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Characterization of Nonuniform Contractions and Expansions with Growth Rates

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Abstract. For a one-sided nonautonomous dynamics defined by a sequence of linear operators, we obtain a complete characterization of (strong) nonuniform exponential contractions and (strong) nonuniform exponential expansions in terms of admissibility of certain pairs of sequence spaces. We allow asymptotic rates of the form $e^{c\rho(n)}$ determined by an arbitrary increasing sequence $\rho(n)$ that tends to infinity. For example, the usual exponential behavior with $\rho(n) = n$ is included as a very special case. As a nontrivial application of our work, we establish the robustness of (strong) nonuniform exponential contractions and (strong) nonuniform exponential expansions, that is, the persistence of those notions under sufficiently small linear perturbations.

Mathematics Subject Classification. Primary 37D99.

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1. Introduction

Given a sequence $(A_m)_m$ of bounded operators on Banach space, we consider the nonautonomous dynamics:

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{N}. \tag{1}$$

The main objective of our paper is to give a complete characterization of the notions of (strong) nonuniform exponential contraction and of (strong) nonuniform exponential expansion for (1) in terms of the admissibility of certain pairs of sequence spaces. We emphasize that in a strong contrast to the existing results:

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1. We consider a nonuniform exponential behavior of the contraction and expansion which includes the usual (uniform) exponential contraction and expansion, but it also allows a “spoiling” of the contraction and expansion along each trajectory as the initial time increases.
2. We allow asymptotic rates of the form $e^{c\rho(n)}$ determined by an arbitrary increasing sequence $\rho(n)$ that tends to infinity, and not only the usual exponential behavior when $\rho(n) = n$.
3. We study strong contractions and expansions, which means that we assume a lower contraction and upper expansion bounds for the exponential behavior.

To the best of our knowledge, we here for the first time establish a characterization, in terms of admissibility, of certain classes of a nonuniform behavior with arbitrary growth rates.

As described in the previous paragraph, the notions studied in this paper include the classical notions of (uniform) exponential contraction and (uniform) exponential expansions as very particular cases. Much of the work in the literature has been devoted to the study of the relationship between (uniform) exponential behavior and the so-called admissibility property. Indeed, the study of the admissibility property goes back to the work of Perron [11] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

in $\mathbb{R}^n$ for any bounded continuous perturbation $f \colon \mathbb{R}_0^+ \to \mathbb{R}^n$. For some of the most relevant early contributions in the area, we refer to the books by Massera and Schäffer [10] (see also [9]), Dalec′kiĭ and Kreĭn [5], and Coppel [4]. We also refer to [8] for some early results in infinite-dimensional spaces. For more recent work, see [3,6,7,12,13] and the references therein.

Certainly, there exist many classes of dynamical systems with a uniform exponential behavior. On the other hand, the requirement of uniformity for the asymptotic behavior is often too stringent for the dynamics. It turns out that the nonuniform exponential behavior, which allows a nonuniform bound on the initial time, is much more typical. In particular, this behavior is ubiquitous in the context of ergodic theory. More precisely, let $f \colon M \to M$ be a diffeomorphism on a smooth manifold and let μ be an f-invariant finite measure on M. This means that $\mu(f^{-1}A) = \mu(A)$ for any measurable set $A \subset M$. In particular, provided that $\log^+\|df\|$ is μ-integrable, one can show that for μ-almost all $x \in M$ if the Lyapunov exponents

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log\|d_x f^n v\|$$

are negative for all $v \neq 0$, then the sequence $A_n(x) = d_{f^n(x)} f$ admits a nonuniform exponential contraction. In fact, the sequence even admits a strong nonuniform exponential contraction.

As mentioned, we also allow asymptotic rates of the form $e^{c\rho(n)}$ determined by an arbitrary increasing sequence $\rho(n)$ that tends to infinity. Thus, the standard exponential behavior when $\rho(n) = n$ as well as the polynomial behavior when $\rho(n) = \log n$ are included as a very particular case. The main motivation are those linear equations for which all Lyapunov exponents are infinite (either $+\infty$ or $-\infty$), and thus to which one is not able, at least without further modifications, to apply the existing stability theory. This gives rise to the notions of (strong) ρ-nonuniform exponential contractions and expansions, which also turn out to be rather common (see [2]).

Hence, taking into account the ubiquity of a nonuniform behavior and the extensive study of the relationship between uniform behavior and admissibility, it is certainly both interesting and important to study the connection between nonuniform behavior and admissibility, and this is precisely the main contribution of our paper. As a nontrivial application of our characterization of (strong) nonuniform contractions and expansions, we establish the robustness of those notions, that is, their stability under sufficiently small linear perturbations.

2. Nonuniform Contractions

Throughout this paper $\rho\colon \mathbb{N} \to \mathbb{R}^+$ is an increasing function satisfying

$$\lim_{n\to+\infty} \rho(n) = +\infty \tag{2}$$

and $X = (X, \|\cdot\|)$ is an arbitrary Banach space. We say that the sequence $(A_m)_{m\in\mathbb{N}}$ of bounded operators on X admits a ρ-nonuniform exponential contraction if there exist constants $D, \lambda > 0$, $\varepsilon \geq 0$, such that

$$\|\mathcal{A}(m,n)\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon\rho(n)} \qquad \text{for } m \geq n, \tag{3}$$

where

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } \ m > n, \\ \mathrm{Id} & \text{if } \ m = n. \end{cases}$$

Given a sequence of norms $\|\cdot\|_n$, $n \in \mathbb{N}$ on X and $\lambda \in \mathbb{R}$, we define

$$Y_\infty = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{N}} : \|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} \left(\|x_n\|_n e^{\lambda\rho(n)} \right) < +\infty \right\}$$

and

$$Y_1 = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{N}} : \|\mathbf{x}\|_1 = \sum_{n=1}^{\infty} \|x_n\|_n e^{\lambda\rho(n)} < +\infty \right\}.$$

It is straightforward to verify that $(Y_\infty, \|\cdot\|_\infty)$ and $(Y_1, \|\cdot\|_1)$ are Banach spaces. Furthermore, Y_∞^0 will denote the closed subspace of Y_∞ that consists of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in Y_\infty$, such that $x_1 = 0$. Similarly, we introduce Y_1^0.

The following is our first result.

Theorem 1. *The sequence $(A_m)_{m\in\mathbb{N}}$ of bounded operators on X admits a ρ-nonuniform exponential contraction if and only if there exists a sequence of norms $\|\cdot\|_n$, $n \in \mathbb{N}$, and $\lambda > 0$, such that:*

1. *For each $y = (y_n)_{n\in\mathbb{N}} \in Y_1^0$, there exists $x = (x_n)_{n\in\mathbb{N}} \in Y_\infty^0$, such that:*
$$x_{n+1} - A_n x_n = y_{n+1}, \quad \text{for every } n \in \mathbb{N}. \tag{4}$$

2. *There exist $C > 0$ and $\delta \geq 0$ such that*

$$\|x\| \leq \|x\|_m \leq Ce^{\delta\rho(m)}\|x\|, \quad \text{for } x \in X \text{ and } m \in \mathbb{N}. \tag{5}$$

Proof. Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential contraction. For $m \in \mathbb{N}$ and $x \in X$, let

$$\|x\|_m = \sup_{n\geq m} \left(\|\mathcal{A}(n,m)x\|e^{\lambda(\rho(n)-\rho(m))}\right).$$

It follows directly from (3) that (5) holds with $C = D$ and $\delta = \varepsilon$. Furthermore, we have

$$\begin{aligned}\|\mathcal{A}(m,n)x\|_m &= \sup_{k\geq m} \left(\|\mathcal{A}(k,m)\mathcal{A}(m,n)x\|e^{\lambda(\rho(k)-\rho(m))}\right)\\ &= e^{-\lambda(\rho(m)-\rho(n))} \sup_{k\geq m} \left(\|\mathcal{A}(k,n)x\|e^{\lambda(\rho(k)-\rho(n))}\right)\\ &\leq e^{-\lambda\left(\rho(m)-\rho(n)\right)}\|x\|_n, \end{aligned} \tag{6}$$

for $m \geq n$. Take now an arbitrary $y = (y_n)_{n\in\mathbb{N}} \in Y_1^0$ and define a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ by

$$x_n = \sum_{k=1}^{n} \mathcal{A}(n,k)y_k, \qquad n \in \mathbb{N}.$$

It follows from (6) that

$$e^{\lambda\rho(n)}\|x_n\|_n \leq e^{\lambda\rho(n)} \sum_{k=1}^{n} e^{-\lambda\left(\rho(n)-\rho(k)\right)}\|y_k\|_k \leq \sum_{k=1}^{\infty} e^{\lambda\rho(k)}\|y_k\|_k = \|\mathbf{y}\|_1,$$

for each $n \in \mathbb{N}$. Since $x_1 = 0$, we conclude that $\mathbf{x} \in Y_\infty^0$. Finally, it is easy to show that (4) holds.

Now, we establish the converse. We define a linear operator $T\colon \mathcal{D}(T) \to Y_1^0$ by

$$(T\mathbf{x})_1 = 0 \quad \text{and} \quad (T\mathbf{x})_{n+1} = x_{n+1} - A_n x_n \tag{7}$$

on the domain $\mathcal{D}(T)$ that consists of all $\mathbf{x} \in Y_\infty^0$, such that $T\mathbf{x} \in Y_1^0$. ☐

Lemma 1. *The linear operator $T\colon \mathcal{D}(T) \to Y_1^0$ is closed.*

Proof of the lemma. Assume that $(\mathbf{x}^k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(T)$ converging to $\mathbf{x} \in Y_\infty^0$, and such that $\mathbf{y}^k = T\mathbf{x}^k$ converges to $\mathbf{y} \in Y_1^0$. Obviously

$$x_n = \lim_{k\to+\infty} x_n^k \quad \text{and} \quad y_n = \lim_{k\to+\infty} y_n^k,$$

for each $n \in \mathbb{N}$. Hence, using the continuity of operators A_n, we obtain

$$x_{n+1} - A_n x_n = \lim_{k\to+\infty} (x_{n+1}^k - A_n x_n^k) = \lim_{k\to+\infty} (T\mathbf{x}^k)_{n+1} = y_{n+1},$$

for $n \in \mathbb{N}$. We conclude that $\mathbf{x} \in \mathcal{D}(T)$ and $T\mathbf{x} = \mathbf{y}$. Therefore, the operator T is closed.

For $\mathbf{x} \in \mathcal{D}(T)$, we consider the graph norm

$$\|\mathbf{x}\|'_\infty = \|\mathbf{x}\|_\infty + \|T\mathbf{x}\|_1. \tag{8}$$

Clearly, the operator

$$T\colon (\mathcal{D}(T), \|\cdot\|'_\infty) \to (Y_1^0, \|\cdot\|_1)$$

is bounded, and for simplicity, we denote it simply by T. Moreover, since T is closed, $(\mathcal{D}(T), \|\cdot\|'_\infty)$ is a Banach space. Furthermore, it follows from (4) that T is invertible. Let G be the inverse of T.

Take an arbitrary $k > 1$, $x \in X$ and define a sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ by $y_k = x$ and $y_n = 0$ for $n \neq k$. Obviously, $\mathbf{y} \in Y_1^0$. Let $\mathbf{x} = G\mathbf{y}$. Then

$$x_n = \begin{cases} \mathcal{A}(n,k)x & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases}$$

Therefore

$$\sup_{n\geq k} \left(\|\mathcal{A}(n,k)x\|_n e^{\lambda\rho(n)}\right) = \|\mathbf{x}\|_\infty \leq \|G\| \cdot \|\mathbf{y}\|_1 = \|G\| \cdot \|x\|_k e^{\lambda\rho(k)}$$

and consequently

$$\|\mathcal{A}(n,k)x\|_n \leq \|G\| e^{-\lambda(\rho(n)-\rho(k))}\|x\|_k, \qquad \text{for } n \geq k > 1.$$

Furthermore, it follows from (5) and the continuity of A_1 that there exists $B > 0$, such that $\|A_1 x\|_2 \leq Be^{-\lambda(\rho(2)-\rho(1))}\|x\|_1$ for each $x \in X$. Hence,

$$\|\mathcal{A}(n,k)x\|_n \leq \|G\| \max\{1, B\} e^{-\lambda(\rho(n)-\rho(k))}\|x\|_k, \qquad \text{for } n \geq k. \tag{9}$$

Finally, it follows from (5) and (9) that

$$\|\mathcal{A}(n,k)x\| \leq C\|G\| \max\{1, B\} e^{-\lambda(\rho(n)-\rho(k))+\delta\rho(k)}\|x\|_k, \qquad \text{for } n \geq k.$$

We conclude that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential contraction and the proof of the theorem is complete. □

Now, we establish the robustness of a nonuniform contractions under sufficiently small linear perturbations.

Theorem 2. *Assume that ρ satisfies:*

(a) *There exists $\kappa \geq 0$, such that for each $c > \kappa$*

$$\sum_{k=1}^{\infty} e^{-c\rho(k)} < +\infty. \tag{10}$$

(b) *There exists $a > 0$, such that*

$$\rho(k+1) - \rho(k) \leq a, \quad \text{for each } k \in \mathbb{N}. \tag{11}$$

Let $(A_m)_{m\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$ be sequences of bounded operators on X, such that:

1. *$(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential contraction.*
2. *There exist constants $c \geq 0$ and $\eta - \varepsilon > \kappa$ such that*

$$\|A_m - B_m\| \leq ce^{-\eta\rho(m+1)}, \quad m \in \mathbb{N}. \tag{12}$$

If c is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential contraction.

Proof. It follows from Theorem 1 that there exists a sequence of norms $\|\cdot\|_m$, $m \in \mathbb{N}$ on X and $\lambda > 0$, such that for each $y = (y_n)_{n\in\mathbb{N}} \in Y_1^0$, there exists $x = (x_n)_{n\in\mathbb{N}} \in Y_\infty^0$, so that (4) holds, and such that (5) holds [with $D = C$ and $\delta = \varepsilon$, as in (3)]. Furthermore, let operator T be defined as in the proof of Theorem 1. We also consider the operator $L\colon \mathcal{D}(L) \to Y_1^0$ by

$$(L\mathbf{x})_1 = 0 \quad \text{and} \quad (L\mathbf{x})_{n+1} = x_{n+1} - B_n x_n \tag{13}$$

on the domain $\mathcal{D}(L)$ that consists of all $\mathbf{x} \in Y_\infty^0$, such that $L\mathbf{x} \in Y_1^0$. For any $\mathbf{x} \in Y_\infty^0$, using (5), (11), and (12), we have

$$\begin{aligned}
\|(T-L)\mathbf{x}\|_1 &= \sum_{k=1}^\infty \|(A_k - B_k)x_k\|_{k+1} e^{\lambda\rho(k+1)} \\
&\le C \sum_{k=1}^\infty e^{\varepsilon\rho(k+1)} \|(A_k - B_k)x_k\| e^{\lambda\rho(k+1)} \\
&\le cCe^{a\lambda} \sum_{k=1}^\infty e^{\varepsilon\rho(k+1)} e^{-\eta\rho(k+1)} \|x_k\|_k e^{\lambda\rho(k)} \\
&\le cCe^{a\lambda} \|\mathbf{x}\|_\infty \sum_{k=1}^\infty e^{(\varepsilon-\eta)\rho(k+1)}.
\end{aligned}$$

Hence, it follows from (10) that $\mathcal{D}(T) = \mathcal{D}(L)$ and

$$\|(T-L)\mathbf{x}\|_1 \le cK\|\mathbf{x}\|'_\infty \tag{14}$$

for any $\mathbf{x} \in \mathcal{D}(T)$ with some constant $K > 0$. Since T is invertible, it follows from (14) that for sufficiently small c, the operator L is invertible. By Theorem 1, the sequence $(B_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential contraction. □

We would like to comment on the conditions (a) and (b) in the statement of Theorem 2. We note that the condition (a) is satisfied with $\kappa = 0$, whenever ρ takes values in $\mathbb{N}$. Indeed, if $\rho\colon \mathbb{N} \to \mathbb{N}$, then we have

$$\sum_{k=1}^\infty e^{-c\rho(k)} \le \sum_{k=\rho(1)}^\infty e^{-ck} < +\infty,$$

for each $c > 0$. Furthermore, we note that (a) holds for polynomial growth rates $\rho(n) = \log n$ with $\kappa = 1$. The condition (b) is a bit more restrictive. However, it is again satisfied both for standard and polynomial growth rates among many others.

3. Nonuniform Expansions

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ of invertible bounded operators on X admits a ρ-nonuniform exponential expansion if there exist constants $D, \lambda > 0$ and $\varepsilon \ge 0$, such that

$$\|\mathcal{A}(m,n)\| \le De^{-\lambda(\rho(n)-\rho(m))+\varepsilon\rho(n)} \quad \text{for } m \le n, \tag{15}$$

where $\mathcal{A}(m,n) = \mathcal{A}(n,m)^{-1}$.

Given a sequence of norms $\|\cdot\|_n$, $n \in \mathbb{N}$ on X and $\lambda \in \mathbb{R}$, let

$$Z_\infty = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{N}} : \|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} (\|x_n\|_n e^{-\lambda\rho(n)}) < +\infty \right\}$$

and

$$Z_1 = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{N}} : \|\mathbf{x}\|_1 = \sum_{n=1}^{\infty} \|x_n\|_n e^{-\lambda\rho(n)} < +\infty \right\}.$$

Then, it is easy to verify that $(Z_\infty, \|\cdot\|_\infty)$ and $(Z_1, \|\cdot\|_1)$ are Banach spaces. Furthermore, Z_∞^0 will denote the closed subspace of Z_∞ that consists of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in Z_\infty$, such that

$$\lim_{n\to\infty} \left(\|x_n\|_n e^{-\lambda\rho(n)} \right) = 0.$$

Finally, Z_1^0 will be the closed subspace of Z_1 that consists of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in Z_1$, such that $x_1 = 0$.

Theorem 3. *The sequence $(A_m)_{m\in\mathbb{N}}$ of bounded invertible operators on X admits a ρ-nonuniform exponential expansion if and only if there exists a sequence of norms $\|\cdot\|_n$, $n \in \mathbb{N}$ and $\lambda > 0$, such that:*

1. *For each $y = (y_n)_{n\in\mathbb{N}} \in Z_1^0$, there exists a unique $x = (x_n)_{n\in\mathbb{N}} \in Z_\infty^0$, such that (4) holds.*
2. *There exist $C > 0$ and $\delta \geq 0$, such that (5) holds.*

Proof. For $m \in \mathbb{N}$ and $x \in X$, let

$$\|x\|_m = \sup_{n\leq m} \left(\|\mathcal{A}(n,m)x\| e^{\lambda(\rho(m)-\rho(n))} \right).$$

It follows directly from (15) that (5) holds with $C = D$ and $\delta = \varepsilon$. Furthermore, we have

$$\begin{aligned} \|\mathcal{A}(m,n)x\|_m &= \sup_{k\leq m} \left(\|\mathcal{A}(k,m)\mathcal{A}(m,n)x\| e^{\lambda(\rho(m)-\rho(k))} \right) \\ &= e^{-\lambda(\rho(n)-\rho(m))} \sup_{k\leq m} \left(\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(n)-\rho(k))} \right) \\ &\leq e^{-\lambda(\rho(n)-\rho(m))} \|x\|_n, \end{aligned} \tag{16}$$

for $m \leq n$. For an arbitrary $y = (y_n)_{n\in\mathbb{N}} \in Z_1^0$, we define a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ by

$$x_n = -\sum_{k=n+1}^{\infty} \mathcal{A}(n,k) y_k, \qquad n \in \mathbb{N}.$$

It follows from (16) that

$$\begin{aligned} e^{-\lambda\rho(n)} \|x_n\|_n &\leq e^{-\lambda\rho(n)} \sum_{k=n+1}^{\infty} e^{-\lambda(\rho(k)-\rho(n))} \|y_k\|_k \\ &\leq \sum_{k=1}^{\infty} e^{-\lambda\rho(k)} \|y_k\|_k = \|\mathbf{y}\|_1, \end{aligned} \tag{17}$$

for each $n \in \mathbb{N}$. Hence, $\mathbf{x} \in Z_\infty$. Furthermore, it follows from the first inequality in (17) that

$$\lim_{n\to\infty} \left(\|x_n\|_n e^{-\lambda\rho(n)}\right) = 0,$$

and therefore, $\mathbf{x} \in Z_\infty^0$. Finally, one can easily verify that (4) holds.

Now, we establish the uniqueness of $\mathbf{x}$. It is sufficient to consider the case when $\mathbf{y} = 0$. Thus, suppose that $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in Z_\infty^0$ satisfies $x_{n+1} = A_n x_n$ for each $n \in \mathbb{N}$. By (16)

$$\begin{aligned}\|x_1\|_1 &= \|\mathcal{A}(1,n)x_n\|_n \\ &\le e^{-\lambda(\rho(n)-\rho(1))}\|x_n\|_n,\end{aligned}$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ yields (see (2)) $x_1 = 0$ and, therefore, $x_n = 0$ for every $n \in \mathbb{N}$.

We now prove the converse. Let $T\colon \mathcal{D}(T) \to Y_1^0$ be the linear operator defined by (7) on the domain $\mathcal{D}(T)$ that consists of all $\mathbf{x} \in Z_\infty^0$, such that $T\mathbf{x} \in Z_1^0$. Proceeding as in the proof of Lemma 1, one can easily prove that T is closed. Consequently, $\mathcal{D}(T)$ becomes a Banach space with respect to the norm $\|\cdot\|'_\infty$ defined as in (8). It follows from the assumptions of the theorem that the operator $T\colon (\mathcal{D}(T), \|\cdot\|'_\infty) \to Y_1^0$ is well-defined, bounded, and invertible. Let G be the inverse of T.

Take an arbitrary $k > 1$, $x \in X$ and define a sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ by $y_k = x$ and $y_n = 0$ for $n \ne k$. Obviously, $\mathbf{y} \in Y_1^0$. Let $\mathbf{x} = G\mathbf{y}$. Then

$$x_n = \begin{cases} -\mathcal{A}(n,k)x & \text{if } n < k, \\ 0 & \text{if } n \ge k. \end{cases}$$

Therefore

$$\sup_{n<k} \left(\|\mathcal{A}(n,k)x\|_n e^{-\lambda\rho(n)}\right) = \|\mathbf{x}\|_\infty \le \|G\| \cdot \|\mathbf{y}\|_1 = \|G\| \cdot \|x\|_k e^{-\lambda\rho(k)}$$

and consequently

$$\|\mathcal{A}(n,k)x\|_n \le \max\{\|G\|, 1\} e^{-\lambda(\rho(k)-\rho(n))}\|x\|_k, \qquad \text{for } n \le k.$$

It follows from (5) that

$$\|\mathcal{A}(n,k)x\| \le \max\{\|G\|, 1\} e^{-\lambda(\rho(k)-\rho(n))+\delta\rho(k)}\|x\|, \qquad \text{for } n \le k.$$

We conclude that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential expansion. □

The following is a version of Theorem 2 for nonuniform expansions.

Theorem 4. *Assume that there exists $\kappa \ge 0$, such that (10) holds for each $c > \kappa$. Furthermore, let $(A_m)_{m\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$ be sequences of bounded invertible operators on X such that:*

1. *$(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential expansion.*
2. *There exist constants $c \ge 0$ and $\eta - \varepsilon > \kappa$, such that (12) holds.*

If c is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential expansion.

Proof. It follows from Theorem 3 that there exists a sequence of norms $\|\cdot\|_m$, $m \in \mathbb{N}$ on X and $\lambda > 0$, such that for each $y = (y_n)_{n\in\mathbb{N}} \in Z_1^0$, there exists $x = (x_n)_{n\in\mathbb{N}} \in Z_\infty^0$, so that (4) holds, and such that (5) holds [with $C = D$ and $\delta = \varepsilon$, as in (15)]. Furthermore, let operator T be defined as in the proof of Theorem 3. We also consider the operator $L\colon \mathcal{D}(L) \to Y_1^0$ defined by (13) on the domain $\mathcal{D}(L)$ that consists of all $\mathbf{x} \in Z_\infty^0$, such that $L\mathbf{x} \in Z_1^0$. For any $\mathbf{x} \in Y_\infty^0$, using (5) and (12), we have

$$\begin{aligned}
\|(T-L)\mathbf{x}\|_1 &= \sum_{k=1}^{\infty} \|(A_k - B_k)x_k\|_{k+1} e^{-\lambda\rho(k+1)} \\
&\le C \sum_{k=1}^{\infty} e^{\varepsilon\rho(k+1)} \|(A_k - B_k)x_k\| e^{-\lambda\rho(k+1)} \\
&\le cC \sum_{k=1}^{\infty} e^{\varepsilon\rho(k+1)} e^{-\eta\rho(k+1)} \|x_k\|_k e^{-\lambda\rho(k)} \\
&\le cC \|\mathbf{x}\|_\infty \sum_{k=1}^{\infty} e^{(\varepsilon-\eta)\rho(k+1)}.
\end{aligned}$$

Hence, we have that $\mathcal{D}(T) = \mathcal{D}(L)$ and (14) holds for any $\mathbf{x} \in \mathcal{D}(T)$ with some constant $K > 0$. Since T is invertible, it follows from (14) that for sufficiently small c, the operator L is invertible. By Theorem 3, the sequence $(B_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential expansion. $\square$

4. Strong Nonuniform Contractions

We now consider the stronger notion of a ρ-exponential contraction. We say that the sequence $(A_m)_{m\in\mathbb{N}}$ of invertible bounded operators on X admits a strong ρ-nonuniform exponential contraction if there exist constants

$$D > 0, \quad \mu \ge \lambda > 0 \quad \text{and } \varepsilon \ge 0$$

such that (3) holds and such that

$$\|\mathcal{A}(m,n)\| \le De^{\mu(\rho(n)-\rho(m))+\varepsilon\rho(n)} \quad \text{for } m \le n, \tag{18}$$

where $\mathcal{A}(m,n) = \mathcal{A}(n,m)^{-1}$. We refer to [1] for some explicit examples of strong nonuniform contractions as well as for nonuniform contractions which are not strong for the usual exponential behavior when $\rho(n) = n$.

Theorem 5. *The sequence $(A_m)_{m\in\mathbb{N}}$ of bounded invertible operators on X admits a strong ρ-nonuniform exponential contraction if and only if there exists a sequence of norms $\|\cdot\|_n$, $n \in \mathbb{N}$ and $\lambda > 0$ such that:*

1. *For each $y = (y_n)_{n\in\mathbb{N}} \in Y_1^0$, there exists $x = (x_n)_{n\in\mathbb{N}} \in Y_\infty^0$ such that (4) holds.*
2. *There exist $C > 0$ and $\delta \ge 0$ such that (5) holds.*

3. *There exist $K, b > 0$, such that*
$$\frac{1}{K} e^{b(\rho(n)-\rho(n+1))} \|x\|_n \leq \|A_n x\|_{n+1}, \tag{19}$$
for every $n \in \mathbb{N}$ and $x \in X$.

Proof. For $m \in \mathbb{N}$ and $x \in X$, let
$$\begin{aligned} \|x\|_m = & \sup_{n \geq m} (\|\mathcal{A}(n,m)x\| e^{\lambda(\rho(n)-\rho(m))}) \\ & + \sup_{n<m} (\|\mathcal{A}(n,m)x\| e^{-\mu(\rho(m)-\rho(n))}). \end{aligned}$$

It follows from (3) and (18) that (5) holds with $C = 2D$ and $\delta = \varepsilon$. Moreover, using $\lambda \leq \mu$, we have
$$\begin{aligned} \|\mathcal{A}(m,n)x\|_m = & \sup_{k \geq m} \left(\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(k)-\rho(m))}\right) \\ & + \sup_{k<m} \left(\|\mathcal{A}(k,n)x\| e^{-\mu(\rho(m)-\rho(k))}\right) \\ \leq & \sup_{k \geq m} \left(\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(k)-\rho(m))}\right) \\ & + \sup_{k<n} \left(\|\mathcal{A}(k,n)x\| e^{-\mu(\rho(m)-\rho(k))}\right) \\ & + \sup_{n \leq k<m} \left(\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(k)-\rho(m))}\right) \\ \leq & \ 2e^{-\lambda(\rho(m)-\rho(n))} \sup_{k \geq n} (\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(k)-\rho(n))}) \\ & + e^{-\mu(\rho(m)-\rho(n))} \sup_{k<n} (\|\mathcal{A}(k,n)x\| e^{-\mu(\rho(n)-\rho(k))}) \\ \leq & \ 2e^{-\lambda(\rho(m)-\rho(n))} \|x\|_n, \end{aligned}$$
for $m \geq n$. Similarly,
$$\begin{aligned} \|\mathcal{A}(m,n)x\|_m \leq & \sup_{k \geq n} (\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(k)-\rho(m))}) \\ & + \sup_{m \leq k<n} (\|\mathcal{A}(k,n)x\| e^{\mu(\rho(k)-\rho(m))}) \\ & + \sup_{k<m} (\|\mathcal{A}(k,n)x\| e^{-\mu(\rho(m)-\rho(k))}) \\ \leq & \ e^{\lambda(\rho(n)-\rho(m))} \sup_{k \geq n} (\|\mathcal{A}(k,n)x\| e^{\lambda(\rho(k)-\rho(n))}) \\ & + 2e^{\mu(\rho(n)-\rho(m))} \sup_{k<n} (\|\mathcal{A}(k,n)x\| e^{-\mu(\rho(n)-\rho(k))}) \\ \leq & \ 2e^{\mu(\rho(n)-\rho(m))} \|x\|_n, \end{aligned}$$
for $m \leq n$. We conclude that
$$\|\mathcal{A}(m,n)x\|_m \leq 2e^{-\lambda(\rho(m)-\rho(n))} \|x\|_n, \quad \text{for } m \geq n \tag{20}$$
and
$$\|\mathcal{A}(m,n)x\|_m \leq 2e^{\mu(\rho(n)-\rho(m))} \|x\|_n, \qquad \text{for } m \leq n. \tag{21}$$

Proceeding as in the proof of Theorem 1, one can prove that (20) implies the first assertion of the theorem. Finally, it follows from (21) that

$$\frac{1}{2}e^{\mu\big(\rho(n)-\rho(n+1)\big)}\|x\|_n \leq \|A_n x\|_{n+1}$$

for every $n \in \mathbb{N}$ and $x \in X$. We conclude that (19) holds with $K = 2$ and $b = \mu$.

To prove the converse, we note that it follows directly from Theorem 1 that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential contraction. Furthermore, it follows from (19) that

$$\|\mathcal{A}(m,n)x\|_m \leq Ke^{b(\rho(n)-\rho(m))}\|x\|_n, \qquad \text{for } m \leq n \text{ and } x \in X.$$

By (5), we have

$$\|\mathcal{A}(m,n)\| \leq Ke^{b(\rho(n)-\rho(m))+\delta\rho(n)}$$

for $m \leq n$. We conclude that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a strong ρ-nonuniform exponential contraction. □

Now, we establish the robustness of strong nonuniform contractions under sufficiently small linear perturbations.

Theorem 6. *Assume that there exist $\kappa \geq 0$, such that (10) holds for $c > \kappa$ and that (11) holds for some $a > 0$. Let $(A_m)_{m\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$ be sequences of bounded invertible operators on X, such that:*

1. *$(A_m)_{m\in\mathbb{N}}$ admits a strong ρ-nonuniform exponential contraction.*
2. *There exist constants $c \geq 0$ and $\eta > 0$ such that (12) holds.*

If c is sufficiently small and η is sufficiently large, then the sequence $(B_m)_{m\in\mathbb{N}}$ admits a strong ρ-nonuniform exponential contraction.

Proof. In view of Theorem 2 and the characterization of strong ρ-nonuniform contractions given by Theorem 5, it is sufficient to show that there exist $K', b' > 0$, such that

$$\frac{1}{K'}e^{b'(\rho(n)-\rho(n+1))}\|x\|_n \leq \|B_n x\|_{n+1}, \tag{22}$$

for every $n \in \mathbb{N}$ and $x \in X$, where $\|\cdot\|_n$ is a sequence of norms given by Theorem 5. Indeed, it follows from (5), (12) and (19) that

$$\begin{aligned}\|B_n x\|_{n+1} &\geq \|A_n x\|_{n+1} - \|(B_n - A_n)x\|_{n+1}\\ &\geq \frac{1}{K}e^{b(\rho(n)-\rho(n+1))}\|x\|_n - cCe^{(\varepsilon-\eta)\rho(n+1)}\|x\|_n\\ &\geq \frac{1}{K}e^{b(\rho(n)-\rho(n+1))}\|x\|_n - cCe^{(\eta-\varepsilon)(\rho(n)-\rho(n+1))}\|x\|_n,\end{aligned}$$

for every $n \in \mathbb{N}$ and $x \in X$. Taking η sufficiently large so that $\eta - \varepsilon \geq b$ and c sufficiently small, we conclude that (22) holds with $b' = b$ and some $K' > 0$. □

5. Strong Nonuniform Expansions

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ of invertible bounded operators on X admits a strong ρ-nonuniform exponential expansion if there exist constants

$$D>0, \quad \mu\ge\lambda>0 \quad \text{and } \varepsilon\ge 0$$

such that (15) holds and such that

$$\|\mathcal{A}(m,n)\|\le De^{\mu(\rho(m)-\rho(n))+\varepsilon\rho(n)} \quad \text{for } m\ge n. \tag{23}$$

Theorem 7. *The sequence $(A_m)_{m\in\mathbb{N}}$ of bounded invertible operators on X admits a strong ρ-nonuniform exponential expansion if and only if there exists a sequence of norms $\|\cdot\|_n$, $n\in\mathbb{N}$, and $\lambda>0$ such that:*

1. *For each $y=(y_n)_{n\in\mathbb{N}}\in Z_1^0$, there exists $x=(x_n)_{n\in\mathbb{N}}\in Z_\infty^0$ such that (4) holds.*
2. *There exist $C>0$ and $\delta\ge 0$ such that (5) holds.*
3. *There exist $K,b>0$ such that*

$$\|A_nx\|_{n+1}\le Ke^{b(\rho(n+1)-\rho(n))}\|x\|_n, \tag{24}$$

for every $n\in\mathbb{N}$ and $x\in X$.

Proof. For $m\in\mathbb{N}$ and $x\in X$, let

$$\begin{aligned}\|x\|_m = &\sup_{n\ge m}\left(\|\mathcal{A}(n,m)x\|e^{-\mu(\rho(n)-\rho(m))}\right)\\ &+\sup_{n<m}\left(\|\mathcal{A}(n,m)x\|e^{\lambda(\rho(m)-\rho(n))}\right).\end{aligned}$$

It follows from (15) and (23) that (5) holds with $C=2D$ and $\delta=\varepsilon$. Furthermore, using $\lambda\le\mu$, we have

$$\begin{aligned}\|\mathcal{A}(m,n)x\|_m &= \sup_{k\ge m}\left(\|\mathcal{A}(k,n)x\|e^{-\mu(\rho(k)-\rho(m))}\right)\\ &\quad+\sup_{k<m}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(\rho(m)-\rho(k))}\right)\\ &\le \sup_{k\ge m}\left(\|\mathcal{A}(k,n)x\|e^{-\mu(\rho(k)-\rho(m))}\right)\\ &\quad+\sup_{k<n}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(\rho(m)-\rho(k))}\right)\\ &\quad+\sup_{n\le k<m}\left(\|\mathcal{A}(k,n)x\|e^{\mu(\rho(m)-\rho(k))}\right)\\ &\le 2e^{\mu(\rho(m)-\rho(n))}\sup_{k\ge n}\left(\|\mathcal{A}(k,n)x\|e^{-\mu(\rho(k)-\rho(n))}\right)\\ &\quad+e^{\lambda(\rho(m)-\rho(n))}\sup_{k<n}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(\rho(n)-\rho(k))}\right)\\ &\le 2e^{\mu(\rho(m)-\rho(n))}\|x\|_n,\end{aligned}$$

for $m \geq n$. Moreover

$$\begin{aligned}\|\mathcal{A}(m,n)x\|_m &\leq \sup_{k\geq n}\left(\|\mathcal{A}(k,n)x\|e^{-\mu(\rho(k)-\rho(m))}\right)\\ &\quad+\sup_{m\leq k<n}\left(\|\mathcal{A}(k,n)x\|e^{-\lambda(\rho(k)-\rho(m))}\right)\\ &\quad+\sup_{k<m}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(\rho(m)-\rho(k))}\right)\\ &\leq e^{-\mu(\rho(n)-\rho(m))}\sup_{k\geq n}\left(\|\mathcal{A}(k,n)x\|e^{-\mu(\rho(k)-\rho(n))}\right)\\ &\quad+2e^{-\lambda(\rho(n)-\rho(m))}\sup_{k<n}\left(\|\mathcal{A}(k,n)x\|e^{\lambda(\rho(n)-\rho(k))}\right)\\ &\leq 2e^{-\lambda(\rho(n)-\rho(m))}\|x\|_n,\end{aligned}$$

for $m \leq n$. We conclude that

$$\|\mathcal{A}(m,n)x\|_m \leq 2e^{\mu(\rho(m)-\rho(n))}\|x\|_n, \quad \text{for } m \geq n \tag{25}$$

and

$$\|\mathcal{A}(m,n)x\|_m \leq 2e^{-\lambda(\rho(n)-\rho(m))}\|x\|_n, \quad \text{for } m \leq n. \tag{26}$$

Proceeding as in the proof of Theorem 3, one can prove that (26) implies the first assertion of the theorem. Finally, it follows from (25) that (24) holds with $K = 2$ and $b = \mu$.

To prove the converse, we note that it follows directly from Theorem 3 that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a ρ-nonuniform exponential expansion. Furthermore, it follows from (19) that

$$\|\mathcal{A}(m,n)x\|_m \leq Ke^{b(\rho(n)-\rho(m))}\|x\|_n, \quad \text{for } m \leq n \text{ and } x \in X.$$

By (5), we have

$$\|\mathcal{A}(m,n)\| \leq Ke^{b(\rho(m)-\rho(n))+\delta\rho(n)}$$

for $m \geq n$. We conclude that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a strong ρ-nonuniform exponential expansion. □

The following is a version of Theorem 6 for strong nonuniform expansions.

Theorem 8. *Assume that there exist $\kappa \geq 0$, such that (10) holds for $c > \kappa$. Let $(A_m)_{m\in\mathbb{N}}$ and $(B_m)_{m\in\mathbb{N}}$ be sequences of bounded invertible operators on X, such that:*

1. *$(A_m)_{m\in\mathbb{N}}$ admits a strong ρ-nonuniform exponential expansion.*
2. *There exist constants $c \geq 0$ and $\eta > \varepsilon$ such that (12) holds.*

If c is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{N}}$ admits a strong ρ-nonuniform exponential contraction.

Proof. In view of Theorem 4 and the characterization of strong ρ-nonuniform expansions given by Theorem 7, it is sufficient to show that there exist $K', b' > 0$, such that

$$\|B_n x\|_{n+1} \leq K'e^{b'(\rho(n+1)-\rho(n))}\|x\|_n, \tag{27}$$

for every $n \in \mathbb{N}$ and $x \in X$, where $\|\cdot\|_n$ is a sequence of norms given by Theorem 5. Indeed, it follows from (5), (12), and (24) that

$$\begin{aligned}\|B_n x\|_{n+1} &\le \|A_n x\|_{n+1} + \|(B_n - A_n)x\|_{n+1} \\ &\le K e^{b(\rho(n+1)-\rho(n))}\|x\|_n + C e^{(\varepsilon-\eta)\rho(n+1)}\|x\|_n \\ &\le K e^{b(\rho(n+1)-\rho(n))}\|x\|_n + cC e^{(\varepsilon-\eta)(\rho(n+1)-\rho(n))}\|x\|_n,\end{aligned}$$

for every $n \in \mathbb{N}$ and $x \in X$. We conclude that (27) holds with $b' = b$ and some $K' = K + cC$. □

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