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A version of a theorem of R. Datko for stability in average

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ABSTRACT

In this note we obtain a version of the well-known theorem of R. Datko for the notion of the exponential stability in average. We consider both cocycles over flows as well as cocycles over maps.

1. Introduction

In the process of extending the Lyapunov operator equation to the case of autonomous systems $x' = Ax$ when the operator A is unbounded, Datko [1] established his famous theorem which asserts that the trajectories of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Hilbert space X exhibit an exponential decay if and only if they stay in $L^2(\mathbb{R}_+, X)$. Since then this theorem became one of the pillars of the modern control theory and has inspired numerous extensions and generalizations. In particular, Pazy [2] proved that the conclusion of Datko's theorem holds if $L^2(\mathbb{R}_+, X)$ is replaced with any $L^p(\mathbb{R}_+, X)$ with $p \in [1, \infty)$. Furthermore, Datko [3] obtained the version of his theorem which deals with the exponential stability of evolution families $\{T(t, s)\}_{t\geq s\geq 0}$ which describe solutions of the variety of differential equations. More precisely, he proved the following result.

Theorem 1. *Let $\{T(t, s)\}_{t\geq s\geq 0}$ be an evolution family on a Banach space X. The following statements are equivalent:*

(1) *there exist $D, \lambda > 0$ such that*

$$\|T(t, s)\| \leq De^{-\lambda(t-s)} \quad \text{for } t \geq s \geq 0;$$

(2) *there exists $p \in [1, \infty)$ such that*

$$\sup_{s\geq 0} \int_s^{\infty} \|T(t, s)x\|^p < \infty \quad \text{for each } x \in X.$$

The first results related to discrete-time evolution families are due to Zabczyk [4].

A major improvement of this ideas is due to Rolewicz [5] who characterized exponential stability of evolution families in terms of the existence of appropriate functions N of two real variables (see [6] for details and further discussion). This approach unified and extended many of the previously known results. The most recent contributions [6,7] deal with obtaining the version of Datko's theorem for the notion of nonuniform exponential stability which was introduced by Barreira and Valls (see [8]). Moreover, in [9] the authors have obtained a certain ergodic version of Datko's theorem.

The main purpose of the present paper is to obtain a version of Datko's theorem for the notion of an exponential stability in average which is a particular case of a more general notion of an exponential dichotomy in average introduced in [10,11] for discrete and continuous time respectively. This notion essentially corresponds to assuming the existence of uniform contraction and uniform expansion along complementary directions but now in average, with respect to a given probability measure. We emphasize that this notion includes the classical concepts of uniform exponential dichotomy (and thus also of uniform exponential stability) as particular cases.

The paper is organized as follows. In Section 2 we recall some basic notions and the concept of an exponential stability in average. In Section 3 we prove the version of Datko's theorem for cocycles over semiflows. Then, in Section 4 we do the same but for cocycles over maps. Finally, in Section 5 we imply those results to the study of the persistence of the notion of the exponential stability in average under small linear perturbations.

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2. Preliminaries

We begin by recalling some well-known notions. Let $\Omega = (\Omega, \mathcal{B}, \mu)$ be a probability space. A measurable map $\varphi: \mathbb{R}_0^+ \times \Omega \to \Omega$ is said to be a *semiflow* on Ω if:

(1) $\varphi(0, \omega) = \omega$ for $\omega \in \Omega$;
(2) $\varphi(t+s, \omega) = \varphi(t, \varphi(s, \omega))$ for $t, s \geq 0$ and $\omega \in \Omega$.

For each $t \geq 0$ we can consider the map $\varphi_t: \Omega \to \Omega$ given by $\varphi_t(x) = \varphi(t, x)$, $x \in \Omega$. Moreover, let X be a Banach space and let $L(X)$ denote the set of all invertible bounded linear operators acting on X. A strongly measurable map $\Phi: \mathbb{R}_0^+ \times \Omega \to L(X)$ (this means that $(t, \omega) \mapsto \Phi(t, \omega)x$ is Bochner measurable for each $x \in X$) is said to be a *cocycle* over φ if:

(1) $\Phi(0, \omega) = \mathrm{Id}$ for $\omega \in \Omega$;
(2) $\Phi(t+s, \omega) = \Phi(t, \varphi_s(\omega))\Phi(s, \omega)$ for $t, s \geq 0$ and $\omega \in \Omega$.

Example 1. In the particular case when the map $t \mapsto \Phi(t, \omega)x$ is of class C^1 for each ω and x the cocycle can be described as follows. Let

$$A(\omega) = \frac{d}{dt}\Phi(t, \omega)\Big|_{t=0}.$$

One can easily verify that the unique solution of the problem

$$x' = A(\varphi_t(\omega))x, \qquad x(0) = x_0$$

is then given by $x(t) = \Phi(t, \omega)x_0$. Note that under the above assumption the map $t \mapsto A(\varphi_t(\omega))x$ is continuous for each ω and x.

Before proceeding, we emphasize that cocycles (over maps and flows) arise naturally in the study of nonautonomous dynamics. For example, smooth ergodic theory builds around the study of the derivative cocycle associated either to map or a flow (see Sections 5 and 6 in [12]). Moreover, cocycles describe solutions of variational equations and Cauchy problems with unbounded coefficients (we refer to Chapter 6 of [13] for detailed discussion). Finally, the notion of a cocycle arises from stochastic differential equations (see Chapter 2 in [14] for details).

Let $\mathcal{F}$ denote the Banach space of all Bochner measurable functions, sometimes simply referred to as measurable functions, $z: \Omega \to X$ such that

$$\|z\|_1 := \int_\Omega \|z(\omega)\| \, d\mu(\omega) < \infty,$$

identified if they are equal to μ-almost everywhere (we note that $\mathcal{F}$ is simply the set of all Bochner integrable functions identified if they are equal to μ-almost everywhere, sometimes denoted by $\mathcal{L}^1_\mu(\Omega, X)$). Given a cocycle Φ over a semiflow φ, we shall always assume that there exist $K, a > 0$ such that

$$\int_\Omega \|\Phi_\omega(t, \tau)z(\omega)\| \, d\mu(\omega) \leq Ke^{a|t-\tau|} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{1}$$

for $z \in \mathcal{F}$ and $t, \tau \geq 0$, where

$$\Phi_\omega(t, s) = \Phi(t, \omega)\Phi(s, \omega)^{-1}.$$

We now introduce the concept of exponential stability in average. We say that the cocycle Φ is *exponentially stable in average* if there exists $D, \lambda > 0$ such that

$$\int_\Omega \|\Phi_\omega(t, s)z(\omega)\| \, d\mu(\omega) \leq De^{-\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \tag{2}$$

for $z \in \mathcal{F}$ and $t \geq s \geq 0$. This notion is a particular case of a more general notion of exponential dichotomy in mean introduced in [11]. We recall that a cocycle Φ is said to admit an *exponential dichotomy in average* if there exist projections $P_\tau: \mathcal{F} \to \mathcal{F}$ for $\tau \geq 0$ such that:

(1) for each $t, \tau \geq 0$ and $z, \bar{z} \in \mathcal{F}$ such that $\bar{z}(\omega) = \Phi_\omega(t, \tau)z(\omega)$ for μ-almost every $\omega \in \Omega$, we have

$$(P_t\bar{z})(\omega) = \Phi_\omega(t, \tau)(P_\tau z)(\omega) \tag{3}$$

for μ-almost every $\omega \in \Omega$;
(2) there exist constants $D, \lambda > 0$ such that for each $z \in \mathcal{F}$, we have

$$\int_\Omega \|\Phi_\omega(t, s)(P_s z)(\omega)\| \, d\mu(\omega) \leq De^{-\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{4}$$

for $t \geq s$ and

$$\int_\Omega \|\Phi_\omega(t, s)(Q_s z)(\omega)\| \, d\mu(\omega) \leq De^{\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{5}$$

for $t \leq s$, where $Q_s = \mathrm{Id} - P_s$.

We note that when $P_t = \mathrm{Id}$, the condition (4) reduces to (2) (while (3) and (5) became trivial) and we recover the notion of exponential stability in average.

Example 2. Any uniformly hyperbolic cocycle admits an exponential dichotomy in average. We recall that a cocycle Φ is *uniformly hyperbolic* if there exist projections $\tilde{P}_t: X \to X$ for $t \in \mathbb{R}$ such that:

(1) for each $t, \tau \geq 0$ and $\omega \in \Omega$, we have

$$P_t\Phi_\omega(t, \tau) = \Phi_\omega(t, \tau)P_\tau;$$

(2) there exist constants $D, \lambda > 0$ such that for each $\omega \in \Omega$, we have

$$\|\Phi_\omega(t, \tau)\tilde{P}_\tau\| \leq De^{-\lambda(t-\tau)}$$

for $t \geq \tau$ and

$$\|\Phi_\omega(t, \tau)\tilde{Q}_\tau\| \leq De^{\lambda(t-\tau)}$$

for $t \leq \tau$, where $\tilde{Q}_t = \mathrm{Id} - \tilde{P}_t$.

Defining projections $P_t: \mathcal{F} \to \mathcal{F}$ for $t \in \mathbb{R}$ by

$$(P_t z)(\omega) = \tilde{P}_t(z(\omega)),$$

we find that each uniformly hyperbolic cocycle admits an exponential dichotomy in average with respect to any probability measure μ on Ω.

The previous example shows that the notion of an exponential dichotomy in average includes the classical notion of uniform hyperbolicity as a particular case.

Example 3. Now we describe examples of cocycles that admit an exponential dichotomy in average but that are not uniformly hyperbolic. Consider a partition $\Omega = \bigcup_{i=0}^N \Omega_i$ of Ω (N may be finite or infinite) with $\mu(\Omega_0) = 0$ and numbers $\lambda_0 = 0$ and $\lambda_i > 0$ for $i \in \mathbb{N}$ with $\inf_{i\in\mathbb{N}} \lambda_i > 0$. We assume that

$$\int_{\Omega_i} \|\Phi_\omega(t, s)(P_s z)(\omega)\| \, d\mu(\omega) \leq De^{-\lambda_i(t-s)} \int_{\Omega_i} \|z(\omega)\| \, d\mu(\omega)$$

for $t \geq s$ and

$$\int_{\Omega_i} \|\Phi_\omega(t, s)(Q_s z)(\omega)\| \, d\mu(\omega) \leq De^{\lambda_i(t-s)} \int_{\Omega_i} \|z(\omega)\| \, d\mu(\omega)$$

for $t \leq s$, for all $z \in \mathcal{F}$ and $i \in \mathbb{N}_0 \cap [0, N]$. Then the cocycle admits an exponential dichotomy in average. If the set Ω_0 is nonempty, then the cocycle is not uniformly hyperbolic. For example, the set

Ω_0 can contain parabolic fixed points or more generally parabolic periodic points. We refer the reader to [12] for many explicit examples of weak hyperbolic behavior that coexists with some parabolic behavior (see in particular Sec. 6.2 for the detailed construction of parabolic horseshoes). We refer to [11] for further discussion.

3. Main result

The following result gives a complete characterization of the notion of an exponential stability in average. It can be regarded as a version of the classical Datko–Pazy results for this notion.

Theorem 2. *The cocycle Φ is exponentially stable in average if and only if there exist $C, p > 0$ such that*

$$\left(\int_{t_0}^{\infty}\left(\int_\Omega \|\Phi_\omega(\tau,t_0)z(\omega)\|\,d\mu(\omega)\right)^p d\tau\right)^{1/p} \le C\int_\Omega \|z(\omega)\|\,d\mu(\omega), \tag{6}$$

for every $t_0 \ge 0$ and $z \in \mathcal{F}$.

Proof. Assume that the cocycle Φ is exponentially stable in average and take an arbitrary $p > 0$. It follows from (2) that

$$\begin{aligned}
&\int_{t_0}^{\infty}\left(\int_\Omega \|\Phi_\omega(\tau,t_0)z(\omega)\|\,d\mu(\omega)\right)^p d\tau \\
&\le \int_{t_0}^{\infty}\left(De^{-\lambda(\tau-t_0)}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\right)^p d\tau \\
&= D^p\left(\int_\Omega \|z(\omega)\|\,d\mu(\omega)\right)^p\int_{t_0}^{\infty} e^{-\lambda p(\tau-t_0)}\,d\tau \\
&= D^p\left(\int_\Omega \|z(\omega)\|\,d\mu(\omega)\right)^p\int_0^{\infty} e^{-\lambda p\tau}\,d\tau \\
&= \frac{D^p}{\lambda p}\left(\int_\Omega \|z(\omega)\|\,d\mu(\omega)\right)^p,
\end{aligned}$$

which immediately implies that (6) holds with $C = \frac{D}{(\lambda p)^{1/p}}$.

We now establish the converse. Assume that there exist $C, p > 0$ such that (6) holds for every $t_0 \ge 0$ and $z \in \mathcal{F}$. Using (1) and (6), we have

$$\begin{aligned}
&\left(\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega)\right)^p\int_{t_0}^{t} e^{-ap(t-s)}\,ds \\
&= \int_{t_0}^{t}\left(\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega)\right)^p e^{-ap(t-s)}\,ds \\
&= \int_{t_0}^{t}\left(\int_\Omega \|\Phi_\omega(t,s)\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega)\right)^p e^{-ap(t-s)}\,ds \\
&\le \int_{t_0}^{t} K^p e^{ap(t-s)}\left(\int_\Omega \|\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega)\right)^p e^{-ap(t-s)}\,ds \\
&\le K^p\int_{t_0}^{\infty}\left(\int_\Omega \|\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega)\right)^p ds \\
&\le K^pC^p\left(\int_\Omega \|z(\omega)\|\,d\mu(\omega)\right)^p,
\end{aligned}$$

for $t \ge t_0 \ge 0$. Since

$$\int_{t_0}^{t} e^{-ap(t-s)}\,ds = \int_0^{t-t_0} e^{-aps}\,ds \ge \int_0^1 e^{-aps}\,ds =: b > 0,$$

for $t \ge t_0 + 1$, we have that

$$\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \le \frac{KC}{b^{1/p}}\int_\Omega \|z(\omega)\|\,d\mu(\omega), \quad \text{for } t \ge t_0+1. \tag{7}$$

On the other hand, it follows from (1) that

$$\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \le Ke^a\int_\Omega \|z(\omega)\|\,d\mu(\omega), \quad \text{for } t \in [t_0, t_0+1]. \tag{8}$$

Combining (7) and (8), we conclude that

$$\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \le L\int_\Omega \|z(\omega)\|\,d\mu(\omega), \quad \text{for } t \ge t_0, \tag{9}$$

where

$$L = \max\left\{\frac{KC}{b^{1/p}}, Ke^a\right\}.$$

Fix now any $t \ge t_0$ and let $s \in [t_0, t]$. It follows from (9) that

$$\begin{aligned}
&\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \\
&= \int_\Omega \|\Phi_\omega(t,s)\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega) \\
&\le L\int_\Omega \|\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega).
\end{aligned}$$

By integrating from t_0 to t and using (6), we obtain

$$\begin{aligned}
&(t-t_0)\left(\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega)\right)^p \\
&\le L^p\int_{t_0}^{t}\left(\int_\Omega \|\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega)\right)^p ds \\
&\le L^p\int_{t_0}^{\infty}\left(\int_\Omega \|\Phi_\omega(s,t_0)z(\omega)\|\,d\mu(\omega)\right)^p ds \\
&\le C^pL^p\left(\int_\Omega \|z(\omega)\|\,d\mu(\omega)\right)^p,
\end{aligned}$$

which implies that

$$(t-t_0)^{1/p}\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \le CL\int_\Omega \|z(\omega)\|\,d\mu(\omega). \tag{10}$$

By adding the inequalities (9) and (10), we conclude that

$$\begin{aligned}
&(1+(t-t_0)^{1/p})\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \\
&\le L(C+1)\int_\Omega \|z(\omega)\|\,d\mu(\omega)
\end{aligned}$$

and thus

$$\int_\Omega \|\Phi_\omega(t,t_0)z(\omega)\|\,d\mu(\omega) \le \frac{L(C+1)}{(1+(t-t_0)^{1/p})}\int_\Omega \|z(\omega)\|\,d\mu(\omega). \tag{11}$$

It follows from (11) that there exists $N \in \mathbb{N}$ such that for all $t \geq t_0 \geq 0$ such that $t - t_0 \geq N$,

$$\int_\Omega \|\Phi_\omega(t, t_0)z(\omega)\| \, d\mu(\omega) \leq \frac{1}{e} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \quad \text{for every } z \in \mathcal{F}. \tag{12}$$

Take now arbitrary $t \geq t_0 \geq 0$ and write $t - t_0$ in the form $t - t_0 = kN + r$, where $k \in \mathbb{N}_0$ and $0 \leq r < N$. By (12),

$$\int_\Omega \|\Phi_\omega(t, t_0)z(\omega)\| \, d\mu(\omega) \leq \frac{1}{e^k} \int_\Omega \|\Phi_\omega(t_0 + r, t_0)z(\omega)\| \, d\mu(\omega),$$

which together with (9) implies that

$$\int_\Omega \|\Phi_\omega(t, t_0)z(\omega)\| \, d\mu(\omega) \leq \frac{L}{e^k} \int_\Omega \|z(\omega)\| \, d\mu(\omega).$$

We conclude that (2) holds with $P_t = \text{Id}$, $D = Le$ and $\lambda = \frac{1}{N}$ and therefore Φ is exponentially stable in average. □

4. Cocycles over maps

In this section we obtain a discrete time version of Theorem 2. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $f : \Omega \to \Omega$ be a measurable map. A measurable map $\mathcal{A} : \mathbb{N}_0 \times \Omega \to L(X)$ is said to be a cocycle over f if $\mathcal{A}(0, \omega) = \text{Id}$ and

$$\mathcal{A}(n + m, \omega) = \mathcal{A}(n, f^m(\omega))\mathcal{A}(m, \omega) \tag{13}$$

for $m, n \in \mathbb{N}_0$ and $\omega \in \Omega$. We write

$$\mathcal{A}_\omega(m, n) = \mathcal{A}(m, \omega)\mathcal{A}(n, \omega)^{-1}.$$

We also consider the map $A = \mathcal{A}(1, \cdot) : \Omega \to L(X)$. Clearly,

$$\mathcal{A}(m, \omega) = \begin{cases} A(f^{m-1}(\omega)) \cdots A(\omega), & m > 0, \\ \text{Id}, & m = 0. \end{cases}$$

We say that the cocycle $\mathcal{A}$ is *exponentially stable in average* if there exists $D, \lambda > 0$ such that

$$\int_\Omega \|\mathcal{A}_\omega(m, n)z(\omega)\| \, d\mu(\omega) \leq De^{-\lambda(m-n)} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \tag{14}$$

for $m \geq n \geq 0$ and $z \in \mathcal{F}$.

The following is a version of Theorem 2 for cocycles over maps. Our proof follows the same strategy as the proof of Theorem 2 with two important distinctions. Namely, in the case of discrete time we do not have to require condition (1) and the analogue of (9) follows directly from the discrete time version of (6) (see (15)). Consequently, the proof is less involved and simpler.

Theorem 3. *The cocycle $\mathcal{A}$ is exponentially stable in average if and only if there exist $C, p > 0$ such that*

$$\left(\sum_{n=m_0}^{\infty} \left(\int_\Omega \|\mathcal{A}_\omega(n, m_0)z(\omega)\| \, d\mu(\omega) \right)^p \right)^{1/p} \leq C \int_\Omega \|z(\omega)\| \, d\mu(\omega), \tag{15}$$

for every $m_0 \in \mathbb{N}_0$ and $z \in \mathcal{F}$.

Proof. We follow closely the proof of Theorem 2. Assume that the cocycle $\mathcal{A}$ is exponentially stable in average and take an arbitrary $p > 0$. It follows from (14) that

$$\begin{aligned} \sum_{n=m_0}^{\infty} \left(\int_\Omega \|\mathcal{A}_\omega(n, m_0)z(\omega)\| \, d\mu(\omega) \right)^p &\leq \sum_{n=m_0}^{\infty} \left(De^{-\lambda(n-m_0)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \right)^p \\ &= D^p \left(\int_\Omega \|z(\omega)\| \, d\mu(\omega) \right)^p \sum_{n=m_0}^{\infty} e^{-\lambda p(n-m_0)} \\ &= D^p \left(\int_\Omega \|z(\omega)\| \, d\mu(\omega) \right)^p \sum_{n=0}^{\infty} e^{-\lambda pn} \\ &= \frac{D^p}{1 - e^{-\lambda p}} \left(\int_\Omega \|z(\omega)\| \, d\mu(\omega) \right)^p, \end{aligned}$$

which immediately implies that (15) holds with $C = \frac{D}{(1-e^{-\lambda p})^{1/p}}$.

Suppose now that there exist $C, p > 0$ such that (15) holds for $m_0 \in \mathbb{N}_0$ and $z \in \mathcal{F}$. It follows directly from (15) that

$$\int_\Omega \|\mathcal{A}_\omega(m, n)z(\omega)\| \, d\mu(\omega) \leq C \int_\Omega \|z(\omega)\| \, d\mu(\omega), \tag{16}$$

for $m \geq n \geq 0$ and $z \in \mathcal{F}$. Take now $m \geq n \geq 0$ and k such that $m \geq k \geq n$. It follows from (16) that

$$\begin{aligned} \int_\Omega \|\mathcal{A}_\omega(m, n)z(\omega)\| \, d\mu(\omega) &= \int_\Omega \|\mathcal{A}_\omega(m, k)\mathcal{A}_\omega(k, n)z(\omega)\| \, d\mu(\omega) \\ &\leq C \int_\Omega \|\mathcal{A}_\omega(k, n)z(\omega)\| \, d\mu(\omega), \end{aligned}$$

which implies that

$$\begin{aligned} (m - n + 1) \left(\int_\Omega \|\mathcal{A}_\omega(m, n)z(\omega)\| \, d\mu(\omega) \right)^p &\leq C^p \sum_{k=n}^{m} \left(\int_\Omega \|\mathcal{A}_\omega(k, n)z(\omega)\| \, d\mu(\omega) \right)^p \\ &\leq C^p \sum_{k=n}^{\infty} \left(\int_\Omega \|\mathcal{A}_\omega(k, n)z(\omega)\| \, d\mu(\omega) \right)^p \\ &\leq C^{2p} \left(\int_\Omega \|z(\omega)\| \, d\mu(\omega) \right)^p. \end{aligned}$$

Hence,

$$\int_\Omega \|\mathcal{A}_\omega(m, n)z(\omega)\| \, d\mu(\omega) \leq \frac{C^2}{(1 + m - n)^{1/p}} \int_\Omega \|z(\omega)\| \, d\mu(\omega),$$

for $m \geq n \geq 0$ and $z \in \mathcal{F}$. One can now proceed as in the proof of Theorem 2 and conclude that the cocycle $\mathcal{A}$ is exponentially stable in average. □

5. Robustness of the stability in average

In this section we use Theorem 3 to establish the persistence of the notion of an exponential stability in average for cocycles over maps under sufficiently small linear perturbations. In [10] we have established similar property for the notion of the exponential dichotomy in average. Although this notion is more general then the notion of stability in average we emphasize that the robustness property of dichotomy does not imply the robustness property for stability.

Theorem 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be a cocycles over f with generators A and B respectively. Furthermore, assume that*

(1) *the cocycle $\mathcal{A}$ is exponentially stable in average;*
(2) *there exists $c > 0$ such that*

$$\|A(\omega) - B(\omega)\| \le c, \quad \text{for a.e. } \omega \in \Omega. \tag{17}$$

Then, if c is sufficiently small, the cocycle $\mathcal{B}$ is exponentially stable in average.

Proof. Let

$$Y = \left\{ \mathbf{z} = (z_n)_{n\ge 0} \subset \mathcal{F} : z_0 = 0 \text{ and } \sum_{n=1}^{\infty} \|z_n\|_1 < \infty \right\}.$$

It is straightforward to verify that Y is a Banach space with respect to the norm

$$\|\mathbf{z}\| = \sum_{n=1}^{\infty} \|z_n\|_1.$$

Moreover, we define a linear operator $T\colon Y \to Y$ by $(T\mathbf{z})_0 = 0$ and

$$(T\mathbf{z})_{m+1}(\omega) = z_{m+1}(\omega) - A(f^m(\omega))z_m(\omega), \quad \text{for } m \ge 0 \text{ and } \omega \in \Omega. \tag{18}$$

It follows from (14) that $\|(T\mathbf{z})_{m+1}\|_1 \le \|z_{m+1}\|_1 + D\|z_m\|_1$ which immediately implies that T is well-defined.

We now claim that T is invertible. Assume that $T\mathbf{z} = 0$ for $\mathbf{z} = (z_m)_{m\ge 0} \in Y$. Then, since $z_0 = 0$ it follows from (18) that $z_m = 0$ for every $m \ge 0$ and thus $\mathbf{z} = 0$. In order to prove that T is onto, take $\mathbf{y} = (y_m)_{m\ge 0} \in Y$ and define $\mathbf{x} = (x_m)_{m\ge 0}$ by

$$x_m(\omega) = \sum_{k=0}^{m} \mathcal{A}_\omega(m,k)y_k(\omega), \quad \text{for } m \ge 0 \text{ and } \omega \in \Omega.$$

It follows from (14) that

$$\|x_m\|_1 \le \sum_{k=1}^{m} De^{-\lambda(m-k)}\|y_k\|_1$$

and thus

$$\sum_{m=1}^{\infty} \|x_m\|_1 \le D\sum_{m=1}^{\infty}\sum_{k=1}^{m} e^{-\lambda(m-k)}\|y_k\|_1 \le \frac{D}{1-e^{-\lambda}}\|\mathbf{y}\|.$$

Therefore, $\mathbf{x} = (x_m)_{m\ge 0} \in Y$ and it is easy to verify that $T\mathbf{x} = \mathbf{y}$. We now introduce an operator $S\colon Y \to Y$ defined by $(T\mathbf{z})_0 = 0$ and

$$(S\mathbf{z})_{m+1}(\omega) = z_{m+1}(\omega) - B(f^m(\omega))z_m(\omega), \quad \text{for } m \ge 0 \text{ and } \omega \in \Omega. \tag{19}$$

It follows directly from (17) that $\|S - T\| \le c$ and thus if c is sufficiently small, S is also invertible. Take now an arbitrary $m_0 \in \mathbb{N}$ and $z \in \mathcal{F}$ and define $\mathbf{y} = (y_m)_{m\ge 0} \in Y$ by $y_{m_0} = z$ and $y_n = 0$ for $n \ne m_0$. Then, there exists $\mathbf{x} = (x_m)_{m\ge 0} \in Y$ such that $S\mathbf{x} = \mathbf{y}$. It follows easily from (19) that

$$x_m(\omega) = \begin{cases} 0 & \text{if } m < m_0; \\ \mathcal{B}_\omega(m,m_0)z(\omega) & \text{if } m \ge m_0. \end{cases}$$

Hence,

$$\|\mathbf{x}\| = \sum_{m=m_0}^{\infty} \int \|\mathcal{B}_\omega(m,m_0)z(\omega)\|\,d\mu(\omega) \le \|S^{-1}\| \cdot \|\mathbf{y}\| = \|S^{-1}\| \cdot \int \|z(\omega)\|\,d\mu(\omega).$$

It follows from Theorem 3 (applied for $p = 1$) that $\mathcal{B}$ is exponentially stable in average. □

We will now formulate the continuous time version of Theorem 4. Given a cocycle Φ over a semiflow φ and an essentially bounded strongly measurable function $B\colon \Omega \to L(X)$, we consider a strongly measurable map $\Psi : \mathbb{R}_0^+ \times \Omega \to L(X)$ satisfying

$$\Psi_\omega(t,s) = \Phi_\omega(t,s) + \int_s^t \Phi_\omega(t,\tau)B(\varphi_\tau(\omega))\Psi_\omega(\tau,s)\,d\tau \tag{20}$$

for $t, s \ge 0$ and μ-almost every $\omega \in \Omega$, where

$\Psi_\omega(t,s) = \Psi(t,\omega)\Psi(s,\omega)^{-1}$.

We shall always assume that Φ is such that Eq. (20) has a unique solution Ψ for any such B. In particular, if the cocycle Φ is continuous in t, then Ψ is unique and is also a cocycle over φ (see for example [14]). This provides a large class of examples.

Example 4. It turns out that there are many examples even under much more restrictive assumptions, although natural in the context of the theory of differential equations. Namely, assume in addition that:

(1) the map $t \mapsto \Phi(t,\omega)x$ is of class C^1 for each ω and x;
(2) the map $t \mapsto B(\varphi_t(\omega))x$ is continuous for each ω and x.

Using also Example 1, one can then easily verify that the unique solution of the problem

$$x' = [A(\varphi_t(\omega)) + B(\varphi_t(\omega))]x, \qquad x(0) = x_0$$

is given by $x(t) = \Psi_\omega(t,0)x_0$, with $\Psi_\omega(t,s)$ specified (uniquely) by (20).

The following is a continuous time version of Theorem 4.

Theorem 5. *Assume that the cocycle Φ is exponentially stable in average. If*

$$c := \operatorname*{ess\,sup}_{\omega\in\Omega} \|B(\omega)\| \tag{21}$$

is sufficiently small, then the cocycle Ψ defined by (20) is also exponentially stable in average.

Theorem 5 can be proved in two ways both of which are easy adaptation of the strategy outlined in the proof of Theorem 4 and already known techniques (and thus the proof is omitted). One can proceed in a similar manner to that in the proof of Theorem 4 by constructing appropriate Y and a continuous time versions of operators T and R from the proof of Theorem 4 (see [11] for the construction of those operators). Alternatively, one can deduce Theorem 5 from Theorem 4 using the standard type of arguments when passing from discrete to continuous time (see [15,16]). We emphasize that the crucial assumption that enables this approach to work is that (1) holds.

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