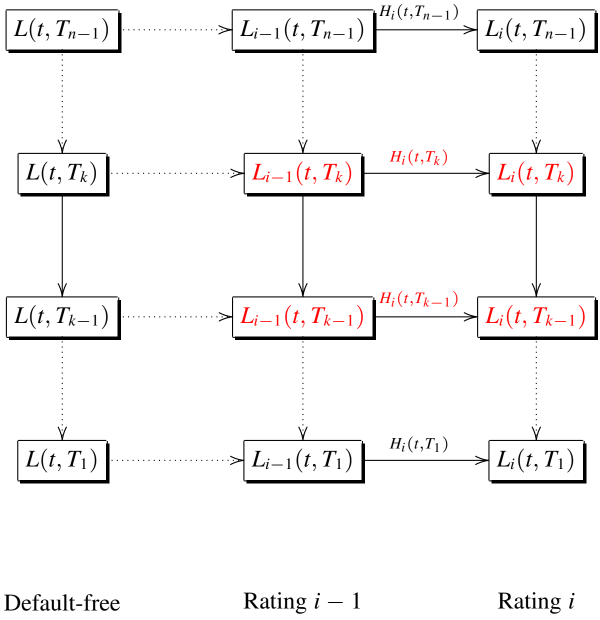


Figure: Connection between subsequent Libor rates

No-arbitrage condition for the rating based model

Recall the defaultable bond price process with fractional recovery of Treasury value q

$$B_C(t,T_k) = \sum_{i=1}^{K-1} B_i(t,T_k)\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} B(t,T_k)\mathbf{1}_{\{C_t=K\}}.$$

No-arbitrage condition for the rating based model

Recall the defaultable bond price process with fractional recovery of Treasury value q

$$B_C(t,T_k) = \sum_{i=1}^{K-1} B_i(t,T_k)\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}}\, B(t,T_k)\mathbf{1}_{\{C_t=K\}}.$$

No-arbitrage: the forward bond price process

$$\frac{B_C(\cdot,T_k)}{B(\cdot,T_j)}$$

must be a $\mathbb{Q}_{T_j}$-local martingale for every $k,j = 1,\ldots,n-1$.

No-arbitrage condition for the rating based model

Recall the defaultable bond price process with fractional recovery of Treasury value q

$$B_C(t,T_k) = \sum_{i=1}^{K-1} B_i(t,T_k)\mathbf{1}_{\{C_t=i\}} + q_{C_{\tau-}} B(t,T_k)\mathbf{1}_{\{C_t=K\}}.$$

Or equivalently: the forward bond price process

$$\frac{B_C(\cdot,T_k)}{B(\cdot,T_j)} = \frac{B_C(\cdot,T_k)}{B(\cdot,T_k)} \underbrace{\frac{B(\cdot,T_j)}{B(\cdot,T_k)}}_{\frac{d\mathbb{Q}_{T_k}}{d\mathbb{Q}_{T_j}}}$$

must be a $\mathbb{Q}_{T_k}$-local martingale for every $k = 1,\dots,n-1$.

We postulate that the forward bond price process is given by

$$\frac{B_C(t,T_k)}{B(t,T_k)} := \sum_{i=1}^{K-1} \underbrace{\prod_{j=1}^{i}\prod_{l=0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)}}_{:=\mathbb{H}(t,T_k,i)} e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} \mathbf{1}_{\{C_t=K\}}$$

$$= \sum_{i=1}^{K-1} \mathbb{H}(t,T_k,i) e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} \mathbf{1}_{\{C_t=K\}}, \qquad (3)$$

where λ_i is some $\mathbb{F}$-adapted process that is integrable on $[0, T^*]$. (go to DFM)

We postulate that the forward bond price process is given by

$$\frac{B_C(t,T_k)}{B(t,T_k)} := \sum_{i=1}^{K-1} \underbrace{\prod_{j=1}^{i}\prod_{l=0}^{k-1} \frac{1}{1+\delta_l H_j(t,T_l)}}_{:=\mathbb{H}(t,T_k,i)} e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} \mathbf{1}_{\{C_t=K\}}$$

$$= \sum_{i=1}^{K-1} \mathbb{H}(t,T_k,i) e^{\int_0^t \lambda_i(s)\mathrm{d}s} \mathbf{1}_{\{C_t=i\}} + q_{C_{\tau\_}} \mathbf{1}_{\{C_t=K\}}, \qquad (3)$$

where λ_i is some $\mathbb{F}$-adapted process that is integrable on $[0,T^*]$. (go to DFM)

Note that this specification is consistent with the definition of H_i which implies the following connection of bond prices and inter-rating spreads:

$$\frac{B_j(t,T_k)}{B_{j-1}(t,T_k)} = \frac{B_j(t,T_{k-1})}{B_{j-1}(t,T_{k-1})} \frac{1}{1+\delta_{k-1}H_j(t,T_{k-1})}$$

and relation

$$\frac{B_i(t,T_k)}{B(t,T_k)} = \frac{B_1(t,T_k)}{B(t,T_k)} \prod_{j=2}^{i} \frac{B_j(t,T_k)}{B_{j-1}(t,T_k)}.$$

Lemma

Let T_k be a tenor date and assume that $H_j(\cdot, T_k)$ are given by (2). The process $\mathbb{H}(\cdot, T_k, i)$ has the following dynamics under $\mathbb{P}_{T_k}$

$$\mathbb{H}(t, T_k, i) = \mathbb{H}(0, T_k, i) \times \mathcal{E}_t\Bigg(\int_0^{\cdot} b^{\mathbb{H}}(s, T_k, i)\mathrm{d}s - \int_0^{\cdot} \sqrt{c_s} \sum_{j=1}^{i} \sum_{l=1}^{k-1} h_j(s-, T_l)\gamma_j(s, T_l)\mathrm{d}W_s^{T_k} + \int_0^{\cdot} \int_{\mathbb{R}^d} \left(\prod_{j=1}^{i} \prod_{l=1}^{k-1} \left(1 + h_j(s-, T_l)(e^{\langle \gamma_j(s,T_l), x\rangle} - 1)\right)^{-1} - 1 \right) \times (\mu - \nu^{T_k})(\mathrm{d}s, \mathrm{d}x) \Bigg),$$

where $b^{\mathbb{H}}(s, T_k, i)$ is the drift term.

No-arbitrage condition

Theorem

Let T_k be a tenor date. Assume that the processes $H_j(\cdot, T_k)$, $j = 1, \dots, K-1$, are given by (2). Then the process $\frac{B_C(\cdot, T_k)}{B(\cdot, T_k)}$ defined in (3) is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$ and filtration $\mathbb{G}$ iff:
for almost all $t \le T_k$ on the set $\{C_t \neq K\}$

$$b^{\mathbb{H}}(t, T_k, C_t) + \lambda_{C_t}(t) = \left(1 - q_{C_t} \frac{e^{-\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t)}\right) \lambda_{C_t K}(t) + \sum_{j=1, j\neq C_t}^{K-1} \left(1 - \frac{\mathbb{H}(t-, T_k, j) e^{\int_0^t \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t) e^{\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}\right) \lambda_{C_t j}(t). \tag{4}$$

No-arbitrage condition

Theorem

Let T_k be a tenor date. Assume that the processes $H_j(\cdot, T_k)$, $j = 1, \dots, K-1$, are given by (2)*. Then the process $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ defined in* (3) *is a local martingale with respect to the forward measure $\mathbb{Q}_{T_k}$ and filtration $\mathbb{G}$ iff:*
for almost all $t \le T_k$ on the set $\{C_t \ne K\}$

$$b^{\mathbb{H}}(t, T_k, C_t) + \lambda_{C_t}(t) = \left(1 - q_{C_t} \frac{e^{-\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t)}\right) \lambda_{C_t K}(t) + \sum_{j=1, j\ne C_t}^{K-1} \left(1 - \frac{\mathbb{H}(t-, T_k, j) e^{\int_0^t \lambda_j(s)\mathrm{d}s}}{\mathbb{H}(t-, T_k, C_t) e^{\int_0^t \lambda_{C_t}(s)\mathrm{d}s}}\right) \lambda_{C_t j}(t). \tag{4}$$

Sketch of the proof: Use the fact that the jump times of the conditional Markov chain C do not coincide with the jumps of any $\mathbb{F}$-adapted semimartingale, use some martingales related to the indicator processes $\mathbf{1}_{\{C_t = i\}}$, $i \in \mathcal{K}$, and stochastic calculus for semimartingales.

Defaultable forward measures

Assume that $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a *true martingale* w.r.t. forward measure $\mathbb{Q}_{T_k}$. (back to DFP)

Defaultable forward measures

Assume that $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a *true martingale* w.r.t. forward measure $\mathbb{Q}_{T_k}$. (back to DFP)

The defaultable forward measure $\mathbb{Q}_{C,T_k}$ for the date T_k is defined on $(\Omega, \mathcal{G}_{T_k})$ by

$$\left.\frac{d\mathbb{Q}_{C,T_k}}{d\mathbb{Q}_{T_k}}\right|_{\mathcal{G}_t} := \frac{B(0,T_k)}{B_C(0,T_k)} \frac{B_C(t,T_k)}{B(t,T_k)}.$$

This corresponds to the choice of $B_C(\cdot, T_k)$ as a numeraire.

Defaultable forward measures

Assume that $\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}$ is a *true martingale* w.r.t. forward measure $\mathbb{Q}_{T_k}$. (back to DFP)

The defaultable forward measure $\mathbb{Q}_{C,T_k}$ for the date T_k is defined on $(\Omega, \mathcal{G}_{T_k})$ by

$$\left.\frac{d\mathbb{Q}_{C,T_k}}{d\mathbb{Q}_{T_k}}\right|_{\mathcal{G}_t} := \frac{B(0,T_k)}{B_C(0,T_k)} \frac{B_C(t,T_k)}{B(t,T_k)}.$$

This corresponds to the choice of $B_C(\cdot,T_k)$ as a numeraire.

Proposition

The defaultable Libor rate $L_C(\cdot,T_k)$ is a martingale with respect to $\mathbb{Q}_{C,T_{k+1}}$ and

$$\left.\frac{d\mathbb{Q}_{C,T_k}}{d\mathbb{Q}_{C,T_{k+1}}}\right|_{\mathcal{G}_t} = \frac{B_C(0,T_{k+1})}{B_C(0,T_k)}(1+\delta_k L_C(t,T_k)).$$

Pricing problems I: Defaultable bond

Proposition

The price of a defaultable bond with maturity T_k and fractional recovery of Treasury value q at time $t \leq T_k$ is given by

$$B_C(t,T_k)\mathbf{1}_{\{C_t \neq K\}} = B(t,T_k)\sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}}\Bigg[\mathbb{E}_{\mathbb{Q}_{T_k}}[1-p_{iK}(t,T_k)|\mathcal{F}_t] + \sum_{j=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{t<\tau\leq T_k\}}\mathbf{1}_{\{C_t=i\}}\mathbf{1}_{\{C_{\tau-}=j\}}q_j|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\Bigg].$$

Pricing problems II: Credit default swap

- consider a maturity date T_m and a defaultable bond with fractional recovery of Treasury value q as the underlying asset
- protection buyer pays a fixed amount S periodically at tenor dates $T_1, \ldots, T_{m-1}$ until default
- protection seller promises to make a payment that covers the loss if default happens:

$$1 - q_{C_{\tau-}}$$

has to paid at T_{k+1} if default occurs in $(T_k, T_{k+1}]$

Pricing problems II: Credit default swap

- consider a maturity date T_m and a defaultable bond with fractional recovery of Treasury value q as the underlying asset
- protection buyer pays a fixed amount S periodically at tenor dates $T_1, \ldots, T_{m-1}$ until default
- protection seller promises to make a payment that covers the loss if default happens:

$$1 - q_{C_{\tau-}}$$

has to paid at T_{k+1} if default occurs in $(T_k, T_{k+1}]$

Proposition

The swap rate S at time 0 is equal to

$$S = \frac{\sum_{k=2}^{m} B(0,T_k) \sum_{j=1}^{K-1} \mathbb{E}_{\mathbb{Q}_{T_k}} \left[(1-q_j)\mathbf{1}_{\{T_{k-1}<\tau\le T_k, C_{\tau-}=j\}}\right]}{\sum_{k=1}^{m-1} B(0,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}\left[1 - p_{iK}(0,T_k)\right]},$$

if the observed class at time zero is i.

Pricing problems III: use of defaultable measures

Proposition

Let Y be a promised $\mathcal{G}_{T_k}$-measurable payoff at maturity T_k of a defaultable contingent claim with fractional recovery q upon default and assume that Y is integrable with respect to $\mathbb{Q}_{T_k}$.
The time-t value of such a claim is given by

$$\pi^t(Y) = B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].$$

Pricing problems III: use of defaultable measures

Proposition

Let Y be a promised $\mathcal{G}_{T_k}$-measurable payoff at maturity T_k of a defaultable contingent claim with fractional recovery q upon default and assume that Y is integrable with respect to $\mathbb{Q}_{T_k}$.
The time-t value of such a claim is given by

$$\pi^t(Y) = B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].$$

Example: a cap on the defaultable forward Libor rate

Pricing problems III: use of defaultable measures

Proposition

Let Y be a promised $\mathcal{G}_{T_k}$-measurable payoff at maturity T_k of a defaultable contingent claim with fractional recovery q upon default and assume that Y is integrable with respect to $\mathbb{Q}_{T_k}$.
The time-t value of such a claim is given by

$$\pi^t(Y) = B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[Y|\mathcal{G}_t].$$

Example: a cap on the defaultable forward Libor rate

The time-t price of a caplet with strike K and maturity T_k on the defaultable Libor rate is given by

$$C_t(T_k, K) = \delta_k B_C(t, T_{k+1})\mathbb{E}_{\mathbb{Q}_{C,T_{k+1}}}[(L_C(T_k, T_k) - K)^+|\mathcal{G}_t]$$

and the price of the defaultable forward Libor rate cap at time $t \leq T_1$ is given as a sum

$$\mathbb{C}_t(K) = \sum_{k=1}^{n} \delta_{k-1} B_C(t, T_k)\mathbb{E}_{\mathbb{Q}_{C,T_k}}[(L_C(T_{k-1}, T_{k-1}) - K)^+|\mathcal{G}_t].$$

- T. Bielecki and M. Rutkowski, *Credit Risk: Modeling, Valuation and Hedging*, Springer, 2002.
- E. Eberlein, W. Kluge, and P. J. Schönbucher, The Lévy Libor model with default risk, *Journal of Credit Risk 2*, 3-42, 2006.
- E. Eberlein and F. Özkan, The Lévy LIBOR model. *Finance and Stochastics 9*, 327-348, 2005.
- R. Elliott, M. Jeanblanc, and M. Yor, On models of default risk, *Mathematical Finance 10*, 179-195, 2000.
- Z. Grbac, *Credit Risk in Lévy Libor Modeling: Rating Based Approach*, Ph.D. Thesis, University of Freiburg, 2009
- J. Jacod and A.N. Shiryaev, *Limit theorems for stochastic processes*, Springer, 2003.
- P. Protter, *Stochastic Integration and Differential Equations*, Springer, 2005.
- K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.