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Mathematical Analysis — *Nonuniform hyperbolicity and one-sided admissibility*, by Luis Barreira, Davor Dragičević and Claudia Valls, communicated on 11 December 2015.[1]

Abstract. — For a general one-sided nonautonomous dynamics defined by a sequence of linear operators, we consider the notion of an exponential dichotomy with respect to a sequence of norms and we characterize it completely in terms of the admissibility of bounded solutions. As a nontrivial application, we establish the robustness of the notion under sufficiently small parameterized perturbations. Moreover, we show that if the perturbations are Lipschitz or of class C^1 on the parameter, then the same happens to the projections onto the stable spaces of the perturbation.

Key words: Admissibility, exponential dichotomies, robustness

Mathematics Subject Classification: 37D25

1. Introduction

We consider the notion of an exponential dichotomy with respect to a sequence of norms for a one-sided nonautonomous dynamics defined by a sequence of linear operators. More precisely, given a sequence $(A_m)_{m \in \mathbb{N}}$ of linear operators acting on a Banach space X, we consider the dynamics

$$v_m = A_{m-1} \dots A_n v_n \quad \text{for } m \geq n \geq 1.$$

The notion of an exponential dichotomy, essentially introduced by Perron in [8], plays a central role in the theory of dynamical systems. Essentially, it corresponds to assume the existence of complementary spaces on which we have either uniform contraction or uniform expansion, with respect to a given norm or norms on the Banach space.

The classical notion of a (uniform) exponential dichotomy *essentially* corresponds to consider a single norm, but this need not always be the case. For example, let $f : M \to M$ be a diffeomorphism of a smooth Riemannian manifold M. Each tangent map

$$d_x f : T_x M \to T_{f(x)} M$$

is a linear operator between $T_x M$ and $T_{f(x)} M$. Now let $\| \cdot \|_x$ be the norm induced by the Riemannian metric on $T_x M$. Writing $X_n = T_{f^n(x)} M$ for $n \in \mathbb{N}$, the map

[1] Presented by J. Palis.

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$d_{f^n(x)} f$ is a linear operator

$$A_n : (X_n, \|\cdot\|'_n) \to (X_{n+1}, \|\cdot\|'_{n+1}),$$

where $\|\cdot\|'_n = \|\cdot\|_{f^n(x)}$. Identifying all tangent spaces with the same Euclidean space $X = \mathbb{R}^n$, where n is the dimension of M, we obtain a sequence of linear operators

$$A_n : (X, \|\cdot\|'_n) \to (X, \|\cdot\|'_{n+1}).$$

So the notion of a (uniform) exponential dichotomy for the nonautonomous dynamics defined by the maps $A_n = d_{f^n(x)} f$ must use the sequence of norms $\|\cdot\|'_n$ instead of a single norm.

On the other hand, our work allows considering arbitrary sequences of norms. For example, if we consider a sequence of Lyapunov norms, then we recover the notion of a nonuniform exponential dichotomy (we refer the reader to [2] for the definitions). A principal motivation for the last notion is that in the context of ergodic theory almost all linear nonautonomous dynamics obtained from the derivative cocycle of a smooth map preserving a finite measure admit a nonuniform exponential dichotomy whenever all corresponding Lyapunov exponents are nonzero. Thus, our work allows considering in a unified manner both uniform and nonuniform exponential behaviors.

Our main aim is to characterize completely the notion of an exponential dichotomy with respect to a sequence of norms in terms of the admissibility of bounded solutions. The study of admissibility goes back to Perron in [8] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

in $\mathbb{R}^n$ for any bounded continuous function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$. It turns out that this property can be used to deduce the stability or the conditional stability under sufficiently small perturbations of a linear equation. There is an extensive literature concerning the relation between admissibility and stability, also in infinite-dimensional spaces. For a detailed list of references, we refer the reader to the book by Chicone and Latushkin [4] and for more recent work to Huy [5]. Moreover, we refer to [1] for related results on $\mathbb{Z}$.

As a nontrivial application of its characterization, we establish the robustness of the notion of an exponential dichotomy with respect to a sequence of norms under sufficiently small parameterized perturbations

$$v_{m+1} = A_m v_m + B_m(\lambda) v_m.$$

Moreover, we show that if the perturbations are Lipschitz or C^1 on the parameter, then the same happens to the projections onto the stable spaces of the perturbation. In the special case of C^1 parameterized perturbations and uniform exponential dichotomies, the robustness was first established in [3] although with a much longer proof based on the use of fixed point problems.

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In the case of continuous time, Johnson and Sell [6] considered exponential dichotomies on $\mathbb{R}$ (in a finite-dimensional space) and showed that for C^k perturbations (including for $k = \infty$ and $k = \omega$), if the perturbation and its derivatives in λ are bounded and equicontinuous (in the parameter), then the projections are of class C^k in λ. Palmer [7] considered the same problem for exponential dichotomies on $\mathbb{R}^+$ and showed that by fixing the null space, for each $k \in \mathbb{N}$ the projections are of class C^k and have bounded Lipschitz derivatives in λ, provided that the perturbation has the same property. See [9] for further developments.

2. Preliminaries

Let $X = (X, \|\cdot\|)$ be a Banach space and let $B(X)$ be the set of all bounded linear operators from X to X. Moreover, let $\|\cdot\|_m$, for $m \in \mathbb{N}$, be a sequence of norms on X such that $\|\cdot\|_m$ is equivalent to $\|\cdot\|$ for each m. Given a sequence $(A_m)_{m\in\mathbb{N}} \subset B(X)$, we define

$$\mathscr{A}(m,n) = \begin{cases} A_{m-1}\dots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n. \end{cases}$$

We say that $(A_m)_{m\in\mathbb{N}}$ admits an *exponential dichotomy* with respect to the sequence of norms $\|\cdot\|_m$ if:

1. there exist projections P_m for $m \in \mathbb{N}$ satisfying

$$A_m P_m = P_{m+1} A_m, \quad m \in \mathbb{N}, \tag{1}$$

   such that each map $A_m \,|\, \ker P_m : \ker P_m \to \ker P_{m+1}$ is invertible;
2. there exist constants $\lambda, D > 0$ such that for every $x \in X$ and $m, n \in \mathbb{N}$ we have

$$\|\mathscr{A}(m,n)P_n x\|_m \le De^{-\lambda(m-n)}\|x\|_n \quad \text{for } m \ge n \tag{2}$$

   and

$$\|\mathscr{A}(m,n)Q_n x\|_m \le De^{-\lambda(n-m)}\|x\|_n \quad \text{for } m \le n, \tag{3}$$

   where $Q_m = \mathrm{Id} - P_m$ and

$$\mathscr{A}(n,m) = (\mathscr{A}(m,n) \,|\, \ker P_n)^{-1} : \ker P_m \to \ker P_n$$

   for $n < m$.

Now let Y be the set of all sequences $\mathbf{x} = (x_m)_{m\in\mathbb{N}}$, with $x_m \in X$ for $m \in \mathbb{N}$, such that

$$\|\mathbf{x}\|_\infty := \sup_{m\in\mathbb{N}} \|x_m\|_m < +\infty.$$

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We note that $(Y, \|\cdot\|_\infty)$ is a Banach space. Moreover, let Y_0 be the set of all $\mathbf{x} \in Y$ such that $x_1 = 0$ and given a closed subspace $Z \subset X$, let Y_Z be the set of all $\mathbf{x} \in Y$ such that $x_1 \in Z$. Clearly, Y_0 and Y_Z are closed subspaces of Y.

We consider the linear operator $T_Z : \mathscr{D}(T_Z) \to Y_0$ defined by

$$(T_Z\mathbf{x})_1 = 0 \quad \text{and} \quad (T_Z\mathbf{x})_{m+1} = x_{m+1} - A_m x_m \quad \text{for } m \in \mathbb{N},$$

on the domain $\mathscr{D}(T_Z)$ composed of those vectors $\mathbf{x} \in Y_Z$ such that $T_Z\mathbf{x} \in Y_0$. We note that T_Z is closed. Indeed, let $(\mathbf{x}^k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{D}(T_Z)$ converging to $\mathbf{x} \in Y_Z$ such that $T_Z\mathbf{x}^k$ converges to $\mathbf{y} \in Y_0$. Then

$$x_{m+1} - A_m x_m = \lim_{k\to+\infty}(x^k_{m+1} - A_m x^k_m) = \lim_{k\to+\infty}(T_Z\mathbf{x}^k)_{m+1} = y_{m+1}$$

for $m \in \mathbb{N}$. This shows that $T_Z\mathbf{x} = \mathbf{y}$ and $\mathbf{x} \in \mathscr{D}(T_Z)$. Therefore, the operator T_Z is closed.

For $x \in \mathscr{D}(T_Z)$ we consider the graph norm

$$\|\mathbf{x}\|_{T_Z} = \|\mathbf{x}\|_\infty + \|T_Z\mathbf{x}\|_\infty.$$

Since T_Z is closed, $(\mathscr{D}(T_Z), \|\cdot\|_{T_Z})$ is a Banach space. Moreover, the operator

$$T_Z : (\mathscr{D}(T_Z), \|\cdot\|_{T_Z}) \to Y_0 \tag{4}$$

is bounded and, for simplicity, from now on we denote it simply by T_Z. In this paper we study the relation between exponential dichotomies and the invertibility of the operators T_Z.

3. Characterization of exponential dichotomies

Our first result ensures that in the presence of an exponential dichotomy at least one of the operators T_Z is invertible.

THEOREM 1. *If the sequence $(A_n)_{n\in\mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$, then for the closed subspace $Z = \operatorname{Im} Q_1$ the operator T_Z is invertible.*

PROOF. We first establish the injectivity of T_Z. Assume that $T_Z\mathbf{x} = 0$ for some $\mathbf{x} \in Y_Z$. Then $x_m = \mathscr{A}(m,1)x_1$ for $m \in \mathbb{N}$. Hence, it follows from (3) that

$$\|Q_1x_1\|_1 = \|\mathscr{A}(1,n)Q_nx_n\|_1 \le De^{-\lambda(n-1)}\|x_n\|_n \le De^{-\lambda(n-1)}\|\mathbf{x}\|_\infty$$

for $n \in \mathbb{N}$. Letting $n \to \infty$ yields that $x_1 = Q_1x_1 = 0$ and hence $\mathbf{x} = 0$. Therefore, T_Z is injective.

Now we show that T_Z is onto. Take a sequence $\mathbf{y} = (y_m)_{m\in\mathbb{N}} \in Y_0$. For $m \in \mathbb{N}$ we define

$$x_m = \sum_{k=1}^{m} \mathscr{A}(m,k)P_k y_k - \sum_{k=m+1}^{\infty} \mathscr{A}(m,k)Q_k y_k.$$

It follows from (2) and (3) that

$$\begin{aligned}\|x_m\|_m &\le D\sum_{k=1}^{m} e^{-\lambda(m-k)}\|y_k\|_k + D\sum_{k=m+1}^{\infty} e^{-\lambda(k-m)}\|y_k\|_k\\ &\le D\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\|\mathbf{y}\|_\infty\end{aligned}$$

for $m \in \mathbb{N}$ (in particular, x_m is well defined). Therefore, $\mathbf{x} = (x_m)_{m\in\mathbb{N}}$ belongs to Y. Moreover, it is straightforward to verify that $T_Z\mathbf{x} = \mathbf{y}$. This completes the proof of the theorem. □

Now we establish the converse of Theorem 1.

Theorem 2. *If for some closed subspace $Z \subset X$ the operator T_Z is invertible, then the sequence $(A_m)_{m\in\mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$.*

Proof. Let $Z \subset X$ be a closed subspace such that the operator T_Z is invertible. For each $n \in \mathbb{N}$, let

$$X(n) = \left\{x \in X : \sup_{m\ge n}\|\mathscr{A}(m,n)x\|_m < +\infty\right\} \quad \text{and} \quad Z(n) = \mathscr{A}(n,1)Z.$$

Clearly, $X(n)$ and $Z(n)$ are subspaces of X.

Lemma 1. *For $n \in \mathbb{N}$, we have*

$$X = X(n) \oplus Z(n). \tag{5}$$

Proof of the lemma. We first take $n > 1$. Given $v \in X$, we define a sequence $\mathbf{y} = (y_m)_{m\in\mathbb{N}}$ by $y_n = v$ and $y_m = 0$ for $m \ne n$. Clearly, $\mathbf{y} \in Y_0$. Hence, there exists $\mathbf{x} \in Y_Z$ such that $T_Z\mathbf{x} = \mathbf{y}$. That is,

$$x_n - A_{n-1}x_{n-1} = v \tag{6}$$

and

$$x_{m+1} = A_m x_m \quad \text{for } m \ne n-1. \tag{7}$$

It follows from (7) that

$$x_m = \mathscr{A}(m,n)x_n \quad \text{for } m \ge n, \quad \text{and} \quad A_{n-1}x_{n-1} = \mathscr{A}(n,1)x_1.$$

Since $\mathbf{x} \in Y_Z$, we conclude that

$$x_n \in X(n) \quad \text{and} \quad A_{n-1}x_{n-1} \in Z(n).$$

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Finally, by (6), we have $v \in X(n) + Z(n)$. Now let $v \in X(n) \cap Z(n)$ and take $z \in Z$ such that $v = \mathscr{A}(n,1)z$. Let $z_m = \mathscr{A}(m,1)z$ for $m \in \mathbb{N}$. Clearly, $\mathbf{z} = (z_m)_{m\in\mathbb{N}} \in Y_Z$ and $T_Z\mathbf{z} = 0$. Since the operator T_Z is invertible, we conclude that $\mathbf{z} = 0$ and thus $v = 0$. This shows that (5) holds for $n > 1$.

Now we establish (5) for $n = 1$. Take $v \in X$ and consider the sequences

$$\mathbf{x}^1 = (v, 0, 0, \dots) \quad \text{and} \quad \mathbf{y}^1 = (0, -A_1 v, 0, 0, \dots). \tag{8}$$

We have

$$x^1_{m+1} - A_m x^1_m = y^1_{m+1} \quad \text{for } m \in \mathbb{N}.$$

Moreover, since $\mathbf{y}^1 \in Y_0$, there exists $\mathbf{x}^2 \in Y_Z$ such that $T_Z\mathbf{x}^2 = \mathbf{y}^1$ and

$$x^1_m - x^2_m = \mathscr{A}(m,1)(v - x^2_1)$$

for $m \in \mathbb{N}$. Therefore, $v - x^2_1 \in X(1)$ and $v \in X(1) + Z$. Now take $v \in X(1) \cap Z$ and let $v_m = \mathscr{A}(m,1)v$ for $m \in \mathbb{N}$. Clearly, $\mathbf{v} = (v_m)_{m\in\mathbb{N}} \in Y_Z$ and $T_Z\mathbf{v} = 0$. Since T_Z is invertible, we have $\mathbf{v} = 0$ and thus $v = 0$. □

Let $P_n : X \to X(n)$ and $Q_n : X \to Z(n)$ be the projections associated to the decomposition in (5), with $P_n + Q_n = \mathrm{Id}$. It follows readily from the definitions that (1) holds.

Lemma 2. *There exists $M > 0$ such that*

$$\|P_1 v\|_1 \le M\|v\|_1 \quad \textit{for } v \in X. \tag{9}$$

Proof of the lemma. Using the notation in the proof of the previous lemma, we have

$$\|Q_1 v\|_1 = \|x^2_1\|_1 \le \|\mathbf{x}^2\|_{T_Z} \le \|T_Z^{-1}\| \cdot \|\mathbf{y}^1\|_\infty = \|T_Z^{-1}\| \cdot \|A_1 v\|_2$$

for $v \in X$. Since A_1 is bounded and the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, there exists a constant $C > 0$ such that $\|A_1 v\|_2 \le C\|v\|_1$ for $v \in X$. Therefore, (9) holds taking $M = 1 + C\|T_Z^{-1}\|$. □

Lemma 3. *The linear operator $A_n \,|\, \ker P_n : \ker P_n \to \ker P_{n+1}$ is invertible for each $n \in \mathbb{N}$.*

Proof of the lemma. We first obtain the injectivity of the operator. Assume that $A_n v = 0$ for $v \in \ker P_n = Z(n)$ and take $z \in Z$ such that $v = \mathscr{A}(n,1)z$. Moreover, let $x_m = \mathscr{A}(m,1)z$ for $m \in \mathbb{N}$. Then $\mathbf{x} = (x_m)_{m\in\mathbb{N}} \in Y_Z$ and $T_Z\mathbf{x} = 0$. Since T_Z is invertible, we conclude that $\mathbf{x} = 0$ and thus $v = 0$.

In order to show that the operator is onto, take $v \in \ker P_{n+1} = Z(n+1)$ and $z \in Z$ such that $v = \mathscr{A}(n+1,1)z$. Clearly, $w = \mathscr{A}(n,1)z \in \ker P_n$ and $A_n w = v$. This shows that $A_n \,|\, \ker P_n$ is onto. □

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Now we obtain the bounds in (2) and (3). Take $n > 1$ and $v \in X$. Moreover, let $\mathbf{x}$ and $\mathbf{y}$ be as in the proof of Lemma 1. We define a family of linear operators

$$B(z) : (\mathscr{D}(T_Z), \|\cdot\|_{T_Z}) \to Y_0$$

for $z \geq 1$ by

$$(B(z)\boldsymbol{v})_1 = 0 \quad \text{and} \quad (B(z)\boldsymbol{v})_{m+1} = \begin{cases} zv_{m+1} - A_m v_m & \text{if } 1 \leq m < n, \\ \frac{1}{z} v_{m+1} - A_m v_m & \text{if } m \geq n. \end{cases}$$

Clearly, $B(1) = T_Z$ and

$$\|(B(z) - T_Z)\boldsymbol{v}\|_\infty \leq (z-1)\|\boldsymbol{v}\|_{T_Z}$$

for $\boldsymbol{v} \in \mathscr{D}(T_Z)$ and $z \geq 1$. Hence, $B(z)$ is invertible whenever $1 \leq z < 1 + 1/\|T_Z^{-1}\|$, in which case

$$\|B(z)^{-1}\| \leq \frac{1}{\|T_Z^{-1}\|^{-1} - (z-1)}.$$

Now take $t \in (0,1)$ such that $1/t < 1 + 1/\|T_Z^{-1}\|$ and let $\mathbf{z} \in Y_Z$ be the unique element such that $B(1/t)\mathbf{z} = \mathbf{y}$. Writing

$$D' = \frac{1}{\|T_Z^{-1}\|^{-1} - (1/t - 1)},$$

we obtain

$$\|\mathbf{z}\|_\infty \leq \|\mathbf{z}\|_{T_Z} = \|B(1/t)^{-1}\mathbf{y}\|_{T_Z} \leq D'\|\mathbf{y}\|_\infty = D'\|v\|_n.$$

For each $m \in \mathbb{N}$, let $x_m^* = t^{|m-n|-1} z_m$ and $\mathbf{x}^* = (x_m^*)_{m \in \mathbb{N}}$. Clearly, $\mathbf{x}^* \in Y_Z$. It is easy to verify that $T_Z \mathbf{x}^* = \mathbf{y}$ and hence $\mathbf{x}^* = \mathbf{x}$. Thus,

$$\|x_m\|_m = \|x_m^*\|_m = t^{|m-n|-1}\|z_m\|_m \leq \frac{D'}{t} t^{|m-n|}\|v\|_n \tag{10}$$

for $m \in \mathbb{N}$. On the other hand, it was shown in the proof of Lemma 1 that $P_n v = x_n$ and $Q_n v = -A_{n-1}x_{n-1}$. Hence, it follows from (7) and (10) that

$$\begin{aligned} \|\mathscr{A}(m,n)P_n v\|_m &= \|\mathscr{A}(m,n)x_n\|_m \\ &= \|x_m\|_m \leq \frac{D'}{t} t^{m-n}\|v\|_n \\ &= \frac{D'}{t} e^{(m-n)\log t}\|v\|_n \end{aligned} \tag{11}$$

for $m \geq n > 1$. Now take $n = 1$. For each $m > 1$ and $v \in X$, we have

$$\|\mathscr{A}(m,1)P_1 v\|_m = \|\mathscr{A}(m,2)P_2 A_1 v\|_m \leq \frac{D'}{t} e^{(m-2)\log t}\|A_1 v\|_2.$$

Therefore,

$$\|\mathscr{A}(m,1)P_1 v\|_m \leq \frac{CD'}{t^2} e^{(m-1)\log t}\|v\|_1 \tag{12}$$

for $v \in X$ and $m > 1$, with $C > 0$ as in the proof of Lemma 2. It follows from (9) that (12) also holds when $m = 1$. Similarly, it follows from (7) and (10) that

$$\|\mathscr{A}(m,n)Q_n v\|_m \leq \frac{D'}{t} e^{(n-m)\log t}\|v\|_n \tag{13}$$

for $v \in X$ and $m < n$ with $n > 1$. By (11), (12) and (13), we conclude that there exists $D > 0$ such that (2) and (3) hold taking $\lambda = \log t$. In other words, the sequence $(A_m)_{m \in \mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$. □

4. Parameterized robustness I

As an application of the characterization of the notion of an exponential dichotomy given by Theorems 1 and 2, we establish the robustness of the notion under sufficiently small parameterized perturbations. More precisely, we consider perturbations that are Lipschitz on the parameter.

Let I be a Banach space (the parameter space) and let $B_n : I \to B(X)$, for $n \in \mathbb{N}$, be continuous functions.

Theorem 3. *Assume that the sequence $(A_m)_{m \in \mathbb{N}}$ admits an exponential dichotomy with respect to a sequence of norms $\|\cdot\|_m$ and that there exist $c, d > 0$ such that*

$$\|B_m(\lambda)x\|_{m+1} \leq c\|x\|_m \tag{14}$$

and

$$\|(B_m(\lambda) - B_m(\mu))x\|_{m+1} \leq d\|\lambda - \mu\| \cdot \|x\|_m \tag{15}$$

for $m \in \mathbb{N}$, $x \in X$ and $\lambda, \mu \in I$. If c is sufficiently small, then for each $\lambda \in I$ the sequence $(A_m + B_m(\lambda))_{m \in \mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$. In addition, one can choose the projections $P_{m,\lambda}$ onto the stable spaces so that the map $\lambda \mapsto P_{m,\lambda}$ is locally Lipschitz for each $m \in \mathbb{N}$.

Proof. By Theorem 1, there exists a closed subspace $Z \subset X$ such that the operator T_Z in (4) is invertible. For each $\lambda \in I$ we consider the sequence

$(A_m + B_m(\lambda))_{m \in \mathbb{N}}$ and the associated operator $T_{Z,\lambda}$. It follows from (14) that

$$\|(T_{Z,\lambda} - T_Z)\mathbf{x}\|_\infty = \sup_{m \in \mathbb{N}} \|B_m(\lambda)x_m\|_{m+1} \le c\|\mathbf{x}\|_\infty \le c\|\mathbf{x}\|_{T_Z} \tag{16}$$

for $\mathbf{x} \in Y_Z$ and $\lambda \in I$. Hence, the domain of the operator $T_{Z,\lambda}$ is $\mathscr{D}(T_Z)$ for $\lambda \in I$. Furthermore, the operator $T_{Z,\lambda} : (\mathscr{D}(T_Z), \|\cdot\|_{T_Z}) \to Y_0$ is bounded. When c is sufficiently small, it follows from (16) that $T_{Z,\lambda}$ is also invertible and thus, the sequence $(A_m + B_m(\lambda))_{m \in \mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$. In addition, it follows from the proof of Lemma 1 that the associated projections $P_{m,\lambda}$ can be chosen so that

$$P_{1,\lambda}v = v - (T_{Z,\lambda}^{-1}\mathbf{y}^1)_1 \quad \text{and} \quad P_{m,\lambda}v = (T_{Z,\lambda}^{-1}\mathbf{y})_m \quad \text{for } m > 1, \tag{17}$$

with the sequence $\mathbf{y}^1$ as in (8).

Before proceeding we establish an auxiliary result.

Lemma 4. *The map $\lambda \mapsto T_{Z,\lambda}$ is Lipschitz.*

Proof of the lemma. It follows from (15) that

$$\begin{aligned}\|(T_{Z,\lambda} - T_{Z,\mu})\mathbf{x}\|_\infty &= \sup_{m \in \mathbb{N}} \|(B_m(\lambda) - B_m(\mu))x_m\|_{m+1} \\ &\le d\|\lambda - \mu\| \sup_{m \in \mathbb{N}} \|x_m\|_m \\ &\le d\|\lambda - \mu\| \cdot \|\mathbf{x}\|_{T_Z}\end{aligned}$$

for $\mathbf{x} \in \mathscr{D}(T_Z)$ and $\lambda, \mu \in I$. Hence,

$$\|T_{Z,\lambda} - T_{Z,\mu}\| \le d\|\lambda - \mu\|,$$

and the desired result follows. □

Now take $m > 1$ and $\lambda \in I$. It follows from Lemma 4 that

$$\|(T_{Z,\mu} - T_{Z,\lambda})T_{Z,\lambda}^{-1}\| < 1 \tag{18}$$

whenever μ is sufficiently close to λ. By (17), we have

$$\begin{aligned}\|P_{m,\lambda}v - P_{m,\mu}v\|_m &= \|((T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1})\mathbf{y})_m\|_m \\ &\le \|(T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1})\mathbf{y}\|_\infty \\ &\le \|(T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1})\mathbf{y}\|_{T_Z} \\ &\le \|T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1}\| \cdot \|\mathbf{y}\|_\infty \\ &= \|T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1}\| \cdot \|v\|_m\end{aligned}$$

for $v \in X$. On the other hand, for any μ satisfying (18) we have

$$\|T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1}\| \le \frac{\|T_{Z,\lambda}^{-1}\|^2 \|T_{Z,\lambda} - T_{Z,\mu}\|}{1 - \|T_{Z,\lambda}^{-1}\| \cdot \|T_{Z,\lambda} - T_{Z,\mu}\|}.$$

Hence, by (17),

$$\|P_{m,\lambda}v - P_{m,\mu}v\|_m \le \frac{\|T_{Z,\lambda}^{-1}\|^2 \|T_{Z,\lambda} - T_{Z,\mu}\|}{1 - \|T_{Z,\lambda}^{-1}\| \cdot \|T_{Z,\lambda} - T_{Z,\mu}\|} \cdot \|v\|_m$$

for $v \in X$. Since the norms $\|\cdot\|_m$ and $\|\cdot\|$ are equivalent, we conclude that the map $\lambda \mapsto P_{m,\lambda}$ is locally Lipschitz.

Finally, we consider the case when $m = 1$. Take $\lambda \in I$. It follows from (17) that

$$\begin{aligned}
\|P_{1,\lambda}v - P_{1,\mu}v\|_1 &= \|((T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1})\mathbf{y}^1)_1\|_1 \\
&\le \|(T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1})\mathbf{y}^1\|_\infty \\
&\le \|(T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1})\mathbf{y}^1\|_{T_Z} \\
&\le \|T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1}\| \cdot \|\mathbf{y}^1\|_\infty \\
&= \|T_{Z,\lambda}^{-1} - T_{Z,\mu}^{-1}\| \cdot \|A_1 v\|_2
\end{aligned}$$

for $v \in X$. Hence, for any μ for which (18) holds we have

$$\|P_{1,\lambda}v - P_{1,\mu}v\|_1 \le \frac{\|T_{Z,\lambda}^{-1}\|^2 \|T_{Z,\lambda} - T_{Z,\mu}\|}{1 - \|T_{Z,\lambda}^{-1}\| \cdot \|T_{Z,\lambda} - T_{Z,\mu}\|} \|A_1 v\|_2$$

for $v \in X$. Since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent to $\|\cdot\|$, we conclude that the map $\lambda \mapsto P_{1,\lambda}$ is locally Lipschitz. This completes the proof of the theorem. □

5. Parameterized robustness II

In this section we establish a smooth version of the parameterized robustness result in Theorem 3. In particular, we establish the smooth dependence of the projections on the parameter.

Let I be a Banach space and let $B_n : I \to B(X)$, for $n \in \mathbb{N}$, be differentiable functions.

Theorem 4. *Assume that the sequence* $(A_m)_{m \in \mathbb{N}}$ *admits an exponential dichotomy with respect to a sequence of norms* $\|\cdot\|_m$ *and that:*

1. *there exist* $c, d > 0$ *such that*

$$\|B_m(\lambda)x\|_{m+1} \le c\|x\|_m$$

and

$$\|[B'_m(\lambda)\mu]x\|_{m+1} \le d\|\mu\| \cdot \|x\|_m \tag{19}$$

for $m \in \mathbb{N}$, $\lambda, \mu \in I$ and $x \in X$;

2. *given $\lambda \in I$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\lambda' \in I$ satisfying $\|\lambda - \lambda'\| \le \delta$ we have*

$$\|([B'_m(\lambda) - B'_m(\lambda')]\mu)x\|_{m+1} \le \varepsilon\|\mu\| \cdot \|x\|_m \tag{20}$$

for $m \in \mathbb{N}$, $\mu \in I$ and $x \in X$.

If c is sufficiently small, then for each $\lambda \in I$ the sequence $(A_m + B_m(\lambda))_{m\in\mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$. In addition, one can choose the projections $P_{m,\lambda}$ onto the stable spaces so that the map $\lambda \mapsto P_{m,\lambda}$ is of class C^1 for each $m \in \mathbb{N}$.

Proof. By Theorem 3, if c is sufficiently small, then for each $\lambda \in I$ the sequence $(A_m + B_m(\lambda))_{m\in\mathbb{N}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$. Now we show that the projections can be chosen so that they depend smoothly on the parameter. We shall use the same notation as in the proof of Theorem 3. The key point in the argument is the following lemma.

Lemma 5. *The map $\lambda \mapsto T_{Z,\lambda}$ is of class C^1.*

Proof of the lemma. We construct explicitly the derivative of $\lambda \mapsto T_{Z,\lambda}$ and show that it is continuous. We define a map $G : I \to B(I, B(\mathscr{D}(T_Z), Y_0))$ by

$$([G(\lambda)\mu]\boldsymbol{v})_1 = 0 \quad \text{and} \quad ([G(\lambda)\mu]\boldsymbol{v})_{m+1} = -[B'_m(\lambda)\mu]v_m \quad \text{for } m \in \mathbb{N},$$

where $B(X, Y)$ denotes the set of all bounded linear operators from X to Y. We claim that $G(\lambda) : I \to B(\mathscr{D}(T_Z), Y_0)$ is a well defined bounded linear operator. The linearity is easy to check. On the other hand, by (19) we have

$$\|[G(\lambda)\mu]\boldsymbol{v}\|_\infty = \sup_{m\in\mathbb{N}} \|[B'_m(\lambda)\mu]v_m\|_{m+1} \le d\|\mu\| \cdot \|\boldsymbol{v}\|_\infty \le d\|\mu\| \cdot \|\boldsymbol{v}\|_{T_Z}$$

for $\mu \in I$ and $\boldsymbol{v} \in Y_Z$. Take $\varepsilon > 0$ and $\delta > 0$ as in inequality (20). A simple computation shows that

$$([T_{Z,\lambda'} - T_{Z,\lambda} - G(\lambda)(\lambda' - \lambda)]\boldsymbol{v})_0 = 0$$

and

$$([T_{Z,\lambda'} - T_{Z,\lambda} - G(\lambda)(\lambda' - \lambda)]\boldsymbol{v})_{m+1} = [B_m(\lambda) - B_m(\lambda') + B'_m(\lambda)(\lambda' - \lambda)]v_m$$

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for $m \in \mathbb{N}$. Since

$$\begin{aligned} &B_m(\lambda') - B_m(\lambda) - B'_m(\lambda)(\lambda' - \lambda) \\ &\quad = \Big(\int_0^1 [B'_m(\lambda + t(\lambda' - \lambda)) - B'_m(\lambda)]\, dt\Big)(\lambda' - \lambda), \end{aligned}$$

it follows from (20) that

$$\begin{aligned} &\|[T_{Z,\lambda'} - T_{Z,\lambda} - G(\lambda)(\lambda' - \lambda)]\boldsymbol{v}\|_\infty \\ &\quad = \sup_{m\in\mathbb{N}} \left\| \Big(\int_0^1 [B'_m(\lambda + t(\lambda' - \lambda)) - B'_m(\lambda)]\, dt\Big)(\lambda' - \lambda) v_m \right\|_{m+1} \\ &\quad \le \varepsilon \|\lambda - \lambda'\| \cdot \|\boldsymbol{v}\|_{T_Z} \end{aligned}$$

for $\boldsymbol{v} \in \mathscr{D}(T_Z)$ and $\lambda' \in I$ such that $\|\lambda - \lambda'\| \le \delta$. Therefore,

$$\frac{\|T_{Z,\lambda'} - T_{Z,\lambda} - G(\lambda)(\lambda' - \lambda)\|}{\|\lambda - \lambda'\|} \le \varepsilon$$

whenever $\|\lambda - \lambda'\| \le \delta$, which shows that $G(\lambda)$ is the derivative of the map $\lambda \mapsto T_{Z,\lambda}$.

In order to show that $\lambda \mapsto G(\lambda)$ is continuous, take $\lambda \in I$ and $\varepsilon > 0$. Moreover, take δ so that (20) holds. We have

$$\begin{aligned} \|([G(\lambda) - G(\lambda')]\mu)\boldsymbol{v}\|_\infty &= \sup_{m\in\mathbb{N}} \|([B'_m(\lambda) - B'_m(\lambda')]\mu) v_m\|_{m+1} \\ &\le \varepsilon \|\mu\| \cdot \|\boldsymbol{v}\|_{T_Z} \end{aligned}$$

for $\mu \in I$ and $\boldsymbol{v} \in \mathscr{D}(T_Z)$. Hence, $\|G(\lambda) - G(\lambda')\| \le \varepsilon$ whenever $\|\lambda' - \lambda\| \le \delta$, which establishes the desired property. □

It follows from the lemma that the map $\lambda \mapsto T_{Z,\lambda}^{-1}$ is smooth. Now we consider projections $P_{m,\lambda}$ as in (17). For $m > 1$, one can write $P_{m,\lambda}$ in the form $P_{m,\lambda} = C_m T_{Z,\lambda}^{-1} D_m$, where D_m is a linear map taking v to $\mathbf{y}$ and C_m is a projection. This shows that $\lambda \mapsto P_{m,\lambda}$ is of class C^1. One can argue in a similar manner for $m = 1$. □

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