Extremogram and ex-periodogram for heavy-tailed time series

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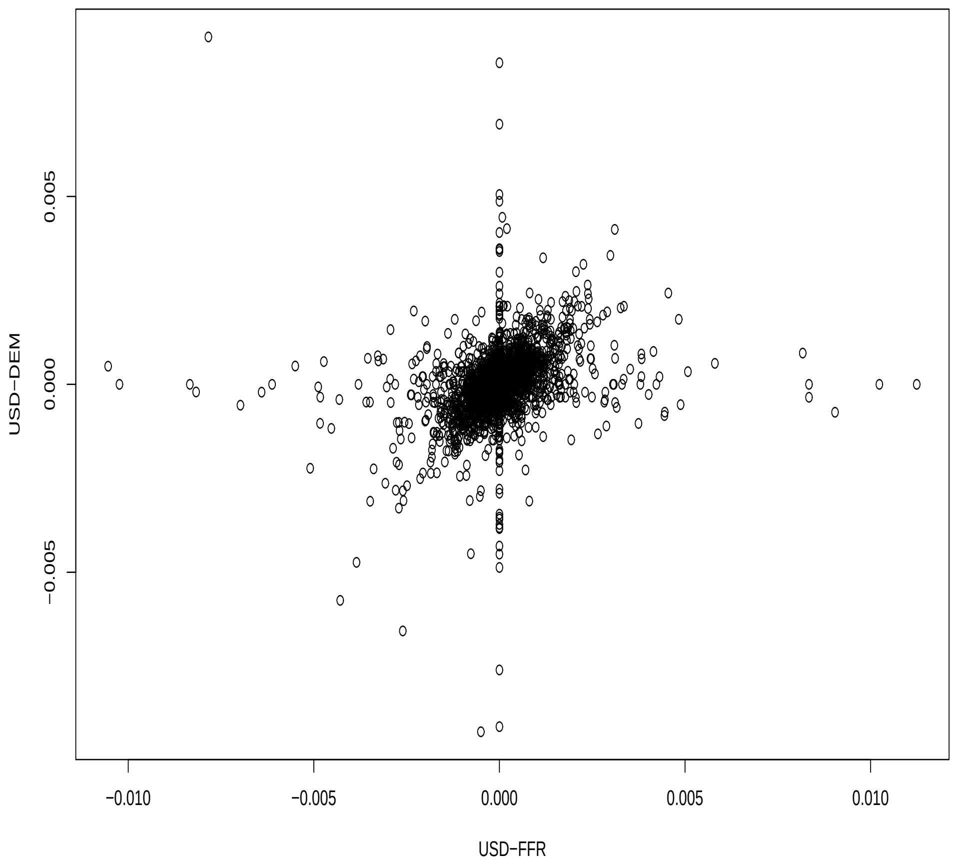


Figure 1. Scatterplot of 5 minute foreign exchange rate log-returns, USD-DEM against USD-FRF.

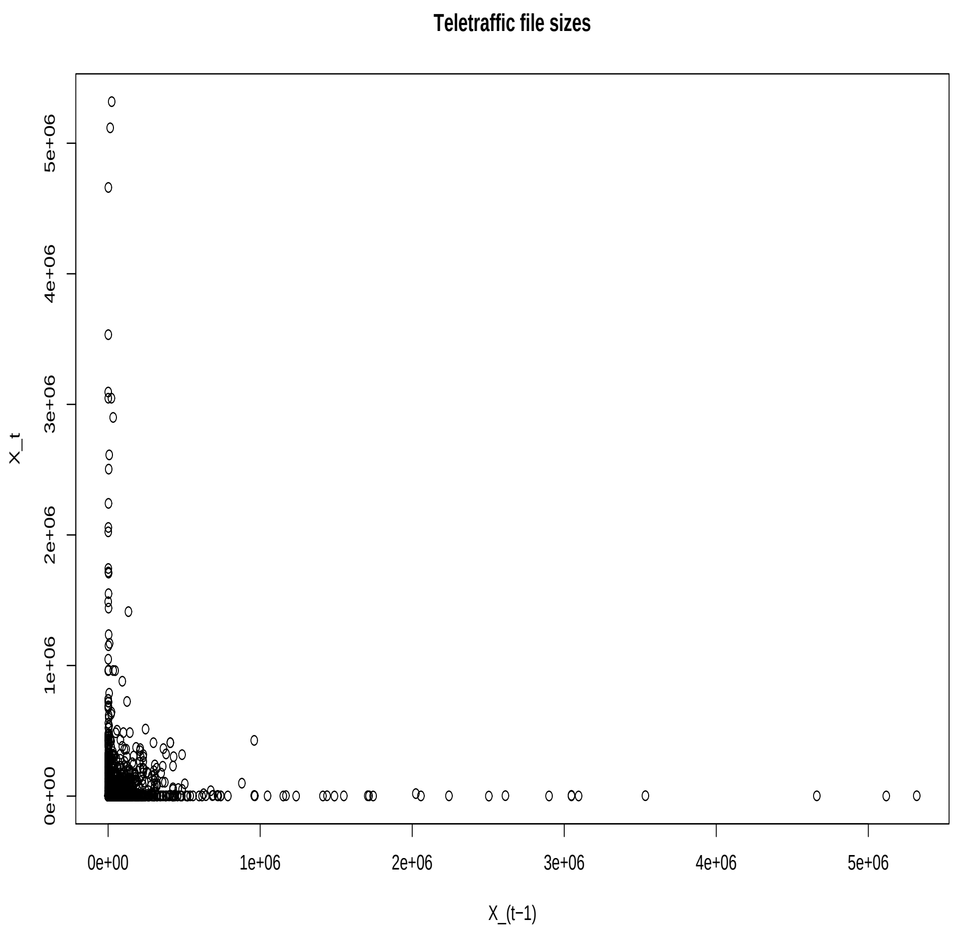


Figure 2. Scatterplot of file sizes of teletraffic data - extremal independence

1. Regularly varying stationary sequences

- **An $\mathbb{R}^d$-valued strictly stationary sequence (X_t) is regularly varying with index $\alpha > 0$ if its finite-dimensional distributions are regularly varying with index α:**
- **For every $k \geq 1$, there exists a non-null Radon measure μ_k in $\overline{\mathbb{R}}_0^{dk}$ such that as $x \to \infty$,**

$$\frac{P(x^{-1}(X_1, \ldots, X_k) \in \cdot)}{P(|X_1| > x)} \xrightarrow{v} \mu_k(\cdot)\,.$$

- **The measures μ_k determine the extremal dependence structure of the finite-dimensional distributions and have the scaling property $\mu_k(tA) = t^{-\alpha}\mu_k(A)$, $t > 0$, for some $\alpha > 0$.**

- **Alternatively,** Basrak, Segers (2009) **for** $\alpha > 0$, $k \geq 0$,

$$P(x^{-1}(X_0, \ldots, X_k) \in \cdot \mid |X_0| > x) \xrightarrow{w} P((Y_0, \ldots, Y_k) \in \cdot)\,,$$

$|Y_0|$ is independent of $(Y_0, \ldots, Y_k)/|Y_0|$ and $P(|Y_0| > y) = y^{-\alpha}$, $y > 1$.

- **For $d = k = 1$, $t > 0$: for some $p, q \geq 0$ such that $p + q = 1$,**

$$\frac{P(x^{-1}X_1 \in (t, \infty))}{P(|X_1| > x)} \to p\,t^{-\alpha} \quad \text{and} \quad \frac{P(x^{-1}X_1 \in (-\infty, -t])}{P(|X_1| > x)} \to q\,t^{-\alpha}\,.$$

Examples.

- IID sequence (Z_t) with regularly varying Z_0.
- Starting from a Gaussian linear process, transform marginals to a student distribution.
- Linear processes e.g. ARMA processes with iid regularly varying noise (Z_t). Rootzén (1978,1983), Davis, Resnick (1985)
- Solutions to stochastic recurrence equation: $X_t = A_t X_{t-1} + B_t$

  Kesten (1973), Goldie (1991)
- GARCH process. $X_t = \sigma_t Z_t$, $\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2$

  Bollerslev (1986), M., Stărică (2000), Davis, M. (1998), Basrak, Davis, M. (2000,2002)
- The simple stochastic volatility model with iid regularly varying noise. Davis, M. (2001)

- **Infinite variance α-stable stationary processes are regularly varying with index $\alpha \in (0,2)$.** Samorodnitsky, Taqqu (1994), Rosiński (1995,2000)

- **Max-stable stationary processes with Fréchet (Φ_α) marginals are regularly varying with index $\alpha > 0$.** de Haan (1984), Stoev (2008), Kabluchko (2009)

2. The extremogram - an analog of the autocorrelation function Davis, M. (2009,2012)

- **For an $\mathbb{R}^d$-valued strictly stationary regularly varying sequence $(\mathbf{X}_t)$ and a Borel set A bounded away from zero the extremogram is the limiting function**

$$\rho_A(h) = \lim_{x\to\infty} P(x^{-1}\mathbf{X}_h \in A \mid x^{-1}\mathbf{X}_0 \in A)$$

$$= \lim_{x\to\infty} \frac{P(x^{-1}\mathbf{X}_0 \in A\,, \quad x^{-1}\mathbf{X}_h \in A)}{P(x^{-1}\mathbf{X}_0 \in A)}$$

$$= \frac{\mu_{h+1}(A \times \overline{\mathbb{R}}_0^{d(h-1)} \times A)}{\mu_{h+1}(A \times \overline{\mathbb{R}}_0^{dh})}\,, \qquad h \geq 0\,.$$

- Since

$$\frac{\mathrm{cov}(I(x^{-1}\mathbf{X}_0 \in A), I(x^{-1}\mathbf{X}_h \in A))}{P(x^{-1}X_0 \in A)}$$

$$= P(x^{-1}\mathbf{X}_h \in A \mid x^{-1}\mathbf{X}_0 \in A) - P(x^{-1}\mathbf{X}_0 \in A)$$

$$\to \rho_A(h)\,, \quad h \geq 0\,,$$

- $(\rho_A(h))$ is the autocorrelation function of a stationary process.

- One can use the notions of classical time series analysis to describe the extremal dependence structure in a strictly stationary sequence.

Examples. Take $A = B = (1, \infty)$. **Tail dependence function**

$$\rho_A(h) = \lim_{x \to \infty} P(X_h > x \mid X_0 > x)\,.$$

- The AR(1) process $X_t = \phi\, X_{t-1} + Z_t$ with iid symmetric regularly varying noise (Z_t) with index α and $\phi \in (-1, 1)$ has the extremogram

$$\rho_A(h) = \text{const}\ \max(0, (\text{sign}(\phi))^h |\phi|^{\alpha h})\,.$$

Short serial extremal dependence

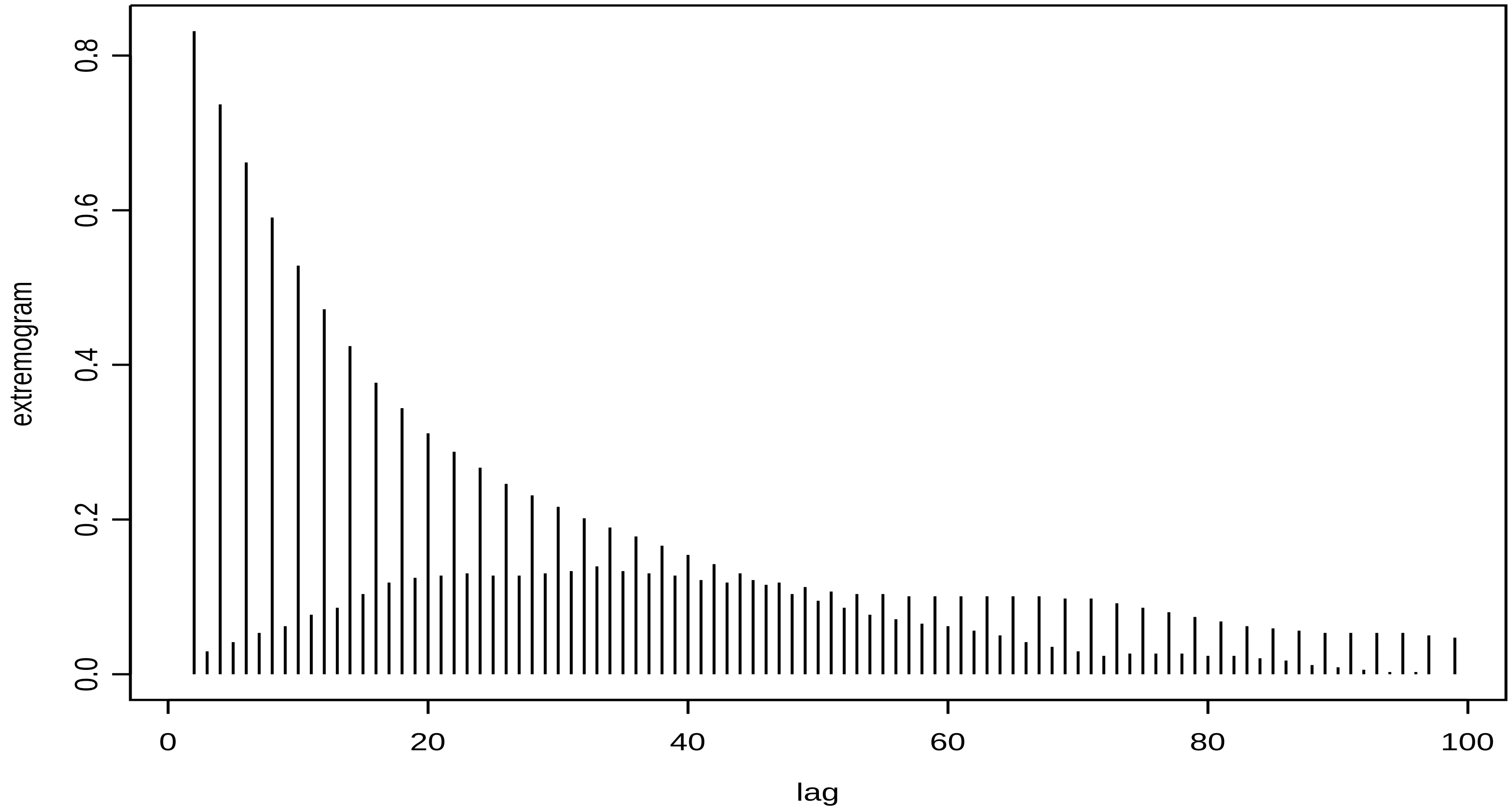


Figure 3. Sample extremogram with $\boldsymbol{A} = \boldsymbol{B} = (\mathbf{1}, \infty)$ for $\mathbf{5}$ minute returns of USD-DEM foreign exchange rates. The extremogram alternates between large values at even lags and small ones at odd lags. This is an indication of AR behavior with negative leading coefficient.

- **The extremogram of a GARCH$(1,1)$ process is not very explicit, but $\rho_A(h)$ decays exponentially fast to zero. This is in agreement with the geometric β-mixing property of GARCH.**

  Short serial extremal dependence

- **The stochastic volatility model with stationary Gaussian $(\log \sigma_t)$ and iid regularly varying (Z_t) with index $\alpha > 0$ has extremogram $\rho_A(h) = 0$ as in the iid case.**

  No serial extremal dependence

- **The extremogram of a linear Gaussian process with index $\alpha > 0$ has extremogram $\rho_A(h) = 0$ as in the iid case.**

  No serial extremal dependence

3. The sample extremogram – an analog of the sample autocorrelation function

- (X_t) regularly varying, $m = m_n \to \infty$ and $m_n/n \to 0$.
- The sample extremogram

$$\widehat{\rho}_A(h) = \frac{\frac{m}{n}\sum_{t=1}^{n-h} I(a_m^{-1}\mathrm{X}_{t+h} \in A, a_m^{-1}\mathrm{X}_t \in A)}{\frac{m}{n}\sum_{t=1}^{n} I(a_m^{-1}\mathrm{X}_t \in A)} = \frac{\widehat{\gamma}_A(h)}{\widehat{\gamma}_A(0)}$$

estimates the extremogram

$$\rho_A(h) = \lim_{n\to\infty} P(x^{-1}X_h \in A \mid x^{-1}X_0 \in A)\,.$$

- $m \to \infty$ and $m/n \to 0$ needed for consistency.
- Pre-asymptotic central limit theory with rate $\sqrt{n/m}$ applies if (X_t) is strongly mixing. Asymptotic covariance matrix is not tractable.

- **These results do not follow from classical time series analysis: the sequences $\big(I(a_m^{-1}\mathrm{X}_t \in A)\big)_{t\le n}$ constitute a triangular array of rowwise stationary sequences.**

- **The quantities a_m are high thresholds, e.g. $P(|X_0| > a_m) \sim m^{-1}$ which typically have to be replaced by empirical quantiles.**

- **Confidence bands: based on permutations of the data or on the stationary bootstrap Politis and Romano (1994).**

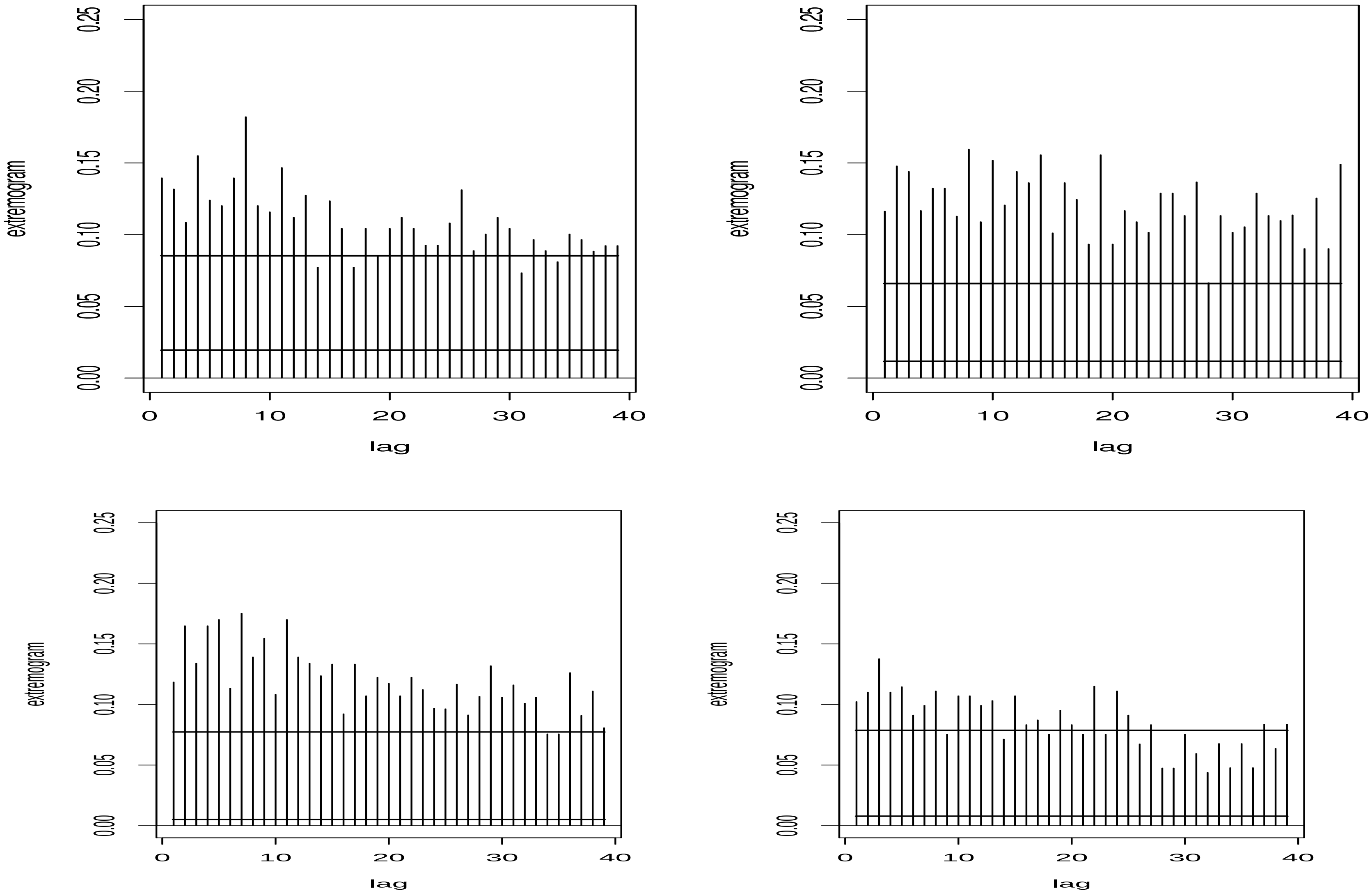


Figure 4. The sample extremogram for the lower tail of the FTSE (top left), S&P500 (top right), DAX (bottom left) and Nikkei. The bold lines represent 95% confidence bands based on random permutations of the data.

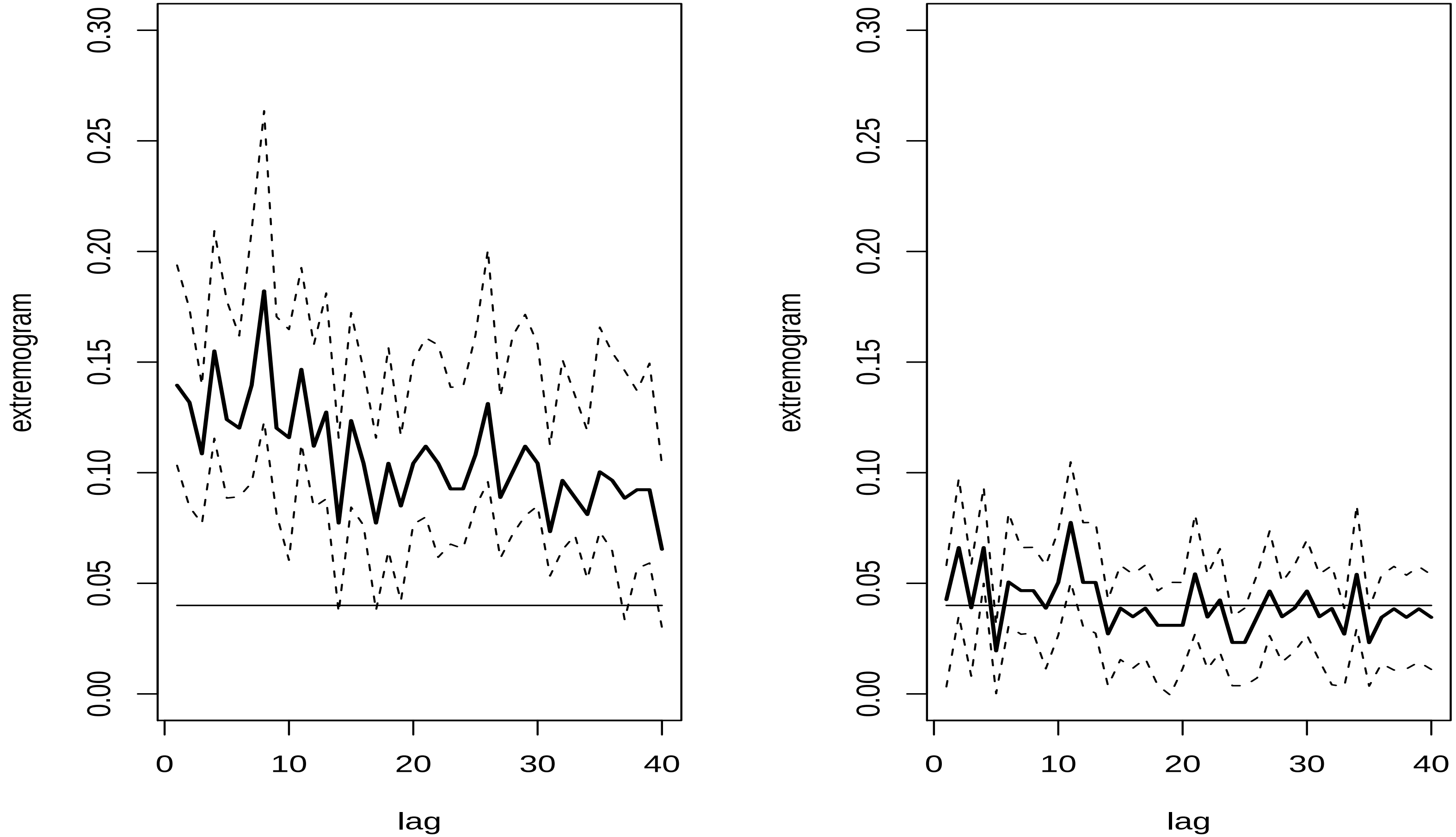


FIGURE 5. Left: 95% bootstrap confidence bands for pre-asymptotic extremogram of 6440 daily FTSE log-returns. Mean block size 200. Right: For the residuals of a fitted GARCH$(1,1)$ model.

4. Cross-extremogram

- Consider a strictly stationary bivariate regularly varying time series $((X_t, Y_t))_{t\in\mathbb{Z}}$.
- For two sets A and B bounded away from 0, the cross-extremogram

$$\rho_{AB}(h) = \lim_{x\to\infty} P(Y_h \in x\,B \mid X_0 \in x\,A)\,, \quad h \geq 0\,,$$

is an extremogram based on the two-dimensional sets $A \times \mathbb{R}$ and $\mathbb{R} \times B$.

- The corresponding sample cross-extremogram for the time series $((X_t, Y_t))_{t\in\mathbb{Z}}$:

$$\widehat{\rho}_{A,B}(h) = \frac{\sum_{t=1}^{n-h} I(Y_{t+h} \in a_{m,Y} B, X_t \in a_{m,X} A)}{\sum_{t=1}^{n} I(X_t \in a_{m,X} A)}\,.$$

5. The extremogram of return times between rare events

- We say that X_t is extreme if $X_t \in xA$ for a set A bounded away from zero and large x.
- If the return times were truly iid, the successive waiting times between extremes should be iid geometric.
- The corresponding return times extremogram

$$\rho_A(h) = \lim_{x\to\infty} P(X_1 \notin xA, \ldots, X_{h-1} \notin xA, X_h \in xA \mid X_0 \in xA)$$

$$= \frac{\mu_{h+1}(A \times (A^c)^{h-1} \times A)}{\mu_{h+1}(A \times \overline{\mathbb{R}}_0^{dh})}, \quad h \geq 0\,.$$

- The return times sample extremogram

$$\widehat{\rho}_A(h) = \frac{\sum_{t=1}^{n-h} I(X_{t+h} \in a_m A, X_{t+h-1} \notin a_m A, \ldots, X_{t+1} \notin a_m A, X_t \in a_m A)}{\sum_{t=1}^{n} I(X_t \in a_m A)},$$

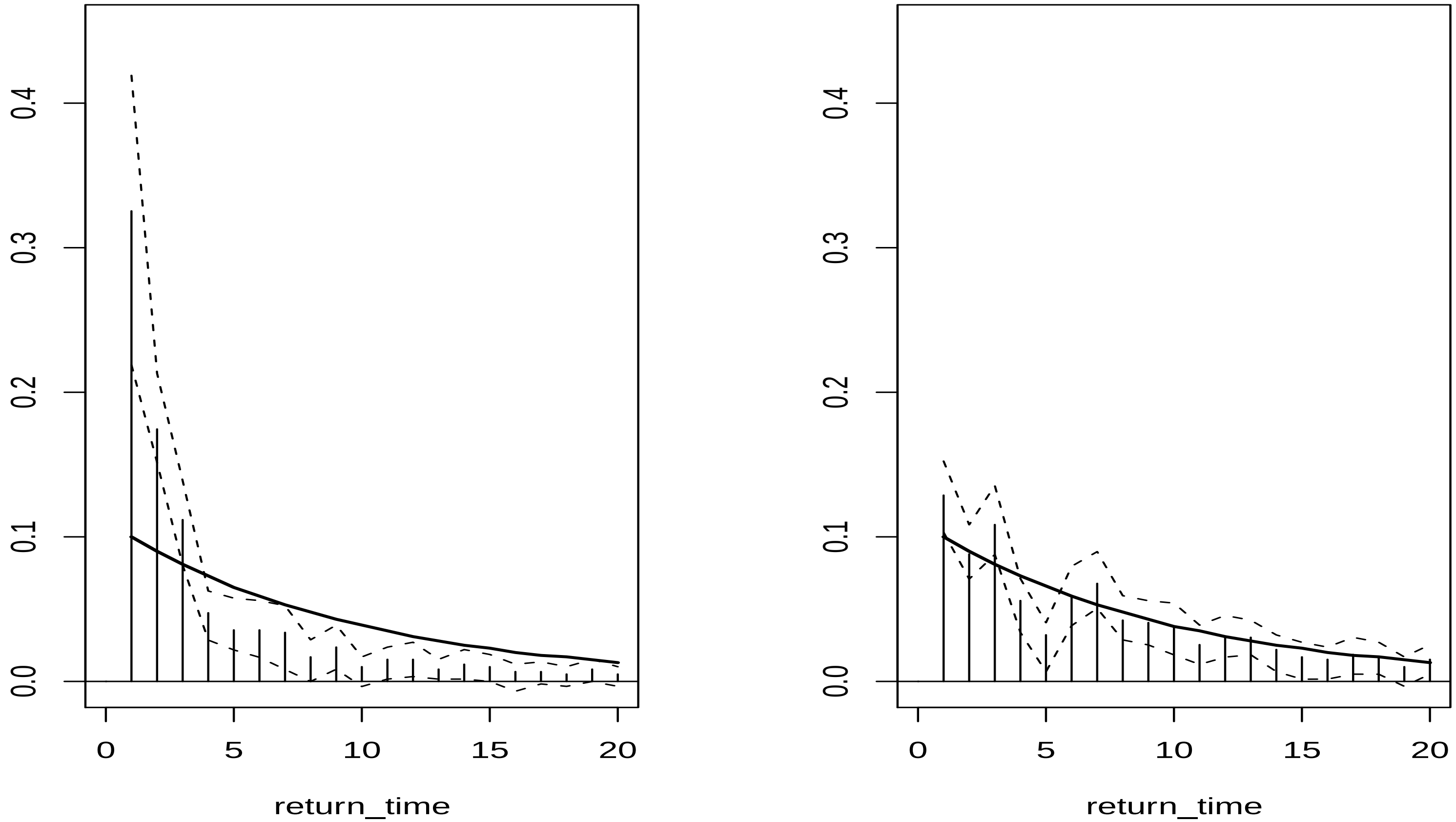


FIGURE 6. Left: Return times sample extremogram for extreme events with $\boldsymbol{A} = \mathbb{R}\backslash[\boldsymbol{\xi_{0.05}}, \boldsymbol{\xi_{0.95}}]$ for the daily log-returns of BAC using bootstrapped confidence intervals (dashed lines), geometric probability mass function (light solid). Right: The corresponding extremogram for the residuals of a fitted GARCH$\mathbf{(1,1)}$ model (right).

6. Frequency domain analysis M. and Zhao (2012,2013)

- The extremogram for a given set A bounded away from zero

$$\rho_A(h) = \lim_{n\to\infty} P(x^{-1}X_h \in A \mid x^{-1}X_0 \in A)\,, \quad h \geq 0\,,$$

is an autocorrelation function.

- Therefore one can define the spectral density for $\lambda \in (0,\pi)$:

$$f_A(\lambda) = 1 + 2\sum_{h=1}^{\infty} \cos(\lambda\, h)\, \rho_A(h) = \sum_{h=-\infty}^{\infty} \mathrm{e}^{-i\,\lambda\,h}\, \rho_A(h)\,.$$

- and its sample analog: the periodogram for $\lambda \in (0,\pi)$:

$$\widehat{f}_{nA}(\lambda) = \frac{I_{nA}(\lambda)}{I_{nA}(0)} = \frac{\frac{m}{n}\Big|\sum_{t=1}^{n} \mathrm{e}^{-it\lambda} I(a_m^{-1}\mathrm{X}_t \in A)\Big|^2}{\frac{m}{n}\sum_{t=1}^{n} I(a_m^{-1}\mathrm{X}_t \in A)}\,.$$

- One has $EI_{nA}(\lambda)/\mu_1(A) \to f_A(\lambda)$ for $\lambda \in (0,\pi)$.
- As in classical time series analysis, $\widehat{f}_{nA}(\lambda)$ is not a consistent estimator of $f_A(\lambda)$: for distinct (fixed or Fourier) frequencies λ_j, and iid standard exponential E_j,

$$(\widehat{f}_{nA}(\lambda_j))_{j=1,\dots,h} \xrightarrow{d} (f_A(\lambda_j)E_j)_{j=1,\dots,h}\,.$$

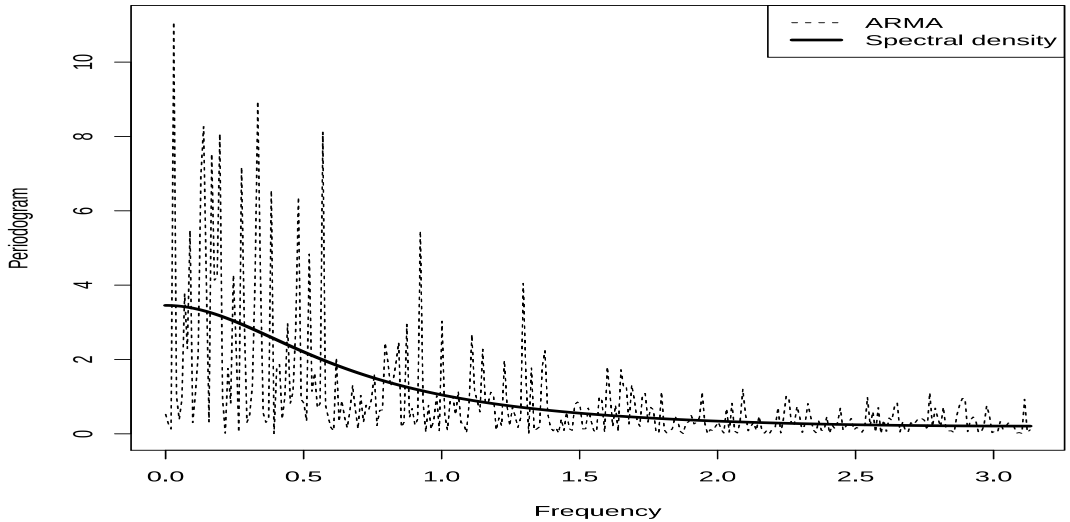


FIGURE 7. Sample extremogram and periodogram for ARMA(1,1) process with student(4) noise. $A = (1, \infty)$

- **Smoothed versions of the periodogram converge to $f(\lambda)$:**

  If $w_n(j) \geq 0$, $|j| \leq s_n \to \infty$, $s_n/n \to 0$, $\sum_{|j|\leq s_n} w_n(j) = 1$ and $\sum_{|j|\leq s_n} w_n^2(j) \to 0$ (e.g. $w_n(j) = 1/(2s_n+1)$) then for any distinct Fourier frequencies λ_j such that $\lambda_j \to \lambda$,

$$\sum_{|j|\leq s_n} w_n(j)\widehat{f}_{nA}(\lambda_j) \xrightarrow{P} f_A(\lambda)\,, \quad \lambda \in (0,\pi)\,.$$

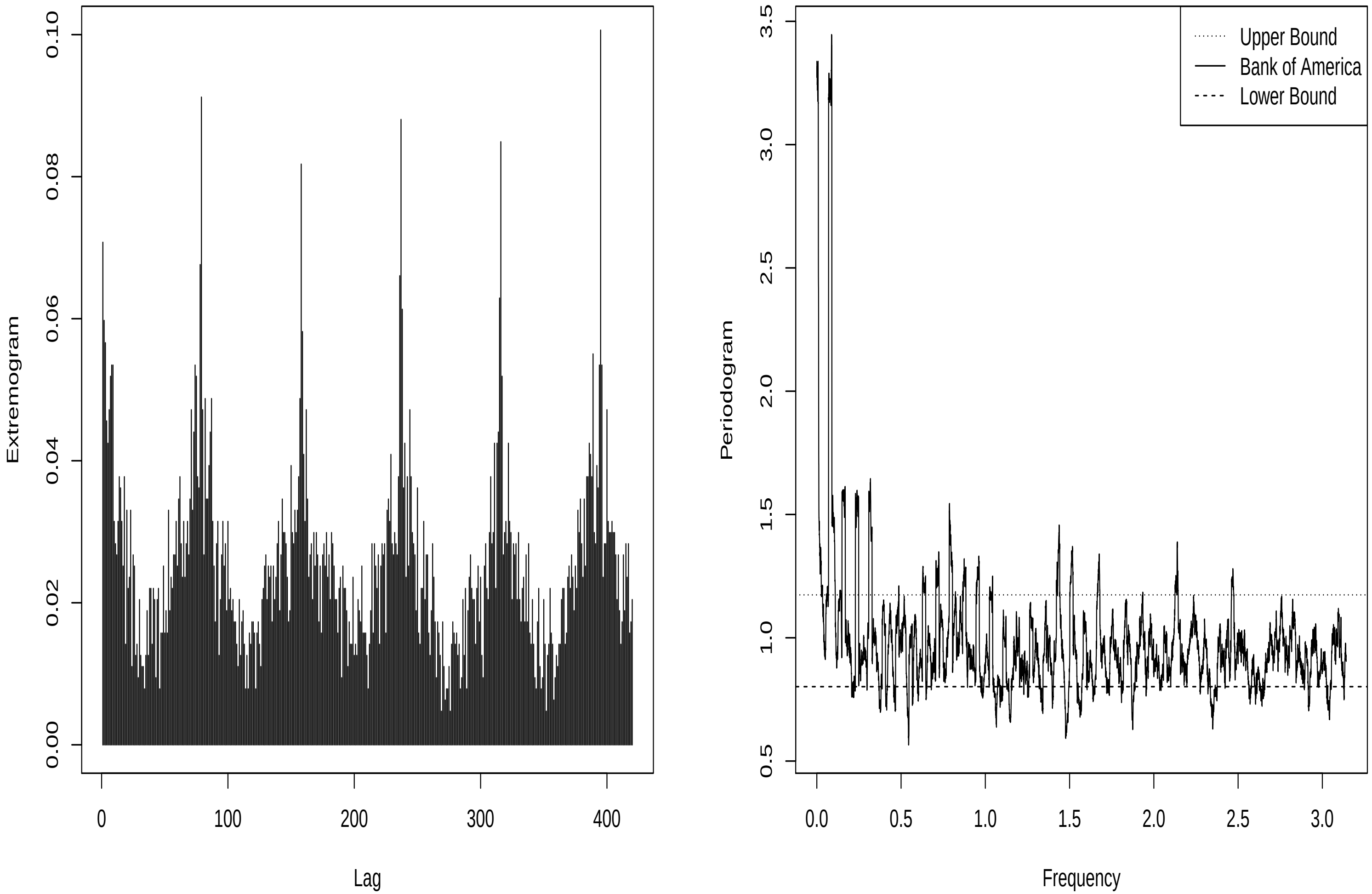


FIGURE 8. Sample extremogram and smoothed periodogram for BAC 5 minute returns. The end-of-the day effects cannot be seen in the corresponding sample autocorrelation function.

7. The integrated periodogram

- The **integrated periodogram**[2]

$$J_{nA}(\lambda) = \int_0^{\lambda} \widehat{f}_{nA}(x)\, g(x)\, dx\,, \quad \lambda \in \Pi = [0,\pi]\,.$$

for a non-negative weight function g is an estimator of the **weighted spectral distribution function**

$$J_{nA}(\lambda) \stackrel{P}{\rightarrow} J_A(\lambda) = \int_0^{\lambda} f_A(x)\, g(x)\, dx\,, \quad \lambda \in \Pi\,.$$

- **Goal.** Use the integrated periodogram for judging whether the extremes in a time series fit a given model.

[2] For practical purposes, one would use a Riemann sum approximation at the Fourier frequencies. The asymptotic theory does not change.

- **Goodness-of-fit tests are based on functional central limit theorems in $\mathbb{C}(\Pi)$:**[3]

$$
\begin{aligned}
&\Big(\frac{n}{m}\Big)^{0.5}\big[J_{nA} - EJ_{nA}\big] \\
= \;& \Big(\frac{n}{m}\Big)^{0.5}\Big[\psi_0\,[\widehat{\gamma}_A(0) - E\widehat{\gamma}_A(0)] + 2\sum_{h=1}^{n-1}\psi_h\,[\widehat{\gamma}_A(h) - E\widehat{\gamma}_A(h)]\Big] \\
\stackrel{d}{\rightarrow}\;& \psi_0 Z_0 + 2\sum_{h=1}^{\infty}\psi_h\,Z_h = G\,,
\end{aligned}
$$

where (Z_h) is a dependent Gaussian sequence and

$$
\psi_h(\lambda) \;=\; \int_0^{\lambda}\cos(h\,x)\,g(x)\,dx\,,\qquad \lambda\in\Pi\,.
$$

[3] not self-normalized, pre-asymptotic

- **Grenander-Rosenblatt test:**

$$(n/m)^{0.5} \sup_{x \in \Pi} |J_{nA}(\lambda) - EJ_{nA}(\lambda)| \xrightarrow{d} \sup_{x \in \Pi} |G(\lambda)| \, .$$

- **ω^2- or Cramér-von Mises test:**

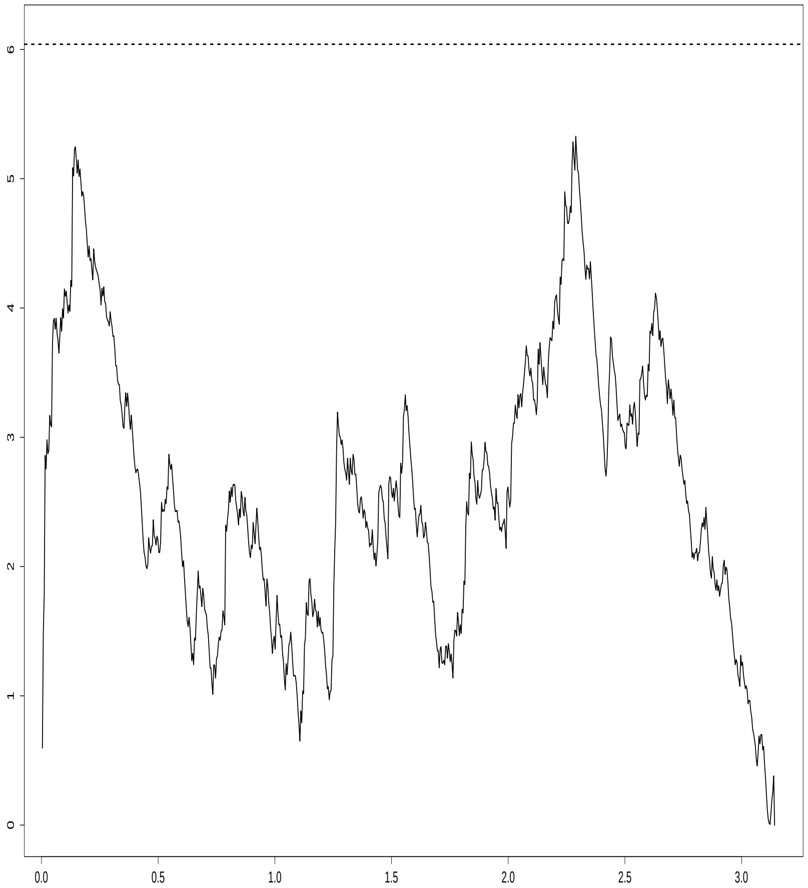
$$(n/m) \int_{\lambda \in \Pi} (J_{nA}(\lambda) - EJ_{nA}(\lambda))^2 \, d\lambda \xrightarrow{d} \int_{\lambda \in \Pi} G^2(\lambda) \, d\lambda \, .$$

- **The distribution of G is not tractable, but the stationary bootstrap allows one to approximate it.**

- **Example.** If (X_t) is iid or a simple stochastic volatility model then $Z_h = 0$ for $h \geq 1$ and the limit process collapses into $G = \psi_0 Z_0$. But in this case

$$n^{0.5}\left[(J_{nA} - EJ_{nA}) - \psi_0\,(\widehat{\gamma}_A(0) - E\widehat{\gamma}_A(0))\,\right] \xrightarrow{d} 2\sum_{h=1}^{\infty} \psi_h\, Z_h\,,$$

for iid (Z_h). For $g \equiv 1$, $\psi_h(\lambda) = \sin(h\lambda)/h$ and limit becomes a Brownian bridge on Π.



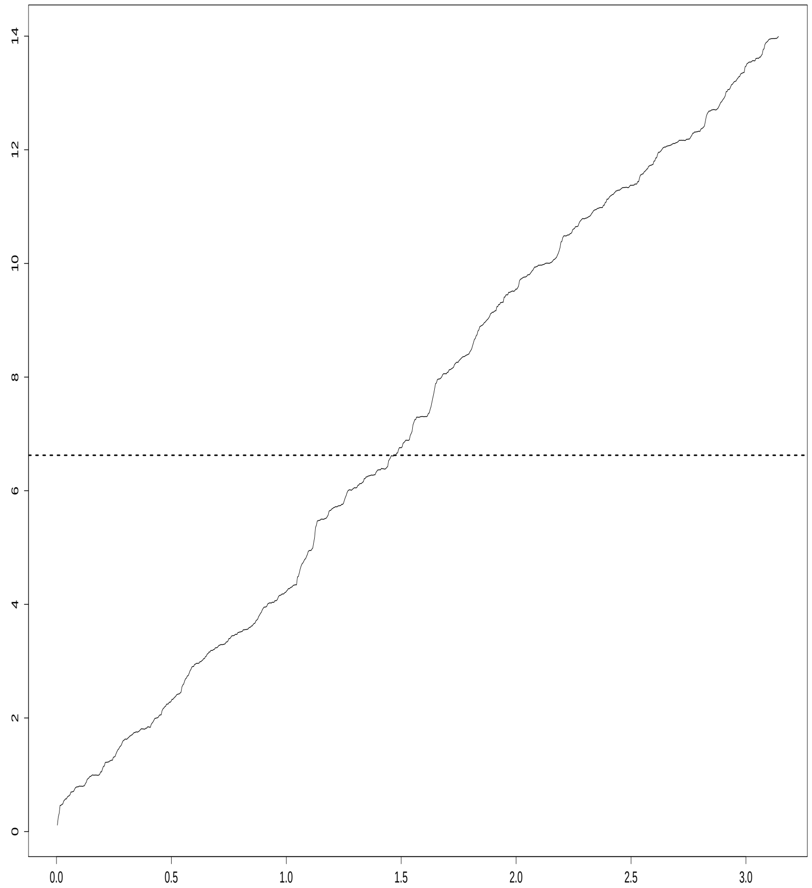


FIGURE 9. Grenander-Rosenblatt test statistic, $\boldsymbol{g \equiv 1}$, for 1560 1-minute Goldman-Sachs log-returns. Left: Under an iid hypothesis. Right: Under GARCH$\boldsymbol{(1,1)}$ hypothesis.