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Results in Mathematics

Nonuniform Stability of Arbitrary Difference Equations

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Abstract. We describe two approaches to study the robustness of the exponential stability of an arbitrary difference equation with finite delay on a Banach space. The first approach is based on a characterization of Perron-type of the exponential stability in terms of the invertibility of a certain linear operator between spaces of bounded sequences. The second approach is based on looking at the dynamics on a higher-dimensional space without delay and so for which one has the cocycle property. We emphasize that none of the results obtained with the two approaches implies the other. More precisely, the second approach allows to obtain the robustness of a weaker notion although also with a stronger hypothesis. We consider the general case of nonuniform exponential stability.

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1. Introduction

Our main aim is to describe two approaches to study the robustness of the exponential stability of an arbitrary difference equation with finite delay on a Banach space. We say that a given type of stability is robust when it persists under sufficiently small perturbations in some appropriate class. We shall consider the class of the linear difference equations with finite delay.

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1.1. Preliminaries

Consider a linear difference equation

$$x(n+1) = \sum_{k=0}^{n} A(n,k)x(k), \quad n \geq 0, \tag{1}$$

where each $A(n,k)$ is a bounded linear operator acting on a Banach space. We say that Eq. (1) has a *finite delay* if there exists $r \in \mathbb{Z}_0^-$ such that $A(n,k) = 0$ for $n > k - r$, that is, if the equation can be written in the form

$$x(n+1) = \sum_{k=\max\{0,n+r\}}^{n} A(n,k)x(k), \quad n \geq 0.$$

Since Eq. (1) is linear, the type of stability of its solutions depends only on the stability of the zero solution. However, there is an important difference between the cases $r = 0$ and $r < 0$. In the first case, writing $A_n = A(n,n)$, the equation becomes

$$x(n+1) = A_n x(n), \quad n \geq 0.$$

Hence,

$$x(n) = S(n,k)x(k), \quad n \geq k, \tag{2}$$

where

$$S(n,k) = A_{n-1}A_{n-2}\dots A_k,$$

and so any type of stability of Eq. (1) can be described in terms of these operators. For example, when $r = 0$ we say that Eq. (1) is *nonuniformly exponentially stable* if there exist constants $D, \lambda, \varepsilon > 0$ such that

$$\|S(n,k)\| \leq De^{-\lambda(n-k)+\varepsilon k}, \quad n \geq k \geq 0. \tag{3}$$

Moreover, we say that Eq. (1) is *uniformly exponentially stable* if (3) holds with $\varepsilon = 0$. The notion of nonuniform exponential stability goes back to seminal work of Pesin in [5,6] (see [1,3] for detailed descriptions of the theory and for many examples). It turns out that the classical uniform exponential stability is a considerable restriction for the dynamics. On the other hand, essentially guided by ergodic theory, one can introduce the much weaker notion of nonuniform exponential stability. For example, any nonautonomous linear differential equation with negative Lyapunov exponents is nonuniformly exponentially stable (see for example [3]).

On the other hand, when $r < 0$ there exist no linear operators $S(n,k)$ satisfying (2). This leads naturally to two types of approaches.

1.2. First Approach

We first consider a notion of exponential stability that tries to imitate as much as possible the one in (3), based on the observation that given an initial condition $x(k)$ at time k there exists a unique solution $x(n)$, for $n \geq k$, of Eq. (1) such that $x(l) = 0$ for $l < k$. Hence, for these initial conditions one can define corresponding operators $S(n, k)$ and introduce a notion of nonuniform exponential stability that relates to (3) (and that is equivalent to (3) when $r = 0$).

The study of the robustness of the notion is based on a characterization of Perron-type of the nonuniform exponential stability in terms of the invertibility of a certain linear operator between spaces of bounded sequences. More precisely, writing $\mathbf{x} = (x(n))_{n \geq 0}$, we consider the operator

$$(R\mathbf{x})(n) = x(n) - \sum_{k=0}^{n-1} A(n-1, k)x(k), \quad n \geq 0,$$

between certain spaces of bounded sequences (see Sect. 2 for details). Up to some technical aspects, we show in Theorems 3 and 4 that R is invertible if and only if the dynamics defined by Eq. (1) is nonuniformly exponentially stable.

This characterization is then used in Theorem 5 to show that the notion of nonuniform exponential stability is robust. See [4] for a related approach in the simpler case of uniform exponential stability.

1.3. Second Approach

The other approach is based on looking at the dynamics in a higher-dimensional space, with the advantage of having the cocycle property. More precisely, a natural strategy (that is widely used for continuous time) corresponds to view the problem in a higher-dimensional space in which the delay r becomes zero. Namely, considering the $(|r| + 1)$-vectors

$$y(n) = \big(x(n), x(n-1), \ldots, x(n-r)\big),$$

there exist linear operators $T(n, k)$ such that $y(n) = T(n, k)y(k)$ for $n \geq k$. This allows one to recover the cocycle property

$$T(n, l)T(l, k) = T(n, k)$$

and so it is natural to introduce a notion of nonuniform exponential stability for Eq. (1) as in (3) with $S(n, k)$ replaced by $T(n, k)$. Using this notion we obtain a second robustness result in Theorem 6.

We emphasize that none of the robustness results in Theorems 5 and 6 implies the other. More precisely, the second approach allows one to obtain the robustness of a weaker notion although also with a stronger hypothesis. The details are given at the end of Sect. 4.

2. Characterization of Stability

Consider the general linear difference Eq. (1), where each $A(n,k)$, for $n \ge k \ge 0$, is a bounded linear operator acting on a Banach space $X = (X, \|\cdot\|)$. Given $n \ge k \ge 0$, we denote by $F(n,k)$ the linear operator such that

$$x(n) = F(n,k)x(k)$$

for all sequences $(x(n))_{n\ge0}$ with $x(l) = 0$ for $l < k$ that satisfy (1). Let $\|\cdot\|_n$, for $n \ge 0$, be a sequence of norms on X such that $\|\cdot\|_n$ is equivalent to $\|\cdot\|$ for each n. We say that Eq. (1) is *exponentially stable with respect to the norms* $\|\cdot\|_n$ if there exist constants $D, \lambda > 0$ such that

$$\|F(n,k)x\|_n \le De^{-\lambda(n-k)}\|x\|_k, \quad x \in X,\ n \ge k \ge 0. \tag{4}$$

The purpose of the norms $\|\cdot\|_n$ is illustrated with the following example.

Example 1. Take $r = 0$ and let

$$A_n = e^{a+\varepsilon(n+1)\cos(n+1)-\varepsilon n\cos n}.$$

Then

$$\begin{aligned} F(m,n) &= e^{a(m-n)+\varepsilon m\cos m-\varepsilon n\cos n} \\ &\le e^{a(m-n)+\varepsilon m+\varepsilon n} \\ &= e^{(a+\varepsilon)(m-n)+2\varepsilon n} \end{aligned} \tag{5}$$

for all $m \ge n$. Now let

$$\begin{aligned} \|x\|_m &= \sup_{k\ge m}\left(\|F(k,m)x\|e^{-(a+\varepsilon)(k-m)}\right) \\ &= \sup_{k\ge m}\left(e^{-\varepsilon(k-m)+\varepsilon k\cos k-\varepsilon m\cos m}\|x\|\right). \end{aligned} \tag{6}$$

Clearly,

$$\|x\| \le \|x\|_m \le e^{2\varepsilon m}\|x\|. \tag{7}$$

Taking $x_m = F(m,n)x_n$, we obtain

$$\begin{aligned} \|x_m\|_m &= \sup_{m\ge n}\left(\|F(k,m)F(m,n)x_n\|e^{-(a+\varepsilon)(k-m)}\right) \\ &= e^{(a+\varepsilon)(m-n)}\sup_{k\ge m}\left(\|F(k,n)x_n\|e^{-(a+\varepsilon)(k-n)}\right) \\ &\le e^{(a+\varepsilon)(m-n)}\sup_{k\ge n}\left(\|F(k,n)x_n\|e^{-(a+\varepsilon)(k-n)}\right) = e^{(a+\varepsilon)(m-n)}\|x_n\|_n, \end{aligned}$$

thus allowing to erase the nonuniform term $e^{2\varepsilon n}$ in the right-hand side of (5). Hence, when $a+\varepsilon < 0$ Eq. (1) is exponentially stable with respect to the norms $\|\cdot\|_m$ introduced in (6).

Moreover, we say that Eq. (1) is *nonuniformly exponentially stable* if there exist constants $D, \lambda, \varepsilon > 0$ such that

$$\|F(n,k)\| \le De^{-\lambda(n-k)+\varepsilon k}, \quad n \ge k \ge 0.$$

Example 1 shows that some nonuniformly exponentially stable equations are exponentially stable with respect to certain norms $\|\cdot\|_n$ satisfying (7). It turns out that the converse always holds.

Proposition 1. *Let $\|\cdot\|_n$, for $n \geq 0$, be a sequence of norms on X such that*

$$\|x\| \leq \|x\|_n \leq Ce^{\varepsilon n}\|x\|, \quad x \in X,\ n \geq 0 \tag{8}$$

for some constants $C, \varepsilon > 0$. If Eq. (1) is exponentially stable with respect to the norms $\|\cdot\|_n$, then Eq. (1) is nonuniformly exponentially stable.

Proof. We have

$$\begin{aligned}\|F(n,k)x\| &\leq \|F(n,k)x\|_n \leq e^{-\lambda(n-k)}\|x\|_k \\ &\leq Ce^{-\lambda(n-k)+\varepsilon k}\|x\|\end{aligned}$$

and so Eq. (1) is nonuniformly exponentially stable. □

Now we consider the vector space

$$l^\infty = \left\{\mathbf{x} = (x(n))_{n\geq 0} \subset X : \sup_{n\geq 0} \|x(n)\|_n < +\infty\right\},$$

which is a Banach space when equipped with the norm $\|\mathbf{x}\| = \sup_{n\geq 0}\|x(n)\|_n$. We define a linear operator $R\colon \mathcal{D}(R) \to l^\infty$ by

$$(R\mathbf{x})(n) = x(n) - \sum_{k=0}^{n-1} A(n-1,k)x(k), \quad n \geq 0, \tag{9}$$

on the domain $\mathcal{D}(R)$ formed by all $\mathbf{x} = (x(n))_{n\geq 0} \in l^\infty$ such that $R\mathbf{x} \in l^\infty$.

Proposition 2. *The operator $R\colon \mathcal{D}(R) \to l^\infty$ is closed.*

Proof. Let $\mathbf{x}^k$ be a sequence in $\mathcal{D}(R)$ converging to $\mathbf{x} \in l^\infty$ such that the sequence $R\mathbf{x}^k$ converges to some $\mathbf{y} \in l^\infty$. Since the norms $\|\cdot\|_n$ are equivalent to the original norm $\|\cdot\|$, we conclude that

$$\lim_{k\to\infty} x^k(n) = x(n) \quad \text{and} \quad \lim_{k\to\infty} (R\mathbf{x}^k)(n) = y(n)$$

on X, for $n \geq 0$. Moreover, since the linear operators $A(n,k)$ are bounded, it follows from (9) that

$$y(n) = \lim_{k\to\infty}(R\mathbf{x}^k)(n) = x(n) - \sum_{k=0}^{n-1} A(n-1,k)x(k),$$

for $n \geq 0$. Hence, $\mathbf{x} \in \mathcal{D}(R)$ and $R\mathbf{x} = \mathbf{y}$. □

For each $\mathbf{x} \in \mathcal{D}(R)$, let

$$\|\mathbf{x}\|_R = \|\mathbf{x}\|_\infty + \|R\mathbf{x}\|_\infty.$$

It follows from Proposition 2 that $(\mathcal{D}(R), \|\mathbf{x}\|_R)$ is a Banach space. Moreover, the linear operator

$$R\colon (\mathcal{D}(R), \|\mathbf{x}\|_R) \to l^\infty$$

is clearly bounded. From now on we denote it simply by R.

Theorem 3. *Let $A(n,k)$, for $n \geq k \geq 0$, be bounded linear operators on X. If Eq. (1) is exponentially stable with respect to some norms $\|\cdot\|_n$, then the operator R is invertible.*

Proof. Clearly, R is one-to-one (simply from solving $R\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero sequence). For the surjectivity, take $\mathbf{y} = (y(n))_{n\geq 0} \in l^\infty$ and let

$$x(n) = \sum_{k=0}^{n} F(n,k)y(k), \quad n \geq 0.$$

It follows from (4) that

$$\|x(n)\|_n \leq \sum_{k=0}^{n} e^{-\lambda(n-k)} \|y(k)\|_k \leq \frac{1}{1-e^{-\lambda}} \|\mathbf{y}\|_\infty$$

for $n \geq 0$. Hence, $\mathbf{x} \in l^\infty$ and one can easily verify that $R\mathbf{x} = \mathbf{y}$. Therefore, R is onto and so it is invertible. ☐

The following result is a partial converse to the previous theorem, with appropriate assumptions on the linear operators $A(n,k)$.

Theorem 4. *Let $A(n,k)$, for $n \geq k \geq 0$, be bounded linear operators on X such that*

$$A(n,k) = 0, \quad n > k - r \tag{10}$$

and

$$\|A(n,k)x\|_n \leq D\|x\|_k, \quad x \in X, \ n \geq k \geq 0, \tag{11}$$

for some numbers $r \in \mathbb{Z}_0^-$ and $D, \varepsilon > 0$. If the operator R is invertible, then Eq. (1) is exponentially stable with respect to the norms $\|\cdot\|_n$.

Proof. Given $\lambda > 0$ and $n \geq 0$, we define a linear operator by

$$(B(\lambda)\mathbf{x})(m) = x(m) - \sum_{k=0}^{m-1} e^{\lambda|n-m|-\lambda|n-k+1|} A(m-1,k)x(k)$$

for $\mathbf{x} = (x(n))_{n\geq 0} \in l^\infty$. In order to estimate the norm of $B(\lambda) - R$ we need the following result.

Lemma 1. *We have*

$$\left|1 - e^{\lambda|n-m|-\lambda|n-k+1|}\right| \leq e^{-\lambda r} - 1 \tag{12}$$

whenever $|k-m-1| \leq -r$.

Proof of the lemma. We first note that

$$\left|1 - e^{\lambda|n-m|-\lambda|n-k+1|}\right| = \frac{|e^{\lambda|n-m|} - e^{\lambda|n-k+1|}|}{e^{\lambda|n-k+1|}}. \tag{13}$$

Now we consider two possibilities. Assume first that $e^{\lambda|n-m|} \ge e^{\lambda|n-k+1|}$. It follows from (13) and the triangle inequality that

$$\begin{aligned}\left|1-e^{\lambda|n-m|-\lambda|n-k+1|}\right| &= \frac{e^{\lambda|n-m|}-e^{\lambda|n-k+1|}}{e^{\lambda|n-k+1|}} \\ &\le \frac{e^{\lambda|n-k+1|+\lambda|k-1-m|}-e^{\lambda|n-k+1|}}{e^{\lambda|n-k+1|}} \\ &= e^{\lambda|k-1-m|}-1,\end{aligned}$$

which shows that (12) holds whenever $|k-m-1| \le -r$.

Now we assume that $e^{\lambda|n-m|} < e^{\lambda|n-k+1|}$. Using again (13) and the triangle inequality, we obtain

$$\begin{aligned}\left|1-e^{\lambda|n-m|-\lambda|n-k+1|}\right| &= \frac{e^{\lambda|n-k+1|}-e^{\lambda|n-m|}}{e^{\lambda|n-k+1|}} \\ &\le \frac{e^{\lambda|n-k+1|}-e^{\lambda|n-m|}}{e^{\lambda|n-m|}} \\ &\le \frac{e^{\lambda|n-m|+\lambda|m-k+1|}-e^{\lambda|n-m|}}{e^{\lambda|n-m|}} \\ &= e^{\lambda|k-1-m|}-1,\end{aligned}$$

which again shows that (12) holds whenever $|k-m-1| \le -r$. □

Now take $\mathbf{x} = (x(n))_{n\ge 0} \in \mathcal{D}(R)$. We have

$$((B(\lambda)-R)\mathbf{x})(m) = \sum_{k=0}^{m-1}\left(1-e^{\lambda|n-m|-\lambda|n-k+1|}\right)A(m-1,k)x(k)$$

for $m \ge 0$. By (10), (11) and (12), we have

$$\begin{aligned}\left\|((B(\lambda)-R)\mathbf{x})(m)\right\|_m &\le \sum_{k=m+r-1}^{m-1}\left|1-e^{\lambda|n-m|-\lambda|n-k+1|}\right| \\ &\quad \cdot \|A(m-1,k)x(k)\|_{m-1} \\ &\le D(e^{-\lambda r}-1)(-r+1)\|\mathbf{x}\|_\infty \\ &\le D(e^{-\lambda r}-1)(-r+1)\|\mathbf{x}\|_R\end{aligned}$$

for $m \ge 0$. Consequently,

$$\|(B(\lambda)-R)\mathbf{x}\|_\infty \le D(e^{-\lambda r}-1)(-r+1)\|\mathbf{x}\|_R \tag{14}$$

for each $\mathbf{x} \in \mathcal{D}(R)$. Taking $\lambda > 0$ such that

$$D(e^{-\lambda r}-1)(-r+1) < \|R^{-1}\|^{-1},$$

it follows from (14) that $B(\lambda)$ is invertible and that

$$\|B(\lambda)^{-1}\| \le K := \frac{1}{\|R^{-1}\|^{-1} - D(e^{-\lambda r}-1)(-r+1)}.$$

We note that K does not depend on n. Now take $v \in X$ and define $\mathbf{y} = (y(m))_{m\geq 0}$ by $y(n) = v$ and $y(m) = 0$ for $m \neq n$. Since $B(\lambda)$ is invertible, there exists $\mathbf{z} = (z(m))_{m\geq 0} \in l^\infty$ such that $B(\lambda)\mathbf{z} = \mathbf{y}$. Let

$$x(m) = e^{-\lambda|n-m|}z(m), \quad m \geq 0.$$

Clearly, $\mathbf{x} = (x(m))_{m\geq 0} \in l^\infty$ and

$$\begin{aligned} x(m) &- \sum_{k=0}^{m-1} A(m-1,k)x(k) \\ &= e^{-\lambda|n-m|}z(m) - \sum_{k=0}^{m-1} e^{-\lambda|n-k+1|}A(m-1,k)z(k) \\ &= e^{-\lambda|n-m|}\left(z(m) - \sum_{k=0}^{m-1} e^{\lambda|n-m|-\lambda|n-k+1|}A(m-1,k)z(k)\right) \\ &= e^{-\lambda|n-m|}(B(\lambda)\mathbf{z})(m) \\ &= e^{-\lambda|n-m|}y(m) = y(m) \end{aligned}$$

for $m \geq 0$ (since $y(m) = 0$ for $m \neq n$), that is, $R\mathbf{x} = \mathbf{y}$. Moreover,

$$\begin{aligned} \|x(m)\|_m &= e^{-\lambda|n-m|}\|z(m)\|_m \\ &\leq e^{-\lambda|n-m|}\|\mathbf{z}\|_\infty \\ &= e^{-\lambda|n-m|}\|B(\lambda)^{-1}\mathbf{y}\|_\infty \\ &\leq Ke^{-\lambda|n-m|}\|\mathbf{y}\|_\infty \\ &= Ke^{-\lambda|n-m|}\|v\|_n. \end{aligned}$$

Again since $y(m) = 0$ for $m \neq n$, we have $x(m) = F(m,n)v$ for $m \geq n$ and so

$$\|F(m,n)v\|_m \leq Ke^{-\lambda(m-n)}\|v\|_n, \quad m \geq n,$$

which shows that Eq. (1) is nonuniformly exponentially stable. □

Note that if (8) holds and there exist constants $D, \varepsilon > 0$ satisfying

$$\|A(n,k)\| \leq De^{-\varepsilon n}, \quad n \geq k \geq 0,$$

then condition (11) holds (with D replaced by CD).

3. Robustness: Method I

In this section we use Theorems 3 and 4 to establish the robustness of the notion of nonuniform exponential stability.

Theorem 5. *Assume that Eq. (1) is exponentially stable with respect to some norms $\|\cdot\|_n$ for some bounded linear operators $A(n,k)$, for $n \geq k \geq 0$, satisfying (10) and (11) for some numbers $r \in \mathbb{Z}_0^-$ and $D > 0$. Moreover, let $B(n,k)$, for $n \geq k \geq 0$, be bounded linear operators on X such that*

$$B(n,k) = 0, \quad n > k - r \tag{15}$$

and

$$\|(B(n,k) - A(n,k))x\|_n \leq c\|x\|_k, \quad x \in X, \ n \geq k \geq 0, \tag{16}$$

for some constant $c > 0$. If c is sufficiently small, then the equation

$$x(n+1) = \sum_{k=0}^{n} B(n,k)x(k) \tag{17}$$

is exponentially stable with respect to the norms $\|\cdot\|_n$.

Proof. By Theorem 3, the operator R defined by (9) is invertible. Now we define a linear operator $S \colon \mathcal{D}(R) \to l^\infty$ by

$$(S\mathbf{x})(n) = x(n) - \sum_{k=0}^{n-1} B(n-1,k)x(k), \quad n \geq 0.$$

It follows from (10), (15) and (16) that

$$\begin{aligned}\|((R-S)\mathbf{x})(n)\|_n &\leq \sum_{k=0}^{n-1} \|(B(n-1,k) - A(n-1,k))x(k)\|_n \\ &\leq c(-r+1)\|\mathbf{x}\|_\infty \\ &\leq c(-r+1)\|\mathbf{x}\|_R\end{aligned}$$

for $n \geq 0$ and $\mathbf{x} = (x(n))_{n\geq 0} \in \mathcal{D}(R)$. Hence,

$$\|R - S\| \leq c(-r+1)$$

and so, for any sufficiently small c the operator S is invertible (since R is invertible). On the other hand, by (11) and (16), we have

$$\|B(n,k)x\|_n \leq (c+D)\|x\|_k, \quad x \in X, \ n \geq k \geq 0.$$

Hence, it follows from Theorem 4 that Eq. (17) is exponentially stable with respect to the norms $\|\cdot\|_n$. □

Note that if (8) holds and there exist constants $D, \varepsilon > 0$ satisfying

$$\|(B(n,k) - A(n,k))\| \leq ce^{-\varepsilon n} \quad n \geq k \geq 0,$$

then condition (16) holds (with c replaced by cC).

4. Robustness: Method II

In this section we give a different perspective of the robustness problem as well as an alternative robustness result.

Given $r \in \mathbb{Z}_0^-$, let Y be the Banach space of all functions $\varphi\colon [r,0]\cap\mathbb{Z} \to X$ equipped with the norm

$$\|\varphi\|_Y = \max\big\{\|\varphi(k)\| : r \le k \le 0\big\}.$$

Moreover, given a function $x\colon C \to X$ on a set $C \subset \mathbb{Z}$ with $n+r,\dots,n \in C$, we define $x_n \in Y$ by $x_n(k) = x(n+k)$ for $k = r,\dots,0$.

We consider linear maps $L_m\colon Y \to X$, for $m \ge 0$, defined by

$$L_m\varphi = \sum_{k=m+r}^{m} A(m,k)\varphi(k-m)$$

for some linear operators $A(m,k)$ and the dynamics

$$x(m+1) = L_m x_m, \quad m \ge 0. \tag{18}$$

Note that Eq. (18) can be written in the form

$$x(m+1) = \sum_{k=m+r}^{m} A(m,k)x(k), \quad m \ge 0.$$

Moreover, when (10) holds, the equation becomes Eq. (1) for $m \ge -r$.

Given $n \ge 0$ and $\varphi \in Y$, there is a unique function $x\colon [n+r,+\infty)\cap\mathbb{Z} \to X$ with $x_n = \varphi$ satisfying (18) for $m \ge n$. We define linear evolution operators $T(m,n)$ on Y by

$$T(m,n)x_n = x_m, \quad m \ge n.$$

Clearly, $T(m,m) = \mathrm{Id}$ and

$$T(l,m)T(m,n) = T(l,n), \quad l \ge m \ge n.$$

We say that Eq. (18) admits a *nonuniform exponentially contraction* if there exist $D, \lambda, \varepsilon > 0$ such that

$$\|T(m,n)\| \le De^{-\lambda(m-n)+\varepsilon n}, \quad m \ge n. \tag{19}$$

Now we consider the dynamics

$$x(m+1) = M_m x_m \tag{20}$$

for some linear operators $M_m\colon Y \to X$, for $m \ge 0$, defined by

$$M_m\varphi = \sum_{k=m+r}^{m} B(m,k)\varphi(k-m)$$

for some linear operators $B(m,k)$.

The following is our alternative robustness result.

Theorem 6. *Assume that Eq. (18) admits a nonuniform exponential contraction. If*

$$\|M_m - L_m\| \leq ce^{-\delta\varepsilon m}, \quad m \geq 0, \tag{21}$$

for some constant $\delta > 1$, then for any sufficiently small $c > 0$, Eq. (20) admits a nonuniform exponential contraction. Moreover, the solution of Eq. (20) with initial condition $x_n = \varphi$ satisfies

$$\|x_m\|_Y \leq 2De^{-\lambda r}e^{-\lambda(m-n)+\varepsilon n}\|\varphi\|_Y, \quad m \geq n. \tag{22}$$

Proof. Let $G_m = M_m - L_m$. We need the following result (see [2, Lemma 1]).

Lemma 2. *The solution of Eq. (20) with $x_n = \varphi$ satisfies*

$$x_m = T(m,n)\varphi + \sum_{j=n}^{m-1} T(m,j+1)(\Gamma G_j x_j), \quad m \geq n, \tag{23}$$

where

$$(\Gamma G_j x_j)(l) = \begin{cases} G_j x_j, & l = 0, \\ 0, & l \neq 0. \end{cases}$$

In view of (23), we consider the operator U defined by

$$(Ux)_m = T(m,n)\varphi + \sum_{k=n}^{m-1} T(m,k+1)(\Gamma G_k x_k), \quad m \geq n, \tag{24}$$

in the space

$$F = \big\{x \colon [n+r,+\infty) \cap \mathbb{Z} \to X : \|x\| \leq 2De^{-\lambda r}\|\varphi\|_Y\big\},$$

equipped with the norm

$$\|x\| = \sup\big\{\|x_m\|_Y e^{-\gamma(m,n)} : m \geq n\big\}, \quad \gamma(m,n) = -\lambda(m-n) + \varepsilon n. \tag{25}$$

One can easily verify that F is a complete metric space.

Note that

$$T(m,n)\varphi(j) = T(m+j,n)\varphi(0)$$

for $j = r, \ldots, 0$. Therefore, for $j = r, \ldots, 0$ it follows from (24) that

$$(Ux)(m+j) = [T(m+j,n)\varphi](0) + \sum_{k=n}^{m-1} T(m+j,k+1)(\Gamma G_k x_k)(0) \tag{26}$$

and thus,

$$\|(Ux)(m+j)\| \leq \|T(m+j,n)\varphi\|_Y + \sum_{k=n}^{m+j-1} \|T(m+j,k+1)\| \cdot \|G_k x_k\|.$$

On the other hand,

$$\|G_m x_m\| \leq \|G_m\| \cdot \|x_m\|_Y \leq ce^{-\delta\varepsilon m}\|x_m\|_Y. \tag{27}$$

By (19) and (27), we have

$$\|(Ux)(m+j)\| \le De^{-\lambda(m+j-n)+\varepsilon n}\|\varphi\|_Y + \sum_{k=n}^{m-1} Dce^{-\lambda(m+j-k-1)+\varepsilon(k+1)}e^{-\delta\varepsilon k}\|x_k\|_Y.$$

Moreover, since

$$\|x_k\|_Y \le 2De^{-\lambda r}e^{\gamma(k,n)}\|\varphi\|_Y$$

we obtain

$$\begin{aligned}\|(Ux)(m+j)\| &\le De^{-\lambda(m+j-n)+\varepsilon n}\|\varphi\|_Y \\ &\quad + 2D^2e^{-\lambda r}c\|\varphi\|_Y \sum_{k=n}^{m-1} e^{-\lambda(m+j-k-1)+\varepsilon(k+1)-\lambda(k-n)+\varepsilon n-\delta\varepsilon k} \\ &\le De^{\gamma(m,n)}e^{-\lambda j}\|\varphi\|_Y \\ &\quad + 2D^2e^{-\lambda r}c\|\varphi\|_Y e^{-\lambda(m+j-n-1)+\varepsilon n}e^{\varepsilon}\sum_{k=n}^{m-1} e^{\varepsilon(1-\delta)k} \\ &\le De^{\gamma(m,n)}e^{-\lambda j}\|\varphi\|_Y + \frac{2D^2e^{-\lambda r+\varepsilon}c}{1-e^{\varepsilon(1-\delta)}}\|\varphi\|_Y e^{\gamma(m,n)} \\ &\le De^{-\lambda r}e^{\gamma(m,n)}\|\varphi\|_Y\left(1+c\mu\right),\end{aligned}$$

where

$$\mu = \frac{2De^{\varepsilon}}{1-e^{\varepsilon(1-\delta)}}.$$

Hence, in view of (25) we have

$$\|Ux\| \le De^{-\lambda r}\|\varphi\|_Y(1+c\mu) \le 2De^{-\lambda r}\|\varphi\|_Y,$$

taking c sufficiently small so that $c\mu < 1$. Therefore, $U(F) \subset F$. Now we show that U is a contraction. By (26) we have

$$\begin{aligned}\|(Ux)(m+j)-(Uy)(m+j)\| &\le \sum_{k=n}^{m-1}\|T(m+j,k+1)\|\cdot\|G_kx_k - G_ky_k\| \\ &\le D\sum_{k=n}^{m-1} e^{-\lambda(m+j-k-1)+\varepsilon(k+1)}\|G_k\|\cdot\|x_k-y_k\|_Y \\ &\le Dc\|x-y\|\sum_{k=n}^{m-1} e^{\gamma(k,n)}e^{-\lambda(m+j-k-1)+\varepsilon(k+1)}e^{-\delta\varepsilon k} \\ &\le Dce^{\varepsilon}\|x-y\|e^{\gamma(m,n)}\sum_{k=n}^{m-1} e^{(1-\delta)\varepsilon k} \\ &\le \frac{Dce^{\varepsilon}}{1-e^{(1-\delta)\varepsilon}}e^{\gamma(m,n)}\|x-y\|,\end{aligned}$$

which shows that

$$\|Ux - Uy\| \leq \frac{Dce^{\varepsilon}}{1 - e^{(1-\delta)\varepsilon}} \|x - y\|.$$

Therefore, U is a contraction in the complete metric space F. Hence, U has a unique fixed point in F, which thus satisfying (22). This completes the proof of the theorem. □

Now we compare Theorems 5 and 6. As we already observed, when (10) holds, Eqs. (1) and (18) coincide for $m \geq -r$. In fact, under that assumption, for any initial condition $\varphi \in Y$ with $\varphi(k)$ for $k \neq 0$, the two equations coincide for $m \geq 0$. On the other hand, one can easily verify that if Eq. (1) is nonuniformly exponentially stable, then Eq. (18) admits a nonuniform exponential contraction. When $r = 0$ the converse also holds. However, when $r > 0$, the converse may not hold (see Example 2 below), but still Theorem 6 shows that for the weaker notion of a nonuniform exponential contraction, if condition (21) holds, then one can still obtain a robustness result. We note that condition (21) is stronger than condition (16). Indeed, if the latter holds, then

$$\begin{aligned} \|(M_m - L_m)\varphi\| &\leq \sum_{k=m+r}^{m} \|A(m,k)\| \cdot \|\varphi(k-m)\| \\ &\leq (-r+1)De^{-\varepsilon m}\|\varphi\|_Y, \end{aligned}$$

while condition (21) requires an exponential decay $e^{-\delta\varepsilon m}$ for some $\delta > 1$. In other words, we require more in Theorem 6 although for a weaker notion. This causes that none of the Theorems 5 and 6 implies the other one.

Example 2. An example of a nonuniform exponential contraction whose dynamics is not nonuniformly exponentially stable is the following. Take $r = 1$,

$$A(k,k) = \begin{cases} e^{m(m-1)}, & k = 2m, \\ e^{-m(m-1)-1}, & k = 2m-1 \end{cases}$$

and $A(k,l) = 0$ for $l < k$. Then $x(k+1) = c_k x(0)$, where

$$c_k = \begin{cases} e^{-m}, & k = 2m, \\ e^{-m^2}, & k = 2m-1. \end{cases}$$

One can easily verify that

$$e^{-k/2} \leq \max\{c_k, c_{k-1}\} \leq e^{(1-k)/2}$$

and so the dynamics admits a nonuniform exponential contraction. On the other hand, $c_{2m}/c_{2m-1} = e^{m(m-1)}$ which shows that the dynamics is not nonuniformly exponentially stable.

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