Dynamics of Extended Gradient Systems

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Is it safe?

The motivating example

Example: Allen-Cahn equation

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\begin{aligned}
u_t &= u_{xx} + u - u^3, \\
u(-L) &= u(L) = 0, \\
u^0 &= u(0), \\
u(x,.) &\in H_0^2([-L, L]).
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Lyapunov function $L(u) = \int_{-L}^{L} \left[\frac{1}{2}u_x^2 + \frac{1}{4}u^4 - \frac{1}{2}u^2\right] dx.$

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La Salle principle: dynamics converges to equilibria

But: It may take **exponentially long time** (a function of L) to get there!

Motivation

Consider the same equation on an unbounded domain, $u \in H^2_{\mathrm{ul}}(\mathbb{R})$

Eckmann, Rougemont (many other authors): A model for coarsening

Polačik 2014,2015: non-equilibria in ω-limit sets! (w.r. to uniform convergence on compact sets)

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- What is the right definition of asymptotics for systems on large domains?
- Can one find bounds on relaxation times independent of the domain size?
- Important special case 1: Lagrangian dynamics with finite and infinite degrees of freedom
- Important special case 2: Some Markov chains on infinite lattices and phase transitions (?)

Content of the talk

Part 1: Description of asymptotics and uniform recurrence

Example: Reaction-diffusion equation, many others

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(Mostly) joint work with Thierry Gallay, Grenoble

Part 1: Abstract definition of extended gradient systems

φ a continuous semiflow on metrizable $\mathcal{X}$

$e(x,t)$-energy, $d(x,t)$-energy dissipation, $f(x,t)$-energy flux fields associated to each (semi)-orbit of φ

$$\begin{aligned} e,d &: \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}, \\ f &: \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N. \end{aligned}$$

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(A1) e, d, f continuous in x, t,

(A2) $e(x,t) = \mathrm{div}_x f(x,t) - d(x,t)$ in the distributional sense,

(A3) $f \leq b(e)d$ **for some strictly increasing** $b : \mathbb{R} \to \mathbb{R}$,

(A4) $d \geq 0$, e bounded from below,

(A5) $d \equiv 0$ if and only if the orbit is in equilibrium.

Example: Reaction-diffusion equation

$$\partial_t u(x,t) = \triangle u(x,t) - V'(u(x,t))$$

- $u : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$,
- $V : \mathbb{R} \to \mathbb{R}_+$ a smooth potential,
- $\mathcal{X} = H^2_{\mathrm{ul}}(\mathbb{R}^N)$ - the uniformly local space

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Axioms hold:

$$\begin{aligned} e &= \frac{1}{2}|\nabla u|^2 + V(u), \\ f &= u_t \nabla u, \\ d &= u_t^2. \end{aligned}$$

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(A3): $|f|^2 = u_t^2 |\nabla u|^2 \leq 2ed$.

A ton of other examples

- Strongly damped wave equation

$$u_{tt} + u_t - \alpha \triangle u_t = \triangle u - V'(u),$$

- The complex Ginzburg-Landau equation, $\alpha = \beta$,

$$u_t = (1 + i\alpha)\triangle u + u - (1 + i\beta)|u|^2 u,$$

- The Landau-Lifshitz-Gilbert Equation, $u : \mathbb{R}^N \to \mathbb{S}^2$,

$$u_t = -u \wedge (u \wedge \triangle u) + \alpha u \wedge \triangle u,$$

- A nonlinear diffusion equation, $a : \mathbb{R} \to (0, \infty)$ smooth:

$$u_t = \mathrm{div}(a(u)\nabla u).$$

Asymptotics and uniform recurrence

Assume $N = 1, 2$, $\mathcal{X}$ compact.

(e.g. a bounded, closed invariant set in $H^2_{\mathrm{ul}}(\mathbb{R}^N)$ equipped with topology of uniform convergence on compact sets)

Theorem (Th. Gallay, S.Sl, 2001, 2015)

1. ω-limit set always contains an equilibrium,

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Counterexamples if $N \geq 3$!

Qualitative bounds and the concept of proof

N=1, $R \leq \sqrt{T}$:

$$\int_0^T \int_{B_R} d(x,t)\,dxdt = O\left(\sqrt{T}\right).$$

N=2, $R \leq T/\log T$:

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Qualitative bounds and the concept of proof

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Idea of the proof:

1. Write integral equation for e, d, f,
2. Factor-out d by (A3), bound e from above,
3. Write ordinary 1-d differential inequality in R for flux $F(R,T)$ integrated over the R-sphere and over $[0,T]$,
4. Solve and check when $F(R,T)$ blows-out for finite R.

Uniformly local convergence to equilibria

Lucky case: two dissipating energies for the same system!!

Assume we can associate to $\mathcal{X}$, φ, two families

e, d, f satisfying (A1-5), (A3): $f^2 \leq \beta_1 ed$,

$\varepsilon, \delta, \psi$ satisfying (A1-5), (A3): $\psi^2 \leq \beta_2 \varepsilon\delta$,

$\varepsilon \leq \gamma d$ for some $\gamma > 0$!

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Theorem: (Th. Gallay, S.Sl, 2014,2015)

Assume all as above, $N = 1$ (possibly $N = 2$ - to be checked). Then for any initial condition, e is uniformly bounded, and δ converges uniformly locally to 0.

If $e, d, f, \varepsilon, \delta, \psi$ are regular enough, dynamics always converges uniformly locally to equilibria (with explicit upper bounds).

Example: Navier-Stokes on a strip

$$
\begin{aligned}
\partial_t u + (u \cdot \nabla) u &= \nu \Delta u - \frac{1}{\rho} \nabla p, \\
\mathrm{div} u &= 0 \ ,
\end{aligned}
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In the strip $u \in \Omega_L = \mathbb{R} \times [0, L]$, $u((x_1, 0), t) = u((x_1, L), t)$.

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In the strip $u \in \Omega_L = \mathbb{R} \times [0, L]$, $u((x_1, 0), t) = u((x_1, L), t)$.

$$\begin{aligned} e(x_1, t) &= \frac{1}{2} \int_0^L |u(x_1, x_2, t)|^2 dx_2 + \frac{M^2}{2}, \\ d(x_1, t) &= \int_0^L |\nabla u(x_1, x_2, t)|^2 dx_2, \\ \varepsilon(x_1, t) &= \frac{1}{2} \int_0^L |\omega(x_1, x_2, t)|^2 dx_2 \ , \\ \delta(x_1, t) &= \int_0^L |\nabla \omega(x_1, x_2, t)|^2 dx_2, \end{aligned}$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$.

Navier-Stokes on the strip: explicit bounds

Theorem: (Th. Gallay, S.Sl, 2015) convergence of $||\hat{u}(., t)||_\infty, ||\omega||_\infty^2$ to 0 as $O(1/\sqrt{t})$.

$\hat{u}(, .t)$ - the "oscillatory" part of u,

$$\hat{u}(x, t) = u(x, t) - \frac{1}{L}\int_0^L u(x, t) dx_2.$$

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In more detail:

1. *(Uniform boundedness of the velocity)* There exists $C > 0$ such that, for all $t \geq 0$,

$$\frac{L}{\nu}\|u(\cdot,t)\|_{L^\infty(\Omega_L)} \leq C\left(R_u + R_\omega + (1+R_\omega)(R_u^2 + R_\omega^2)\right),$$

where $R_u = \frac{L}{\nu}\|u_0\|_{L^\infty(\Omega_L)}$, $R_\omega = \frac{L^2}{\nu}\|\omega_0\|_{L^\infty(\Omega_L)}$.

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Navier-Stokes on the strip: further bounds

2. *(Uniform decay of the vorticity)* There exists $C > 0$ such that, for all $t > 0$,

$$\left(\frac{L^2}{\nu}\,\|\omega(\cdot,t)\|_{L^\infty(\Omega_L)}\right)^2 \;\le\; C(1+R_\omega)(R_u^2+R_\omega^2)\frac{L}{\sqrt{\nu t}}\;.$$

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3. *(Exponential convergence to a shear flow)* Assume $||\omega_0||_{L^\infty} < 4\pi^2$. For any $\gamma < 2\pi^2$, we have

$$\frac{L}{\nu}\,\|u(\cdot,t)-u_\infty(\cdot,t)\|_{L^\infty(\Omega_L)} \;=\; O\left(\exp\left(-\frac{\gamma\nu t}{L^2}\right)\right)\;,$$

where

$$u_\infty(x,t) \;=\; \begin{pmatrix} c \\ m(x_1,t)\end{pmatrix}\;,\qquad p(x,t) \;=\; 0\;, \tag{1}$$

with $c \in \mathbb{R}$ and $\partial_t m + c\partial_1 m - \nu\partial_1^2 m = O(exp^{-2\gamma\nu t/L^2})$ as $t \to \infty$.

An entropy reaction-diffusion equation

Unucky case: no two dissipating energies / invariant manifold not flat

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Example: Entropy reaction-diffusion equation (Mielke, Haskovec, 2015), $a, b > 0$:

$$\begin{aligned} u_t &= a u_{xx} + k(v^2 - u) \\ v_t &= b v_{xx} + 2k(u - v^2) \end{aligned}$$

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It is an extended gradient system!

$$\begin{aligned} F(u) &= u \log u - u + 1 \\ e &= F(u) + F(v) + e_0 \\ f &= a(\log u) u_x + b(\log v) v_x \\ d &= a\frac{u_x^2}{u} + b\frac{v_x^2}{v} + k(v^2 - u)\log\frac{v^2}{u} \end{aligned}$$

Convergence to an invariant manifold

Theorem (Th. Gallay, S.Sl, work in progress)
Assume $\mathcal{M}$ is an invariant normally hyperbolic subset of $\mathcal{X}$ (i.e. normally hyperbolic for any restriction to a finite domain).
Assume $\mathcal{X} \setminus \mathcal{M}$ contains no equilibria.
Then the dynamics converges uniformly locally to $\mathcal{M}$.

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Assume $\mathcal{X} \setminus \mathcal{M}$ contains no equilibria.
Then the dynamics converges uniformly locally to $\mathcal{M}$.

Also bounds on relaxation times, depending on energy dissipation away from $\mathcal{M}$

Proof: Adapting ideas of Otto, Reznikoff (JDE 2004) + extended dynamical systems techniques

Corollary: The entropy reaction diffusion equation converges uniformly locally to $v^2 = u$.

An ODE toy example

$$x_t = -\nabla E(y) + f(y,t)$$

- $x(t) \in \mathbb{R}^N$,
- $E : \mathbb{R}^N \to \mathbb{R}$ smooth, $E \geq 0$,
- $E(0) = 0$,
- $|f(x,t)| \leq \delta$, f smooth.

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Stronger result, sufficient and necessary condition: Connected component of $\{x, E(x) = 0\}$ containing 0 does not intersect $\{x = 1\}$ (Proof: Morse-Sard theorem).

Pohožaev Theorem

$A : \mathcal{X} \to \mathbb{R}$,

$\mathcal{X}$ a real, separable, reflexive Banach space

A is Fréchet, if it is C^2 (in the sence of Fréchet derivatives), dimension of Ker $D^2 A(h)$ (as an operator from $\mathcal{X}$ to $\mathcal{X}^*$ for a fixed h) is finite dimensional.

The critical value of A is any value $x \in \mathbb{R}$, such that there exists $h \in \mathcal{X}$ so that $A(h) = x$, $DA(h) \equiv 0$.

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Theorem (Morse-Sard-Pohožaev) Assume that Ker $D^2A(h) \leq m < \infty$ for any $h \in \mathcal{X}$, and let $k \geq \max\{m, 2\}$. Then the set of critical values of A has Lebesgue measure 0.

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Tool to bound dissipation from below away from zero!

Example: Lagrangian dynamics

We say that $L : \mathbb{R}^N/\mathbb{Z}^N \times \mathbb{R}^N \times \mathbb{R}$, $L(q, v, t)$ is a time-dependent Tonelli Lagrangian on a torus, if

1. L is C^2,
2. L is strictly convex in the fibers, i.e. $\partial^2 L/\partial v^2$ is positive definite,
3. L is superlinear in each fiber, i.e.

$$\lim_{||v||\to\infty} \frac{L(x, v, t)}{||v||} = +\infty,$$

4. L is time-periodic, i.e. $L(x, v, t) = L(x, v, t + 1)$.

Euler-Lagrange equations $\delta A(q) = 0$:

$$L_{vv}(q, q_t, t)q_{tt} - L_x(q, q_t, t) + L_{qv}(q, q_t, t)q_t + L_{vt}(q, q_t, t) = 0.$$

Formally gradient dynamics of the action

We write $q_s = -\delta A(q)0$ as

$$q_s = L_{vv}(q, q_t, t)q_{tt} - L_x(q, q_t, t) + L_{qv}(q, q_t, t)q_t + L_{vt}(q, q_t, t).$$

We change for $q : \mathbb{R} \to \mathbb{R}^N$:

$$q_s = q_{tt} + L_{vv}^{-1}(q, q_t, t)\left(-L_x(q, q_t, t) + L_{qv}(q, q_t, t)q_t + L_{vt}(q, q_t, t)\right)$$

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Corollary: Every bounded invariant set of the formally gradient dynamics of the action contains a Euler-Lagrange orbit.

Arnold's example

V. Arnold, 1964: $q = (u, v) \in \mathbb{R}^2$:

$$
\begin{aligned}
L(u, v, u_t, v_t, t) &= u_t^2/2 + v_t^2/2 + V(u, v, t) \\
V(u, v, t) &= \varepsilon \left[1 - \cos(u)\right] \left[1 - \mu(\cos(v) + cos(t))\right]
\end{aligned}
$$

Arnold's example

V. Arnold, 1964: $q = (u, v) \in \mathbb{R}^2$:

$$\begin{aligned} L(u, v, u_t, v_t, t) &= u_t^2/2 + v_t^2/2 + V(u, v, t) \\ V(u, v, t) &= \varepsilon \left[1 - \cos(u)\right] \left[1 - \mu(\cos(v) + cos(t))\right] \end{aligned}$$

- V.I. Arnold, Instability of dynamical systems with several degrees of freedom, Soviet. Math. Dokl. **5** (1964), 581-585.
- Introduced "Arnold's diffusion" - construction of "wandering" orbits in nonintegrable Hamiltonian systems
- "The details of the proof must be formidable, although the idea of the proof is clearly outlined." - J. Moser, Mathematical Reviews, 1965
- "Perhaps it is difficult to find a 4 page paper that has generated so much." - Rafael de la Llave, International Congress of Mathematicians, Hyderabad, 2010

Extended gradient dynamics approach

$$
\begin{aligned}
u_s &= u_{tt} - \varepsilon \sin(u) \left[1 - \mu(\cos(v) + cos(t))\right], \\
v_s &= v_{tt} - \varepsilon\mu \left[1 - \cos(u)\right] \sin(v).
\end{aligned}
$$

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Theorem: Construction of

- "Diffusive" orbits of the Euler-Lagrange flow for a wide range of ε, μ,
- Orbits dense in the momentum space,
- Positive-entropy invariant measures.

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Theorem: Construction of

- "Diffusive" orbits of the Euler-Lagrange flow for a wide range of ε, μ,
- Orbits dense in the momentum space,
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Proof:

- Invariant set by energy - energy dissipation - flux equality "between jumps"
- Lower bound on dissipation by Pohožaev theorem
- Upper bound on flux away from "jumps" by hyperbolicity argument.

To be continued ... (B. Rabar talk)

Thank you!