Publ. Math. Debrecen
Manuscript

Exponential dichotomies in average for flows and admissibility

By Luis Barreira, Davor Dragičević and Claudia Valls

Abstract. For a measurable cocycle over a flow or a semiflow acting on L^1 functions, we consider the general notion of an exponential dichotomy in average and we characterize it in terms of an admissibility property. As a nontrivial application, we establish in a simple manner the robustness of the notion under sufficiently small linear perturbations, both for cocycles over a flow and a semiflow.

1. Introduction

1.1. Exponential behavior. The notion of an exponential dichotomy, essentially introduced by Perron in [17], is central in several parts of the theory of differential equations and dynamical systems. It essentially corresponds to assuming the existence of uniform contraction and uniform expansion along complementary directions. In particular, the existence of an exponential dichotomy implies that there are stable and unstable invariant manifolds for any sufficiently small nonlinear perturbation. The consequent local instability of the trajectories, together with the nontrivial recurrence caused by the presence of a finite invariant measure, is one of the main mechanisms for the occurrence of stochastic behavior. We refer the reader to the books [2, 4, 5, 11, 23] for details and further references.

On the other hand, the existence of an exponential dichotomy is a strong

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Mathematics Subject Classification: Primary: 37D99.
Key words and phrases: admissibility, exponential dichotomies in average, robustness.
L.B. and C.V. were supported by FCT/Portugal through UID/MAT/04459/2013. D.D. was supported in part by an Australian Research Council Discovery Project DP150100017, Croatian Science Foundation under the project IP-2014-09-2285 and by the University of Rijeka research grant 13.14.1.2.02.

requirement and particularly in view of its central role in the theory, it is of interest to look for more general types of hyperbolic behavior.

In this paper, for a measurable cocycle over a flow or a semiflow acting on L^1 functions, we consider the more general notion of an *exponential dichotomy in average*. As it happens with the classical notion of an exponential dichotomy and some of its variants, it essentially corresponds to assuming the existence of uniform contraction and uniform expansion along complementary directions but now in average, with respect to a given probability measure. This includes as a special case any (linear) cocycle over a measure-preserving flow with nonzero Lyapunov exponents almost everywhere, by considering families of Lyapunov norms along each trajectory (see [4] for details). On the other hand, the notion includes many dynamics for which the exponential behavior is uniform on a set of positive measure but which may even fail to be exponential on the complement of that set (due to the presence of some parabolic behavior, such as in Example 3).

We emphasize that we do not require the measure μ to be invariant and thus we are not able to use results from ergodic theory.

1.2. Admissibility. Our main objective is to characterize the notion of an exponential dichotomy in average in terms of an admissibility property.

The study of admissibility goes back to pioneering work of Perron in [17] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t) \tag{1}$$

in $\mathbb{R}^n$ for any bounded continuous function $f\colon \mathbb{R}_0^+ \to \mathbb{R}^n$ (where $\mathbb{R}_0^+ = [0, +\infty)$). This property can be used to deduce the stability or the conditional stability under sufficiently small perturbations of a linear equation. More precisely, the following result was establish by Perron in [17], for $n \times n$ matrices $A(t)$ varying continuously with $t \geq 0$.

Theorem 1. *If equation* (1) *has at least one bounded solution in* $\mathbb{R}_0^+$ *for each bounded continuous function* f *and the equation* $x' = A(t)x$ *has* $k \leq n$ *bounded linearly independent solutions, then for each* $r > 0$ *there exists* $\delta > 0$ *such that if* g *is a continuous function satisfying*

$$\|g(t,x)\| \leq \delta \quad \text{and} \quad \|g(t,x) - g(t,y)\| \leq \delta \|x - y\|$$

for $t \geq 0$ *and* $x, y \in \mathbb{R}^n$ *with* $\|x\|, \|y\| < r$*, then the equation*

$$x' = A(t)x + g(t,x)$$

has a k*-parameter family of bounded solutions. If in addition* $g(t,0) = 0$ *for* $t \geq 0$*, then all these solutions tend to zero when* $t \to +\infty$.

The assumption in Theorem 1 is called the admissibility of the pair of spaces in which we take the perturbation f and look for the solution x. The theorem is probably the first step in the literature towards a study of the relation between admissibility and the notions of stability and conditional stability. We note that one can consider the admissibility of many other pairs of spaces, where we respectively take the perturbation and look for the solution. For the same reason, in order to refer to the type of relation studied in our paper we shall only use loosely the expression "admissibility", instead of introducing linear operators and then using them only once.

There is an extensive literature concerning the relation between admissibility and stability, also in infinite-dimensional spaces. For some of the most relevant early contributions in the area we refer to the books by Massera and Schäffer [14] (see also [13]) and by Dalec'kiĭ and Kreĭn [10]. For many related references we refer the reader to [4, 5, 7].

In order to obtain a criterion for the existence of an exponential dichotomy one can also use a Fredholm alternative for the nonlinear perturbations. In particular, related work is due to Palmer [16] for ordinary differential equations, Lin [12] for functional equations, Blázquez [6], Rodrigues and Silveira [21], Zeng [24] and Zhang [25] for parabolic evolution equations, and Chow and Leiva [8], Sacker and Sell [22] and Rodrigues and Ruas-Filho [20] for abstract evolution equations.

1.3. A nontrivial application: robustness. As a nontrivial application of the characterization of the notion of an exponential dichotomy in average, we establish in a simple manner its robustness under sufficiently small linear perturbations, both for cocycles over a flow and a semiflow.

Due to the central role played by the notion of an exponential dichotomy, it is crucial to understand whether it persists under linear perturbations. This is the so-called robustness problem. An exponential dichotomy associated to a linear equation $x' = A(t)x$ is said to be *robust* in some class of linear perturbations if for any arbitrarily small perturbation $B(t)$ in that class, the equation

$$x' = [A(t) + B(t)]x$$

still admits an exponential dichotomy. The study of robustness has a long history. In particular, it was discussed by Massera and Schäffer [13], Coppel [9] and Dalec'kiĭ and Kreĭn [10]. For more recent works we refer the reader to [8, 15, 18, 19] and the references therein. Moreover, we refer to [3] for related results in the case of discrete time.

2. Basic notions

2.1. Cocycles. Let $\Omega = (\Omega, \mu)$ be a probability space. A measurable map $\varphi\colon \mathbb{R}_0^+ \times \Omega \to \Omega$ is said to be a *semiflow* on Ω if:

(1) $\varphi(0, \omega) = \omega$ for $\omega \in \Omega$;

(2) $\varphi(t+s, \omega) = \varphi(t, \varphi(s, \omega))$ for $t, s \geq 0$ and $\omega \in \Omega$.

We also consider the measurable maps $\varphi_t = \varphi(t, \cdot)$. We shall always assume that φ_t is invertible for $t \geq 0$.

Moreover, let X be a Banach space and let $L(X)$ be the set of all invertible bounded linear operators acting on X. A strongly measurable map $\Phi\colon \mathbb{R}_0^+ \times \Omega \to L(X)$ (this means that $(t, \omega) \mapsto \Phi(t, \omega)x$ is Bochner measurable for each $x \in X$) is said to be a *cocycle* over φ if:

(1) $\Phi(0, \omega) = \mathrm{Id}$ for $\omega \in \Omega$;

(2) $\Phi(t+s, \omega) = \Phi(t, \varphi_s(\omega))\Phi(s, \omega)$ for $t, s \geq 0$ and $\omega \in \Omega$.

Example 1. In the particular case when the map $t \mapsto \Phi(t, \omega)x$ is of class C^1 for each ω and x the cocycle can be described as follows. Let

$$A(\omega) = \frac{d}{dt}\Phi(t, \omega)\Big|_{t=0}.$$

One can easily verify that the unique solution of the problem

$$x' = A(\varphi_t(\omega))x, \quad x(0) = x_0$$

is then given by $x(t) = \Phi(t, \omega)x_0$. Note that under the above assumption the map $t \mapsto A(\varphi_t(\omega))x$ is continuous for each ω and x.

2.2. Exponential dichotomies in average. Let $\mathcal{F}$ be the Banach space of all Bochner measurable functions, sometimes simply referred to as measurable functions, $z\colon \Omega \to X$ such that

$$\|z\|_1 := \int_\Omega \|z(\omega)\| \, d\mu(\omega) < \infty,$$

identified if they are equal μ-almost everywhere (we note that $\mathcal{F}$ is simply the set of all Bochner integrable functions identified if they are equal μ-almost everywhere, sometimes denoted by $\mathcal{L}_\mu^1(\Omega, X)$). Given a cocycle Φ, we shall always assume in the paper that there exist $K, a > 0$ such that

$$\int_\Omega \|\Phi_\omega(t, \tau)z(\omega)\| \, d\mu(\omega) \leq Ke^{a|t-\tau|} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{2}$$

for $z \in \mathcal{F}$ and $t, \tau \geq 0$, where

$$\Phi_\omega(t, s) = \Phi(t, \omega)\Phi(s, \omega)^{-1}.$$

A cocycle Φ is said to admit an *exponential dichotomy in average* if there exist projections $P_\tau \colon \mathcal{F} \to \mathcal{F}$ for $\tau \geq 0$ such that:

(1) for each $t, \tau \geq 0$ and $z, \bar{z} \in \mathcal{F}$ such that $\bar{z}(\omega) = \Phi_\omega(t, \tau)z(\omega)$ for μ-almost every $\omega \in \Omega$, we have

$$(P_t\bar{z})(\omega) = \Phi_\omega(t, \tau)(P_\tau z)(\omega) \tag{3}$$

for μ-almost every $\omega \in \Omega$;

(2) there exist constants $D, \lambda > 0$ such that for each $z \in \mathcal{F}$, we have

$$\int_\Omega \|\Phi_\omega(t, s)(P_s z)(\omega)\| \, d\mu(\omega) \leq De^{-\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{4}$$

for $t \geq s$ and

$$\int_\Omega \|\Phi_\omega(t, s)(Q_s z)(\omega)\| \, d\mu(\omega) \leq De^{\lambda(t-s)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{5}$$

for $t \leq s$, where $Q_s = \mathrm{Id} - P_s$.

Example 2. Any uniformly hyperbolic cocycle admits an exponential dichotomy in average. We recall that a cocycle Φ is *uniformly hyperbolic* if there exist projections $\tilde{P}_t \colon X \to X$ for $t \in \mathbb{R}$ such that:

(1) for each $t, \tau \geq 0$ and $\omega \in \Omega$, we have

$$P_t\Phi_\omega(t, \tau) = \Phi_\omega(t, \tau)P_\tau;$$

(2) there exist constants $D, \lambda > 0$ such that for each $\omega \in \Omega$, we have

$$\|\Phi_\omega(t, \tau)\tilde{P}_\tau\| \leq De^{-\lambda(t-\tau)}$$

for $t \geq \tau$ and

$$\|\Phi_\omega(t, \tau)\tilde{Q}_\tau\| \leq De^{\lambda(t-\tau)}$$

for $t \leq \tau$, where $\tilde{Q}_t = \mathrm{Id} - \tilde{P}_t$.

Defining projections $P_t \colon \mathcal{F} \to \mathcal{F}$ for $t \in \mathbb{R}$ by

$$(P_t z)(\omega) = \tilde{P}_t(z(\omega)),$$

we find that each uniformly hyperbolic cocycle admits an exponential dichotomy in average with respect to any probability measure μ on Ω.

Example 3. Now we describe examples of cocycles that admit an exponential dichotomy in average but that are not uniformly hyperbolic. Consider a partition $\Omega = \bigcup_{i=0}^N \Omega_i$ of Ω (N may be finite or infinite) with $\mu(\Omega_0) = 0$ and numbers $\lambda_0 = 0$ and $\lambda_i > 0$ for $i \in \mathbb{N}$ with $\inf_{i\in\mathbb{N}} \lambda_i > 0$. We assume that

$$\int_{\Omega_i} \|\Phi_\omega(t,s)(P_s z)(\omega)\| \, d\mu(\omega) \le De^{-\lambda_i(t-s)} \int_{\Omega_i} \|z(\omega)\| \, d\mu(\omega)$$

for $t \ge s$ and

$$\int_{\Omega_i} \|\Phi_\omega(t,s)(Q_s z)(\omega)\| \, d\mu(\omega) \le De^{\lambda_i(t-s)} \int_{\Omega_i} \|z(\omega)\| \, d\mu(\omega)$$

for $t \le s$, for all $z \in \mathcal{F}$ and $i \in \mathbb{N}_0 \cap [0,N]$. Then the cocycle admits an exponential dichotomy in average. If the set Ω_0 is nonempty, then the cocycle is not uniformly hyperbolic. For example, the set Ω_0 can contain parabolic fixed points or more generally parabolic periodic points.

2.3. Auxiliary spaces. We will need two additional Banach spaces. Namely, let $Y = (Y, \|\cdot\|_\infty)$ be the set of all functions $x\colon \mathbb{R}_0^+ \to \mathcal{F}$ such that

$$\|x\|_\infty = \sup_{t\ge 0} \|x(t)\|_1 < +\infty$$

and $W = (W, \|\cdot\|_\infty)$ the set of all Bochner measurable functions $y\colon \mathbb{R}_0^+ \times \Omega \to X$ such that

$$\|y\|_\infty = \operatorname*{ess\,sup}_{t\ge 0} \int_\Omega \|y(t,\omega)\| \, d\mu(\omega) < +\infty,$$

identified if they are equal (Lebesgue$\times\mu$)-almost everywhere.

3. Admissibility in $\mathbb{R}_0^+$

3.1. Characterization of exponential dichotomies. In this section we obtain a complete characterization of the notion of an exponential dichotomy in average in terms of an admissibility property.

Our first result shows that the existence of a exponential dichotomy in average yields an admissibility property. For simplicity of the notation we shall write

$$x(t)(\omega) = x(t,\omega) = x_t(\omega).$$

Theorem 2. *Let Φ be a cocycle over a semiflow. If Φ admits an exponential dichotomy in average, then for each $y \in W$ there exists a unique $x \in Y$ with $x_0 \in \operatorname{Im} Q_0$ satisfying*

$$x_t(\omega) = \Phi_\omega(t,\tau)x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,s)y_s(\omega)\,ds \tag{6}$$

for $t \ge \tau \ge 0$ and μ-almost every $\omega \in \Omega$.

Proof. Take $y \in W$. We first show that the integral in (6) is well defined for μ-almost every $\omega \in \Omega$. Indeed,

$$\begin{aligned}\int_\Omega \int_\tau^t \|\Phi_\omega(t,s)y_s(\omega)\|\,ds\,d\mu(\omega) &= \int_\tau^t \int_\Omega \|\Phi_\omega(t,s)y_s(\omega)\|\,d\mu(\omega)\,ds \\ &\le \int_\tau^t Ke^{a|t-s|} \int_\Omega \|y_s(\omega)\|\,d\mu(\omega)\,ds \\ &\le K\|y\|_\infty \int_\tau^t e^{a|t-s|}\,ds < +\infty.\end{aligned}$$

Therefore, $\int_\tau^t \|\Phi_\omega(t,s)y_s(\omega)\|\,ds < +\infty$ for μ-almost every $\omega \in \Omega$.

For $t \ge 0$ and $\omega \in \Omega$, we define

$$x_t(\omega) = \int_0^t \Phi_\omega(t,\tau)(P_\tau y_\tau)(\omega)\,d\tau - \int_t^\infty \Phi_\omega(t,\tau)(Q_\tau y_\tau)(\omega)\,d\tau.$$

It follows from (4) and (5) that

$$\begin{aligned}&\int_\Omega \int_0^t \|\Phi_\omega(t,\tau)(P_\tau y_\tau)(\omega)\|\,d\tau\,d\mu(\omega) \\ &\quad + \int_\Omega \int_t^\infty \|\Phi_\omega(t,\tau)(Q_\tau y_\tau)(\omega)\|\,d\tau\,d\mu(\omega) \\ &= \int_0^t \int_\Omega \|\Phi_\omega(t,\tau)(P_\tau y_\tau)(\omega)\|\,d\mu(\omega)\,d\tau \\ &\quad + \int_t^\infty \int_\Omega \|\Phi_\omega(t,\tau)(Q_\tau y_\tau)(\omega)\|\,d\mu(\omega)\,d\tau \\ &\le D\|y\|_\infty \left(\int_0^t e^{-\lambda(t-\tau)} d\tau + \int_t^\infty e^{-\lambda(\tau-t)} d\tau \right) \\ &= D\left(\frac{1}{\lambda} + \frac{1}{\lambda}\right)\|y\|_\infty\end{aligned} \tag{7}$$

for $t \ge 0$. This shows that x_t is well defined for every $t \ge 0$ and it also follows from (7) that $x = (x_t)_{t \ge 0} \in Y$. Moreover, given $t \ge \tau \ge 0$, we have

$$\begin{aligned}
x_t(\omega) &= \int_\tau^t \Phi_\omega(t,s) y_s(\omega)\, ds - \int_\tau^t \Phi_\omega(t,s)(P_s y_s)(\omega)\, ds \\
&\quad - \int_\tau^t \Phi_\omega(t,s)(Q_s y_s)(\omega)\, ds + \int_0^t \Phi_\omega(t,s)(P_s y_s)(\omega)\, ds \\
&\quad - \int_t^\infty \Phi_\omega(t,s)(Q_s y_s)(\omega)\, ds \\
&= \int_\tau^t \Phi_\omega(t,s) y_s(\omega)\, ds + \int_0^\tau \Phi_\omega(t,s)(P_s y_s)(\omega)\, ds \\
&\quad - \int_\tau^\infty \Phi_\omega(t,s)(Q_s y_s)(\omega)\, ds \\
&= \Phi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,s) y_s(\omega)\, ds
\end{aligned}$$

for μ-almost every $\omega \in \Omega$. This establishes (6). Moreover, it follows from (7) that $x \in Y$. Finally, by (3), we have $x_0 \in \operatorname{Im} Q_0$.

Now we show that x is the unique function in Y with $x_0 \in \operatorname{Im} Q_0$ and satisfying (6). We note that it is sufficient to consider the case when $y = 0$. Take $x \in Y$ with $x_0 \in \operatorname{Im} Q_0$ and let $x_t(\omega) = \Phi_\omega(t,\tau) x_\tau(\omega)$ for $t \ge \tau \ge 0$. It follows from (3) and (5) that

$$\begin{aligned}
\int_\Omega \|(Q_0 x_0)(\omega)\|\, d\mu(\omega) &= \int_\Omega \|\Phi_\omega(0,t)(Q_t x_t)(\omega)\|\, d\mu(\omega) \\
&\le De^{-\lambda t} \int_\Omega \|x_t(\omega)\|\, d\mu(\omega) \\
&\le De^{-\lambda t} \|x\|_\infty
\end{aligned}$$

for $t \ge 0$. Letting $t \to \infty$ we obtain $x_0 = Q_0 x_0 = 0$ and hence $x = 0$. This completes the proof of the theorem. □

Now we establish the converse of Theorem 2.

Theorem 3. *For a cocycle Φ over a semiflow, assume that there exists a closed subspace $Z \subset \mathcal{F}$ such that for each $y \in W$ there exists a unique $x \in Y$ with $x_0 \in Z$ satisfying* (6)*. Then Φ admits an exponential dichotomy in average.*

Proof. Let Y_Z be the set of all $x \in Y$ such that $x(0) \in Z$. Clearly, Y_Z is a closed subspace of Y. Moreover, let $H \colon \mathcal{D}(H) \to W$ be the linear operator defined by $Hx = y$ on the domain $\mathcal{D}(H)$ formed by all $x \in Y_Z$ for which there exists $y \in W$ satisfying (6).

Lemma 1. *The operator H is well defined.*

Proof of the lemma. Assume that there exist $x \in Y_Z$ and $y^1, y^2 \in W$ such that both pairs (x, y^1) and (x, y^2) satisfy (6). For each $t \geq \tau \geq 0$, we have

$$\int_\tau^t \Phi_\omega(t,s)(y_s^1(\omega) - y_s^2(\omega))ds = 0 \tag{8}$$

for μ-almost every $\omega \in \Omega$. Now take $t_0 \geq 0$. Applying $\Phi_\omega(t_0, t)$ to (8), we obtain

$$\frac{1}{n}\int_\tau^{\tau+\frac{1}{n}} \Phi_\omega(t_0,s)(y_s^1(\omega) - y_s^2(\omega))ds = 0 \tag{9}$$

for $\tau \geq 0$ and μ-almost every $\omega \in \Omega$. We also show that for μ-almost every $\omega \in \Omega$, the map $s \mapsto \Phi_\omega(t_0, s)(y_s^1(\omega) - y_s^2(\omega))$ is locally integrable. Let I be any finite interval containing t_0. It follows from (3) that

$$\begin{aligned}
&\int_\Omega \int_I \|\Phi_\omega(t_0,s)(y_s^1(\omega) - y_s^2(\omega))\| \, ds \, d\mu(\omega) \\
&= \int_I \int_\Omega \|\Phi_\omega(t_0,s)(y_s^1(\omega) - y_s^2(\omega))\| \, d\mu(\omega) \, ds \\
&\leq \int_I Ke^{a|t_0-s|} \int_\Omega \|y_s^1(\omega) - y_s^2(\omega)\| \, d\mu(\omega) \, ds \\
&\leq \|y^1 - y^2\|_\infty \int_I Ke^{a|t_0-s|} ds < \infty.
\end{aligned}$$

Therefore,

$$\int_I \|\Phi_\omega(t_0,s)(y_s^1(\omega) - y_s^2(\omega))\| \, ds < \infty$$

for $t_0 \geq 0$ and μ-almost every $\omega \in \Omega$. Finally, letting $n \to \infty$ in identity (9) yields that $y^1 = y^2$. □

Lemma 2. *The operator $H \colon \mathcal{D}(H) \to W$ is closed.*

Proof of the lemma. Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(H)$ converging to $x \in Y_Z$ such that $y^n = Hx^n$ converges to $y \in W$. For each $t \geq \tau \geq 0$, there exists a subsequence p_n such that

$$x_t^{p_n}(\omega) \to x_t(\omega) \quad \text{and} \quad \Phi_\omega(t,\tau)x_\tau^{p_n}(\omega) \to \Phi_\omega(t,\tau)x_\tau(\omega)$$

when $n \to \infty$, for μ-almost every $\omega \in \Omega$. Moreover,

$$\begin{aligned}
&\int_\Omega \left\| \int_\tau^t \Phi_\omega(t,s) y_s^n(\omega)\, ds - \int_\tau^t \Phi_\omega(t,s) y_s(\omega)\, ds \right\| d\mu(\omega) \\
&\le K \int_\Omega \int_\tau^t e^{a|t-s|} \|y_s^n(\omega) - y_s(\omega)\|\, ds\, d\mu(\omega) \\
&\le K e^{a|t-\tau|} \int_\tau^t \int_\Omega \|y_s^n(\omega) - y_s(\omega)\|\, d\mu(\omega)\, ds \\
&\le K e^{a|t-\tau|} \|y^n - y\|_\infty (t-\tau).
\end{aligned}$$

Since y_n converges to y in W, there exists a subsequence q_n of p_n such that

$$\lim_{n\to\infty} \int_\tau^t \Phi_\omega(t,s) y_s^{q_n}(\omega)\, ds = \int_\tau^t \Phi_\omega(t,s) y_s(\omega)\, ds$$

for μ-almost every $\omega \in \Omega$. Therefore,

$$\begin{aligned}
x_t(\omega) - \Phi_\omega(t,\tau) x_\tau(\omega) &= \lim_{n\to\infty} \big(x_t^{q_n}(\omega) - \Phi_\omega(t,\tau) x_\tau^{q_n}(\omega) \big) \\
&= \lim_{n\to\infty} \int_\tau^t \Phi_\omega(t,s) y_s^{q_n}(\omega)\, ds \\
&= \int_\tau^t \Phi_\omega(t,s) y_s(\omega)\, ds
\end{aligned}$$

for μ-almost every $\omega \in \Omega$ and (6) holds. Hence, $Hx = y$ and $x \in \mathcal{D}(H)$. □

It follows from the closed graph theorem that the operator H has a bounded inverse $G\colon W \to Y$. For each $\tau \ge 0$, let

$$\mathcal{F}_\tau^s = \left\{ z \in \mathcal{F} : \sup_{t\ge\tau} \int_\Omega \|\Phi_\omega(t,\tau) z(\omega)\|\, d\mu(\omega) < +\infty \right\} \tag{10}$$

and let $\mathcal{F}_\tau^u$ be the set of all functions $z \in \mathcal{F}$ for which there exists $\bar{z} \in Z$ such that

$$z(\omega) = \Phi_\omega(\tau, 0) \bar{z}(\omega) \quad \text{for} \quad \omega \in \Omega.$$

One can easily verify that $\mathcal{F}_\tau^s$ and $\mathcal{F}_\tau^u$ are subspaces of $\mathcal{F}$.

Lemma 3. *For $\tau \ge 0$, we have $\mathcal{F} = \mathcal{F}_\tau^s \oplus \mathcal{F}_\tau^u$.*

Proof of the lemma. Take $z \in \mathcal{F}$ and $\tau \ge 0$. We define $y\colon \mathbb{R}_0^+ \times \Omega \to X$ by the formula

$$y(t,\omega) = \chi_{[\tau,\tau+1]}(t) \Phi_\omega(t,\tau) z(\omega). \tag{11}$$

By (2) we have

$$\operatorname*{ess\,sup}_{t\ge 0}\int_\Omega \|y(t,\omega)\|\,d\mu(\omega) = \operatorname*{ess\,sup}_{t\in[\tau,\tau+1]}\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega)$$
$$\le Ke^a\int_\Omega \|z(\omega)\|\,d\mu(\omega) < +\infty$$

and $y \in W$. Hence, there exists $x \in Y_Z$ such that $Hx = y$. It follows from (6) that

$$x_t(\omega) = \Phi_\omega(t,\tau)(z(\omega) + x_\tau(\omega)) \tag{12}$$

for $t \ge \tau + 1$ and μ-almost every $\omega \in \Omega$. Similarly,

$$x_t(\omega) = \Phi_\omega(t,\tau)x_\tau(\omega) \tag{13}$$

for $0 \le t \le \tau$ and μ-almost every $\omega \in \Omega$. Now we define $z_1, z_2 \in \mathcal{F}$ by

$$z_1(\omega) = x_\tau(\omega) \quad \text{and} \quad z_2(\omega) = z(\omega) + x_\tau(\omega) \tag{14}$$

for $\omega \in \Omega$. Since $x_0 \in Z$, it follows from (13) that $z_1 \in \mathcal{F}^u_\tau$. Moreover, it follows from (2) and (12) that $z_2 \in \mathcal{F}^s_\tau$. Finally, since $z = z_2 - z_1$ we have $z \in \mathcal{F}^s_\tau + \mathcal{F}^u_\tau$.

In order to show that the spaces form a direct sum, take $z \in \mathcal{F}^s_\tau \cap \mathcal{F}^u_\tau$ and let $\bar{z} \in Z$ be such that $z(\omega) = \Phi_\omega(\tau,0)\bar{z}(\omega)$ for $\omega \in \Omega$. We define $x = (x_t)_{t\ge 0}$ by $x_t(\omega) = \Phi_\omega(t,0)\bar{z}(\omega)$. It follows from (2) and (10) that $x \in Y_Z$. Moreover, $Hx = 0$. Since H is invertible, we have $x = 0$ and thus $z = 0$. ☐

Let $P_\tau\colon \mathcal{F} \to \mathcal{F}^s_\tau$ and $Q_\tau\colon \mathcal{F} \to \mathcal{F}^u_\tau$ be the projections associated with the decomposition $\mathcal{F} = \mathcal{F}^s_\tau \oplus \mathcal{F}^u_\tau$.

Lemma 4. *The projections P_t satisfy condition* (3).

Proof of the lemma. Take $z, \bar{z} \in \mathcal{F}$ and $t \ge \tau$ such that

$$\bar{z}(\omega) = \Phi_\omega(t,\tau)z(\omega)$$

for μ-almost every $\omega \in \Omega$. It follows directly from the definitions that $\omega \mapsto \Phi_\omega(t,\tau)(P_\tau z)(\omega)$ belongs to $\mathcal{F}^s_t$ and $\omega \mapsto \Phi_\omega(t,\tau)(Q_\tau z)(\omega)$ belongs to $\mathcal{F}^u_t$. This readily implies that condition (3) is satisfied. ☐

Lemma 5. *There exists $M > 0$ such that*

$$\int_\Omega \|(P_\tau z)(\omega)\|\,d\mu(\omega) \le M\int_\Omega \|z(\omega)\|\,d\mu(\omega) \tag{15}$$

for $z \in \mathcal{F}$ and $\tau \ge 0$.

Proof of the lemma. Using the notation in the proof of Lemma 3, it follows from (2) that

$$\begin{aligned}\int_\Omega \|(P_\tau z)(\omega)\|\,d\mu(\omega) &= \int_\Omega \|z(\omega)+x_\tau(\omega)\|\,d\mu(\omega)\\ &\le \int_\Omega \|z(\omega)\|\,d\mu(\omega)+\|x\|_\infty\\ &= \int_\Omega \|z(\omega)\|\,d\mu(\omega)+\|Gy\|_\infty\\ &\le \int_\Omega \|z(\omega)\|\,d\mu(\omega)+\|G\|\cdot\|y\|_\infty\\ &\le (1+K\|G\|e^a)\int_\Omega \|z(\omega)\|\,d\mu(\omega).\end{aligned}$$

Hence, inequality (15) holds with $M=1+K\|G\|e^a$. □

Now we establish the exponential bounds.

Lemma 6. *There exist constants $D,\lambda>0$ such that*

$$\int_\Omega \|\Phi_\omega(t,\tau)(P_\tau z)(\omega)\|\,d\mu(\omega)\le De^{-\lambda(t-\tau)}\int_\Omega \|z(\omega)\|\,d\mu(\omega) \tag{16}$$

for $z\in\mathcal{F}$ and $t\ge\tau\ge 0$.

Proof of the lemma. Take $z\in\mathcal{F}_\tau^s$ and define a function $\varphi\colon\mathbb{R}_0^+\to\mathbb{R}$ by

$$\varphi(t)=\begin{cases}0, & 0\le t\le\tau,\\ t-\tau, & \tau\le t\le\tau+1,\\ 1, & \tau+1\le t.\end{cases}$$

Moreover, define $x=(x_t)_{t\ge0}$ and $y\colon\mathbb{R}_0^+\times\Omega\to Y$ by

$$x_t(\omega)=\varphi(t)\Phi_\omega(t,\tau)z(\omega)\quad\text{and}\quad y(t,\omega)=\chi_{[\tau,\tau+1]}(t)\Phi_\omega(t,\tau)z(\omega). \tag{17}$$

One can easy verify that $x\in Y_Z$, $y\in W$ and $Hx=y$. Moreover,

$$\begin{aligned}&\sup\left\{\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega): t\in[\tau+1,+\infty)\right\}\\ &=\sup\left\{\int_\Omega \|\varphi(t)\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega): t\in[\tau+1,+\infty)\right\}\\ &=\sup\left\{\int_\Omega \|x_t(\omega)\|\,d\mu(\omega): t\in[\tau+1,+\infty)\right\}\end{aligned}$$

$$
\begin{aligned}
&\le \|x\|_\infty = \|Gy\|_\infty \le \|G\| \cdot \|y\|_\infty \\
&= \|G\| \sup\left\{ \int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\| \, d\mu(\omega) : t \in [\tau, \tau+1] \right\} \\
&\le K\|G\|e^a \int_\Omega \|z(\omega)\| \, d\mu(\omega),
\end{aligned}
$$

using (2) in the last inequality. Using again (2), we obtain

$$
\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\| \, d\mu(\omega) \le C \int_\Omega \|z(\omega)\| \, d\mu(\omega) \quad \text{for} \quad t \ge \tau, \tag{18}
$$

where $C = Ke^a \max\{1, \|G\|\}$.

Now we show that there exists an integer $N \in \mathbb{N}$ such that for each $\tau \ge 0$ and $z \in \mathcal{F}_\tau^s$, we have

$$
\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\| \, d\mu(\omega) \le \frac{1}{2} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \quad \text{for} \quad t - \tau \ge N. \tag{19}
$$

Take $t_0 > \tau$ such that

$$
\int_\Omega \|\Phi_\omega(t_0,\tau)z(\omega)\| \, d\mu(\omega) > \frac{1}{2} \int_\Omega \|z(\omega)\| \, d\mu(\omega).
$$

It follows from (18) that

$$
\frac{1}{2C} \int_\Omega \|z(\omega)\| \, d\mu(\omega) < \int_\Omega \|\Phi_\omega(s,\tau)z(\omega)\| \, d\mu(\omega) \le C \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{20}
$$

for $\tau \le s \le t_0$. Now we consider the functions

$$
y(t,\omega) = \chi_{[\tau,t_0]}(t)\Phi_\omega(t,\tau)z(\omega)
$$

and

$$
v_t(\omega) = \Phi_\omega(t,\tau)z(\omega) \int_0^t \chi_{[\tau,t_0]}(s) \, ds,
$$

for $t \ge 0$ and $\omega \in \Omega$. One can easily verify that $v = (v_t)_{t \ge 0} \in Y_Z$, $y \in W$ and $Hv = y$. Therefore,

$$
\|G\| \sup\left\{ \int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\| \, d\mu(\omega) : t \in [\tau, t_0] \right\} \ge \|G\| \cdot \|y\|_\infty \ge \|v\|_\infty
$$

and it follows from (20) that

$$\begin{aligned} C\|G\| \int_\Omega \|z(\omega)\| \, d\mu(\omega) &\ge \int_\Omega \|v_{t_0}(\omega)\| \, d\mu(\omega) \\ &\ge (t_0-\tau) \int_\Omega \|\Phi_\omega(t_0,\tau) z(\omega)\| \, d\mu(\omega) \\ &\ge \frac{1}{2C}(t_0-\tau) \int_\Omega \|z(\omega)\| \, d\mu(\omega). \end{aligned}$$

Therefore, property (19) holds taking $N > 2C^2\|G\|$. Now take $t \ge \tau$ and write $t-\tau = kN + r$, with $k \in \mathbb{N}$ and $0 \le r < N$. By (15), (18) and (19), we obtain

$$\begin{aligned} \int_\Omega \|\Phi_\omega(t,\tau)(P_\tau z)(\omega)\| \, d\mu(\omega) &= \int_\Omega \|\Phi_\omega(\tau+kN+r,\tau)(P_\tau z)(\omega)\| \, d\mu(\omega) \\ &\le \frac{1}{2^k} \int_\Omega \|\Phi_\omega(\tau+r,\tau)(P_\tau z)(\omega)\| \, d\mu(\omega) \\ &\le \frac{C}{2^k} \int_\Omega \|(P_\tau z)(\omega)\| \, d\mu(\omega) \\ &\le 2CMe^{-(t-\tau)\log 2/N} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \end{aligned} \tag{21}$$

for $x \in X$. Inequality (16) holds taking $D = 2CM$ and $\lambda = \log 2/K$. □

Lemma 7. *There exist constants $D, \lambda > 0$ such that*

$$\int_\Omega \|\Phi_\omega(t,\tau)(Q_\tau z)(\omega)\| \, d\mu(\omega) \le De^{-\lambda(\tau-t)} \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{22}$$

for $z \in \mathcal{F}$ and $0 \le t \le \tau$.

Proof of the lemma. We first show that there exists $L > 0$ such that

$$\int_\Omega \|\Phi_\omega(t,0)z(\omega)\| \, d\mu(\omega) \le L \int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\| \, d\mu(\omega) \tag{23}$$

for $\tau \ge t \ge 0$ and $z \in Z$. Given $\tau > 0$ and $z \in Z$, for a sufficiently small $h > 0$ let $\psi \colon \mathbb{R}_0^+ \to \mathbb{R}$ be a smooth function supported on $[0,\tau]$ such that $\psi = 1$ on $[0, \tau-h]$ and $\sup_{t\ge 0}|\psi'(t)| \le 2$. For $t \ge 0$ and $\omega \in \Omega$, we define

$$x_t(\omega) = \psi(t)\Phi_\omega(t,0)z(\omega) \quad \text{and} \quad y(t,\omega) = \psi'(t)\Phi_\omega(t,0)z(\omega).$$

One can easily verify that $x = (x_t)_{t\ge 0} \in Y_Z$, $y \in W$ and $Hx = y$. Hence,

$$\begin{aligned} &\sup\left\{ \int_\Omega \|\Phi_\omega(t,0)z(\omega)\| \, d\mu(\omega) : t \in [0,\tau-h] \right\} \\ &= \sup\left\{ \int_\Omega \|\psi(t)\Phi_\omega(t,0)z(\omega)\| \, d\mu(\omega) : t \in [0,\tau-h] \right\} \\ &= \sup\left\{ \int_\Omega \|x_t(\omega)\| \, d\mu(\omega) : t \in [0,\tau-h] \right\} \end{aligned}$$

$$
\begin{aligned}
&\le \|x\|_\infty = \|Gy\|_\infty \le \|G\| \cdot \|y\|_\infty \\
&= \|G\| \sup \left\{ \int_\Omega \|\psi'(t)\Phi_\omega(t,0)z(\omega)\| \, d\mu(\omega) : t \in [\tau - h, \tau] \right\} \\
&\le 2\|G\| \sup \left\{ \int_\Omega \|\Phi_\omega(t,0)z(\omega)\| \, d\mu(\omega) : t \in [\tau - h, \tau] \right\} \\
&\le 2Ke^{ah}\|G\| \int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\| \, d\mu(\omega),
\end{aligned}
$$

using (2) in the last inequality. Finally, letting $h \to 0$, we conclude that (23) holds taking $L = 2K\|G\|$.

Now we show that there exists an integer $N \in \mathbb{N}$ such that for each $\tau \ge 0$ and $z \in Z$, we have

$$
\int_\Omega \|\Phi_\omega(t,0)z(\omega)\| \, d\mu(\omega) \ge 2 \int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\| \, d\mu(\omega) \quad \text{for} \quad t - \tau \ge N. \tag{24}
$$

Take $t_0 > \tau$ such that

$$
\int_\Omega \|\Phi_\omega(t_0,0)z(\omega)\| \, d\mu(\omega) < 2 \int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\| \, d\mu(\omega).
$$

It follows from (23) that

$$
\begin{aligned}
\frac{1}{2L} \int_\Omega \|\Phi_\omega(t_0,0)z(\omega)\| \, d\mu(\omega) &< \int_\Omega \|\Phi_\omega(s,0)z(\omega)\| \, d\mu(\omega) \\
&\le L \int_\Omega \|\Phi_\omega(t_0,0)z(\omega)\| \, d\mu(\omega)
\end{aligned} \tag{25}
$$

for $\tau \le s \le t_0$. Now we consider the functions

$$
y(t,\omega) = -\chi_{[\tau,t_0]}(t)\Phi_\omega(t,0)z(\omega)
$$

and

$$
v_t(\omega) = \Phi_\omega(t,0)z(\omega) \int_t^\infty \chi_{[\tau,t_0]}(s) \, ds,
$$

for $t \ge 0$ and $\omega \in \Omega$. One can easily verify that $v = (v_t)_{t \ge 0} \in Y_Z$, $y \in W$ and $Hv = y$. Therefore, using (25) we obtain

$$
\|v\|_\infty = \|Gy\|_\infty \le \|G\| \cdot \|y\|_\infty \le L\|G\| \int_\Omega \|\Phi_\omega(t_0,0)z(\omega)\| \, d\mu(\omega).
$$

Hence,

$$
\begin{aligned}
L\|G\| \int_\Omega \|\Phi_\omega(t_0,0)z(\omega)\|\,d\mu(\omega) &\ge \|v\|_\infty \\
&\ge (t_0-\tau)\int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\|\,d\mu(\omega) \\
&\ge \frac{1}{2L}(t_0-\tau)\int_\Omega \|\Phi_\omega(t_0,0)z(\omega)\|\,d\mu(\omega)
\end{aligned}
$$

and (24) holds taking $N > 2L^2\|G\|$.

Now take $t \ge \tau \ge 0$ and write $t-\tau = kN + r$, with $k \in \mathbb{N}$ and $0 \le r < N$. By (23) and (24), we obtain

$$
\begin{aligned}
\int_\Omega \|\Phi_\omega(t,0)z(\omega)\|\,d\mu(\omega) &= \int_\Omega \|\Phi_\omega(\tau+kN+r,0)z(\omega)\|\,d\mu(\omega) \\
&\ge 2^k \int_\Omega \|\Phi_\omega(\tau+r,0)z(\omega)\|\,d\mu(\omega) \\
&\ge \frac{2^k}{L}\int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\|\,d\mu(\omega)
\end{aligned}
$$

for $z \in Z$. Hence,

$$
\int_\Omega \|\Phi_\omega(t,0)z(\omega)\|\,d\mu(\omega) \le 2Le^{-(\tau-t)\log 2/N}\int_\Omega \|\Phi_\omega(\tau,0)z(\omega)\|\,d\mu(\omega) \tag{26}
$$

for $0 \le t \le \tau$ and $z \in Z$. Writing $Q(\tau)z$ in the form $\Phi_\omega(\tau,0)z$, it follows from (15) and (26) that inequality (22) holds taking $D = 2(1+M)L$ and $\lambda = \log 2/N$. $\square$

It follows readily from Lemmas 4, 6 and 7 that the cocycle Φ admits an exponential dichotomy in average. $\square$

3.2. Robustness as an application. As a nontrivial application, in this section we establish the robustness of the notion of an exponential dichotomy in average using the characterization given in the former section.

Given a cocycle Φ over a semiflow φ and an essentially bounded strongly measurable function $B\colon \Omega \to L(X)$, we consider a strongly measurable map $\Psi\colon \mathbb{R}_0^+ \times \Omega \to L(X)$ satisfying

$$
\Psi_\omega(t,s) = \Phi_\omega(t,s) + \int_s^t \Phi_\omega(t,\tau)B(\varphi_\tau(\omega))\Psi_\omega(\tau,s)\,d\tau \tag{27}
$$

for $t,s \ge 0$ and μ-almost every $\omega \in \Omega$, where

$$
\Psi_\omega(t,s) = \Psi(t,\omega)\Psi(s,\omega)^{-1}.
$$

We shall always assume that Φ is such that equation (27) has a unique solution Ψ for any such B. In particular, if the cocycle Φ is continuous in t, then Ψ is unique and is also a cocycle over φ (see for example [1]). This provides a large class of examples.

Example 4. It turns out that there are many examples even under much more restrictive assumptions, although natural in the context of the theory of differential equations. Namely, assume in addition that:

(1) the map $t \mapsto \Phi(t,\omega)x$ is of class C^1 for each ω and x;

(2) the map $t \mapsto B(\varphi_t(\omega))x$ is continuous for each ω and x.

Using also Example 1, one can then easily verify that the unique solution of the problem

$$x' = [A(\varphi_t(\omega)) + B(\varphi_t(\omega))]x, \quad x(0) = x_0$$

is given by $x(t) = \Psi_\omega(t,0)x_0$, with $\Psi_\omega(t,s)$ specified (uniquely) by (27).

Now we establish the robustness of the notion of an exponential dichotomy in average. In comparison to proofs of the robustness for other notions in the literature, the present proof must be considered simple. This is made possible precisely by the characterization of the notion of an exponential dichotomy in average in terms of an admissibility property.

Theorem 4. *Assume that the cocycle Φ admits an exponential dichotomy in average. If*

$$c := \operatorname*{ess\,sup}_{\omega\in\Omega} \|B(\omega)\| \tag{28}$$

is sufficiently small, then the cocycle Ψ defined by (27) *also admits an exponential dichotomy in average.*

Proof. We first show that there exist $K', a' > 0$ such that

$$\int_\Omega \|\Psi_\omega(t,\tau)z(\omega)\|\, d\mu(\omega) \le K' e^{a'|t-\tau|} \int_\Omega \|z(\omega)\|\, d\mu(\omega) \tag{29}$$

for $z \in \mathcal{F}$ and $t, \tau \ge 0$. By (2), we have

$$\begin{aligned}
&\int_\Omega \|\Psi_\omega(t,\tau)z(\omega)\|\, d\mu(\omega) \\
&= \int_\Omega \left\| \Phi_\omega(t,\tau)z(\omega) + \int_\tau^t \Phi_\omega(t,s)B(\varphi_s(\omega))\Psi_\omega(s,\tau)z(\omega)\, ds \right\| d\mu(\omega) \\
&\le Ke^{a(t-\tau)}\|z\|_1 + Kc \int_\tau^t e^{a(t-s)} \int_\Omega \|\Psi_\omega(s,\tau)z(\omega)\|\, d\mu(\omega)\, ds \\
&\le Ke^{a(t-\tau)}\|z\|_1 + cK \int_\tau^t e^{a(t-s)} \int_\Omega \|\Psi_\omega(t,\tau)z(\omega)\|\, d\mu(\omega)\, ds
\end{aligned}$$

for $z \in \mathcal{F}$ and $t \geq \tau$. Hence, the function

$$\psi(t) = e^{-at} \int_\Omega \|\Psi_\omega(t,\tau) z(\omega)\| \, d\mu(\omega)$$

satisfies

$$\psi(t) \leq K\psi(\tau) + cK \int_\tau^t \psi(s) \, ds$$

and it follows from Gronwall's lemma that

$$\psi(t) \leq K\psi(\tau) e^{cK(t-\tau)}$$

for $t \geq \tau$. This establishes property (29) for $t \geq \tau$. A similar argument can be used for $t \leq \tau$ and so property (2) holds for the cocycle Ψ.

Since the cocycle Φ admits an exponential dichotomy in average, by Theorem 2 there exists a closed subspace $Z \subset \mathcal{F}$ such that for each $y \in W$ there exists a unique $x \in Y_Z$ such that (6) holds. Let L be the linear operator associated to the cocycle Ψ, defined by $Lx = y$ on the domain $\mathcal{D}(L)$ formed by all $x \in Y_Z$ for which there exists $y \in W$ such that

$$x_t(\omega) = \Psi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Psi_\omega(t,s) y_s(\omega) \, ds$$

for $t \geq \tau \geq 0$ and μ-almost every $\omega \in \Omega$. Proceeding in a similar manner to that in the proof of Lemma 1, one can show that L is well defined. For each $x = (x_t)_{t\geq 0} \in Y_Z$ and $y \in W$ such that $Lx = y$, we have

$$\begin{aligned}
x_t(\omega) &= \Psi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Psi_\omega(t,s) y_s(\omega) \, ds \\
&= \Phi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,s) B(\varphi_s(\omega)) \Psi_\omega(s,\tau) x_\tau(\omega) \, ds \\
&\quad + \int_\tau^t \Phi_\omega(t,s) y_s(\omega) \, ds + \int_\tau^t \int_s^t \Phi_\omega(t,r) B(\varphi_r(\omega)) \Psi_\omega(r,s) y_s(\omega) \, dr \, ds \\
&= \Phi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,r) B(\varphi_r(\omega)) \Psi_\omega(r,\tau) x_\tau(\omega) \, dr \\
&\quad + \int_\tau^t \Phi_\omega(t,s) y_s(\omega) \, ds + \int_\tau^t \int_\tau^r \Phi_\omega(t,r) B(\varphi_r(\omega)) \Psi_\omega(r,s) y_s(\omega) \, ds \, dr \\
&= \Phi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,s) y_s(\omega) \, ds \\
&\quad + \int_\tau^t \Phi_\omega(t,r) B(\varphi_r(\omega)) \left(\Psi_\omega(r,\tau) x_\tau(\omega) + \int_\tau^r \Psi_\omega(r,s) y_s(\omega) \, ds \right) dr \\
&= \Phi_\omega(t,\tau) x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,r) \big(y_r(\omega) + B(\varphi_r(\omega)) x_r(\omega) \big) \, dr,
\end{aligned}$$

that is,

$$x_t(\omega) = \Phi_\omega(t,\tau)x_\tau(\omega) + \int_\tau^t \Phi_\omega(t,r)\big(y_r(\omega) + B(\varphi_r(\omega))x_r(\omega)\big)\,dr \tag{30}$$

for $t \ge \tau$. Now we introduce a linear operator $R\colon Y_Z \to W$ by

$$(Rx)(t,\omega) = B(\varphi_t(\omega))x_t(\omega).$$

It follows from (28) that

$$\int_\Omega \|B(\varphi_t(\omega))x_t(\omega)\|\,d\mu(\omega) \le c\int_\Omega \|x_t(\omega)\|\,d\mu(\omega) \le c\|x\|_\infty \tag{31}$$

for $t \ge 0$ and so the operator R is well defined and bounded. Moreover, it follows from (30) that $\mathcal{D}(H) = \mathcal{D}(L)$ and $H = L + R$. For $x \in \mathcal{D}(H)$ we consider the graph norm

$$\|x\|'_\infty = \|x\|_\infty + \|Hx\|_\infty.$$

Clearly, the operator

$$H\colon (\mathcal{D}(H), \|\cdot\|'_\infty) \to (Y, \|\cdot\|_\infty)$$

is bounded and for simplicity we denote it simply by H. Since H is closed, $(\mathcal{D}(H), \|\cdot\|'_\infty)$ is a Banach space. By (31) we have

$$\|(H-L)x\|_\infty = \|Rx\|_\infty \le c\|x\|_\infty \le c\|x\|'_\infty \tag{32}$$

for $x \in \mathcal{D}(H)$. On the other hand, by Theorem 2, the operator H is invertible and hence, it follows from (32) that if c is sufficiently small, then L is also invertible. Applying Theorem 3 yields that the cocycle Ψ admits an exponential dichotomy in average. □

4. Admissibility in $\mathbb{R}$

4.1. Preliminaries. Again let (Ω,μ) be a probability space. A measurable map $\varphi\colon \mathbb{R}\times\Omega \to \Omega$ is said to be a *flow* on Ω if:

(1) $\varphi(0,\omega) = \omega$ for $\omega \in \Omega$;

(2) $\varphi(t+s,\omega) = \varphi(t,\varphi(s,\omega))$ for $t,s \in \mathbb{R}$ and $\omega \in \Omega$.

Now let X be a Banach space. A strongly measurable map $\Phi\colon \mathbb{R}\times\Omega \to L(X)$ is said to be a *cocycle* over φ if:

(1) $\Phi(0,\omega) = \mathrm{Id}$ for $\omega \in \Omega$;

(2) $\Phi(t+s,\omega) = \Phi(t,\varphi_s(\omega))\Phi(s,\omega)$ for $t,s \in \mathbb{R}$ and $\omega \in \Omega$.

We shall always assume that there exist $K, a > 0$ such that (2) holds for $z \in \mathcal{F}$ and $t, \tau \in \mathbb{R}$. A cocycle Φ is said to admit an *exponential dichotomy in average* if there exist projections $P_\tau \colon \mathcal{F} \to \mathcal{F}$ for $\tau \in \mathbb{R}$ such that:

(1) for each $t, \tau \in \mathbb{R}$ and $z, \bar{z} \in \mathcal{F}$ such that $\bar{z}(\omega) = \Phi_\omega(t,\tau)z(\omega)$ for μ-almost every $\omega \in \Omega$, property (3) holds;

(2) there exist constants $D, \lambda > 0$ such that for every $z \in \mathcal{F}$ properties (4) and (5) hold, respectively, for $t \geq s$ and $t \leq s$.

4.2. Characterization of exponential dichotomies. Let $Y = (Y, \|\cdot\|_\infty)$ be the set of all functions $x \colon \mathbb{R} \to \mathcal{F}$ such that

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|_1 < +\infty$$

and $W = (W, \|\cdot\|_\infty)$ the set of all Bochner measurable functions $y \colon \mathbb{R} \times \Omega \to X$ such that

$$\|y\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \int_\Omega \|y(t,\omega)\|\, d\mu(\omega) < \infty,$$

identified if they are equal (Lebesgue$\times\mu$)-almost everywhere. One can easily verify that both Y and W are Banach spaces.

We first show that the existence of an exponential dichotomy in average yields an admissibility property.

Theorem 5. *Let Φ be a cocycle over a flow. If Φ admits an exponential dichotomy in average, then for each $y \in W$ there exists a unique $x \in Y$ such that* (6) *holds for $t \geq \tau$ and μ-almost every $\omega \in \Omega$.*

Proof. We proceed in a similar manner to that in the proof of Theorem 2. Take $y \in W$. For $t \in \mathbb{R}$ and $\omega \in \Omega$, we define

$$x_t(\omega) = \int_{-\infty}^t \Phi_\omega(t,\tau)(P_\tau y_\tau)(\omega)\, d\tau - \int_t^\infty \Phi_\omega(t,\tau)(Q_\tau y_\tau)(\omega)\, d\tau.$$

As in the proof of Theorem 2, one can show that $x = (x_t)_{t\in\mathbb{R}} \in Y$ satisfies (6) for $t \geq \tau$ and μ-almost every $\omega \in \Omega$.

In order to establish the uniqueness of x, take $x = (x_t)_{t\in\mathbb{R}}$ such that $x_t(\omega) = \Phi_\omega(t,\tau)x_\tau(\omega)$ for $t \geq \tau$ and μ-almost every $\omega \in \Omega$. It follows from (3) and (4) that

$$\begin{aligned}\|P_\tau x_\tau\|_1 &= \int_\Omega \|\Phi_\omega(\tau,t)(P_t x_t)(\omega)\|\,d\mu(\omega)\\ &\le De^{-\lambda(\tau-t)}\int_\Omega \|x_t(\omega)\|\,d\mu(\omega)\\ &\le De^{-\lambda(\tau-t)}\|x\|_\infty\end{aligned}$$

for $t \le \tau$. Letting $t \to -\infty$ yields that $P_\tau x_\tau = 0$. Similarly, $Q_\tau x_\tau = 0$ and hence $x_\tau = 0$. Since $\tau \in \mathbb{R}$ is arbitrary, we conclude that $x = 0$. □

Now we establish the converse of Theorem 5.

Theorem 6. *For a cocycle Φ over a flow, if for each $y \in W$ there exists a unique $x \in Y$ satisfying* (6), *then Φ admits an exponential dichotomy in average.*

Proof. Let H be the linear operator defined by $Hx = y$ on the domain $\mathcal{D}(H)$ formed by all $x \in Y$ for which there exists $y \in W$ satisfying (6). Proceeding as in the proofs of Lemmas 1 and 2, one can show that H is a well defined closed linear operator. Hence, by the closed graph theorem, the operator H has a bounded inverse $G\colon W \to Y$.

For each $\tau \in \mathbb{R}$, let

$$\mathcal{F}_\tau^s = \left\{ z \in \mathcal{F} : \sup_{t \ge \tau} \int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) < +\infty \right\}$$

and

$$\mathcal{F}_\tau^u = \left\{ z \in \mathcal{F} : \sup_{t \le \tau} \int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) < +\infty \right\}.$$

Clearly, $\mathcal{F}_\tau^s$ and $\mathcal{F}_\tau^u$ are subspaces of $\mathcal{F}$.

Lemma 8. *For $\tau \in \mathbb{R}$, we have $\mathcal{F} = \mathcal{F}_\tau^s \oplus \mathcal{F}_\tau^u$.*

Proof of the lemma. Take $z \in \mathcal{F}$ and $\tau \in \mathbb{R}$. We define $y\colon \mathbb{R} \times \Omega \to X$ by (11). Proceeding as in the proof of Lemma 3, one can show that $y \in W$. Hence, there exists $x \in Y$ such that $Hx = y$ and proceeding as in (14) yields that $z \in \mathcal{F}_\tau^s + \mathcal{F}_\tau^u$.

Now take $z \in \mathcal{F}_\tau^s \cap \mathcal{F}_\tau^u$. We define $x = (x_t)_{t\in\mathbb{R}}$ by $x_t(\omega) = \Phi_\omega(t,\tau)z(\omega)$. One can easily verify that $x \in Y$ and $Hx = 0$. Since H is invertible, we have $x = 0$ and thus $z = 0$. □

Let $P_\tau\colon \mathcal{F} \to \mathcal{F}_\tau^s$ and $Q_\tau\colon \mathcal{F} \to \mathcal{F}_\tau^u$ be the projections associated to the decomposition $\mathcal{F} = \mathcal{F}_\tau^s \oplus \mathcal{F}_\tau^u$. Again property (3) holds and proceeding as in the proof of Lemma 5 we find that there exists $M > 0$ such that (15) holds for $z \in \mathcal{F}$ and $\tau \in \mathbb{R}$. It remains to establish the exponential bounds.

Lemma 9. *There exist constants $D, \lambda > 0$ such that* (16) *holds for $z \in \mathcal{F}$ and $t \ge \tau$.*

Proof of the lemma. Take $z \in \mathcal{F}_\tau^s$ and define a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} 0, & t \le \tau, \\ t-\tau, & \tau \le t \le \tau+1, \\ 1, & \tau+1 \le t. \end{cases}$$

Moreover, let $x = (x_t)_{t \in \mathbb{R}}$ and $y\colon \mathbb{R} \times \Omega \to Y$ be as in (17). One can verify that $x \in Y$, $y \in W$ and $Hx = y$. Proceeding as in the proof of Lemma 6 yields that (18) holds, where $C = Ke^a \max\{1, \|G\|\}$.

Now we show that there exists an integer $N \in \mathbb{N}$ such that property (19) holds for each $\tau \in \mathbb{R}$ and $z \in \mathcal{F}_\tau^s$. We define

$$y(t, \omega) = \chi_{[\tau, t_0]}(t) \Phi_\omega(t, \tau) z(\omega)$$

and

$$v_t(\omega) = \Phi_\omega(t, \tau) z(\omega) \int_{-\infty}^t \chi_{[\tau, t_0]}(s)\, ds.$$

One can verify that $v = (v_t)_{t \in \mathbb{R}} \in Y$, $y \in W$ and $Hv = y$. Proceeding as in the proof of Lemma 6 yields that

$$C\|G\| \int_\Omega \|z(\omega)\|\, d\mu(\omega) \ge \frac{1}{2C}(t_0 - \tau) \int_\Omega \|z(\omega)\|\, d\mu(\omega),$$

which implies that property (19) holds taking $N > 2C^2\|G\|$. Proceeding as in (21) yields inequality (16) taking $D = 2CM$ and $\lambda = \log 2/K$. $\square$

Lemma 10. *There exist constants $D, \lambda > 0$ such that* (22) *holds for $z \in \mathcal{F}$ and $t \le \tau$.*

Proof of the lemma. Take $z \in \mathcal{F}_\tau^u$ and define a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} 1, & t \le \tau-1, \\ \tau-t, & \tau-1 \le t \le \tau, \\ 0, & \tau \le t. \end{cases}$$

Moreover, we define $x = (x_t)_{t \in \mathbb{R}}$ and $y\colon \mathbb{R} \times \Omega \to Y$ by

$$x_t(\omega) = \varphi(t) \Phi_\omega(t, \tau) z(\omega) \quad \text{and} \quad y(t, \omega) = -\chi_{[\tau-1, \tau]}(t) \Phi_\omega(t, \tau) z(\omega).$$

One can easily verify that $x \in Y$, $y \in W$ and $Hx = y$. Hence,

$$\begin{aligned}
&\sup\left\{\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) : t \in (-\infty, \tau-1]\right\}\\
&= \sup\left\{\int_\Omega \|\varphi(t)\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) : t \in (-\infty, \tau-1]\right\}\\
&= \sup\left\{\int_\Omega \|x_t(\omega)\|\,d\mu(\omega) : t \in (-\infty, \tau-1]\right\}\\
&\le \|x\|_\infty = \|Gy\|_\infty \le \|G\|\cdot\|y\|_\infty\\
&= \|G\|\sup\left\{\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) : t \in [\tau-1, \tau]\right\}\\
&\le K\|G\|e^a \int_\Omega \|z(\omega)\|\,d\mu(\omega),
\end{aligned}$$

using (2) in the last inequality. Therefore, using again (2), we obtain

$$\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) \le C\int_\Omega \|z(\omega)\|\,d\mu(\omega) \quad \text{for} \quad t \le \tau, \tag{33}$$

where $C = Ke^a \max\{1, \|G\|\}$.

Now we show that there exists an integer $N \in \mathbb{N}$ such that for each $\tau \in \mathbb{R}$ and $z \in \mathcal{F}_\tau^u$, we have

$$\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) \le \frac{1}{2}\int_\Omega \|z(\omega)\|\,d\mu(\omega) \quad \text{for} \quad t \le \tau - N. \tag{34}$$

Take $t_0 < \tau$ such that

$$\int_\Omega \|\Phi_\omega(t_0,\tau)z(\omega)\|\,d\mu(\omega) > \frac{1}{2}\int_\Omega \|z(\omega)\|\,d\mu(\omega).$$

It follows from (33) that

$$\frac{1}{2C}\int_\Omega \|z(\omega)\|\,d\mu(\omega) < \int_\Omega \|\Phi_\omega(s,\tau)z(\omega)\|\,d\mu(\omega) \le C\int_\Omega \|z(\omega)\|\,d\mu(\omega), \tag{35}$$

for $t_0 \le s \le \tau$. Now we consider the functions

$$y(t,\omega) = \chi_{[t_0,\tau]}(t)\Phi_\omega(t,\tau)z(\omega)$$

and

$$v_t(\omega) = -\Phi_\omega(t,\tau)z(\omega)\int_t^\infty \chi_{[t_0,\tau]}(s)\,ds.$$

One can verify that $v=(v_t)_{t\in\mathbb{R}}\in Y$, $y\in W$ and $Hv=y$. Therefore,

$$\|G\|\sup\left\{\int_\Omega \|\Phi_\omega(t,\tau)z(\omega)\|\,d\mu(\omega) : t\in[t_0,\tau]\right\}\ge \|G\|\cdot\|y\|_\infty\ge\|v\|_\infty.$$

Hence, it follows from (35) that

$$\begin{aligned}C\|G\|\int_\Omega\|z(\omega)\|\,d\mu(\omega)&\ge\int_\Omega\|v_{t_0}(\omega)\|\,d\mu(\omega)\\&\ge(\tau-t_0)\int_\Omega\|\Phi_\omega(t_0,\tau)z(\omega)\|\,d\mu(\omega)\\&\ge\frac{1}{2C}(\tau-t_0)\int_\Omega\|z(\omega)\|\,d\mu(\omega)\end{aligned}$$

and property (34) holds taking $N>2C^2\|G\|$.

Now take $t\le\tau$ and write $\tau-t=kN+r$, with $k\in\mathbb{N}$ and $0\le r<N$. By (15), (33) and (35), we obtain

$$\begin{aligned}\int_\Omega\|\Phi_\omega(t,\tau)(Q_\tau z)(\omega)\|\,d\mu(\omega)&=\int_\Omega\|\Phi_\omega(\tau-kN-r,\tau)(Q_\tau z)(\omega)\|\,d\mu(\omega)\\&\le\frac{1}{2^k}\int_\Omega\|\Phi_\omega(\tau-r,\tau)(Q_\tau z)(\omega)\|\,d\mu(\omega)\\&\le\frac{C}{2^k}\int_\Omega\|(Q_\tau z)(\omega)\|\,d\mu(\omega)\\&\le 2C(1+M)e^{-(\tau-t)\log 2/N}\int_\Omega\|z(\omega)\|\,d\mu(\omega),\end{aligned}$$

for $x\in X$. Taking $D=2C(1+M)$ and $\lambda=\log 2/K$ yields (22). □

This completes the proof of the theorem. □

Let Φ be a cocycle over a flow φ and let $B\colon\Omega\to L(X)$ be an essentially bounded strongly measurable function. We consider the cocycle Ψ satisfying (27) for $t,s\in\mathbb{R}$ and μ-almost every $\omega\in\Omega$ (again we assume that it is uniquely defined).

Theorem 7. *Assume that the cocycle Φ admits an exponential dichotomy in average. If the constant c in* (28) *is sufficiently small, then the cocycle Ψ also admits an exponential dichotomy in average.*

The proof is analogous to the proof of Theorem 4 (using Theorems 5 and 6 instead of Theorems 2 and 3) and so it is omitted.

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Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal

E-mail: barreira@math.tecnico.ulisboa.pt

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia

E-mail: ddragicevic@math.uniri.hr

Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal

E-mail: cvalls@math.tecnico.ulisboa.pt