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Characterization of nonuniform exponential trichotomies for flows ☆

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ABSTRACT

For an evolution family, we characterize completely the notion of a nonuniform exponential trichotomy in terms of admissibility properties. As a nontrivial application, we establish the robustness of the notion in a very simple manner. We also obtain corresponding results for a strong nonuniform exponential trichotomy. We emphasize that both notions are ubiquitous in the context of ergodic theory. Moreover, we develop a corresponding theory for the notion of a nonuniformly partially hyperbolic set, in which case one considers simultaneously a collection of trajectories instead of a single one. In particular, this required first developing an appropriate theory for nonuniformly hyperbolic sets, which is of independent interest.

1. Introduction

Our main aim is to give a complete characterization of the notion of a *nonuniform* exponential trichotomy in terms of admissibility properties. The study of admissibility goes back to Perron in [13] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

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in a finite-dimensional space $\mathbb{R}^n$ for any bounded continuous function f. One can also consider other spaces, both where we take the perturbations and where we look for the solutions. In this paper, we take locally integrable measurable perturbations f satisfying

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s)\|_s \, ds < +\infty$$

and look for solutions x such that

$$\sup_{t\in\mathbb{R}} \|x(t)\|_t < +\infty,$$

where $\|\cdot\|_t$ is an appropriate family of norms. When the norms are independent of t we recover the "classical" case of a uniform exponential behavior. We emphasize that the need to consider different spaces for the perturbations and for the solutions is not due to considering a *nonuniform* exponential behavior. Instead, it is unavoidable whenever the original linear dynamics does not have the bounded growth property. Indeed, the problem already occurs for a uniform exponential behavior (see [6]), although it never occurs in the case of discrete time.

For some of the early contributions to the study of admissibility properties in connection with stability and conditional stability, we refer to the books by Massera and Schäffer [11] and by Dalec'kiĭ and Kreĭn [7]. See [10] for some early results in infinite-dimensional spaces. For further references, we refer the reader to [5,9].

We also consider the notion of a *strong* nonuniform exponential trichotomy and we give a corresponding characterization in terms of admissibility properties. In this case there are both lower and upper exponential bounds on the stable and unstable directions, instead of only along the central direction. As an application of our results, we give simple proofs of the robustness of the notions of a nonuniform exponential trichotomy and of a strong nonuniform exponential trichotomy under sufficiently small linear perturbations.

A principal motivation to consider the notions of a nonuniform exponential trichotomy and of a strong nonuniform exponential trichotomy is that from the point of view of ergodic theory, that is, for an autonomous differential equation whose flow preserves a finite measure, almost all linear variational equations have a nonuniform exponential behavior. For example, this happens on any compact energy level of a Hamiltonian system with respect to the Liouville measure.

Finally, we obtain a corresponding characterization of the notion of a nonuniformly partially hyperbolic set. Essentially this corresponds to consider various trajectories simultaneously instead of a single one. We emphasize that the notion of a nonuniformly partially hyperbolic set arises naturally in the context of smooth ergodic theory (see [3]). Before obtaining this characterization, we first develop an appropriate theory for nonuniformly hyperbolic sets, which is of independent interest on its own. For related work in the case of discrete time, we refer the reader to Mather [12] and Dragičević and Slijepčević [8].

2. Preliminaries

Let $X = (X, \|\cdot\|)$ be a Banach space and let $B(X)$ be the set of all bounded linear operators acting on X. A family $T(t,\tau)$, with $t,\tau \in \mathbb{R}$, of linear operators in $B(X)$ is called an *evolution family* if:

1. $T(t,t) = \mathrm{Id}$ for $t \in \mathbb{R}$;
2. $T(t,s)T(s,\tau) = T(t,\tau)$ for $t,s,\tau \in \mathbb{R}$;
3. for each $t,\tau \in \mathbb{R}$ and $x \in X$, the maps $s \mapsto T(t,s)x$ and $s \mapsto T(s,\tau)x$ are continuous.

We say that an evolution family $T(t,\tau)$ admits a *nonuniform exponential trichotomy* if there exist projections $P_t^i\colon X \to X$ for $i \in \{1,2,3\}$ and $t \in \mathbb{R}$ satisfying

$$P_t^1 + P_t^2 + P_t^3 = \mathrm{Id}, \quad T(t,\tau)P_\tau^i = P_t^i T(t,\tau)$$

for $t,\tau \in \mathbb{R}$ and $i \in \{1,2,3\}$, and there exist constants

$$D > 0, \quad 0 \le a < b, \quad 0 \le c < d, \quad \varepsilon \ge 0$$

such that

$$\|T(t,\tau)P_\tau^1\| \le De^{-d(t-\tau)+\varepsilon|\tau|}, \quad \|T(t,\tau)P_\tau^3\| \le De^{a(t-\tau)+\varepsilon|\tau|} \tag{1}$$

for $t,\tau \in \mathbb{R}$ with $t \ge \tau$ and

$$\|T(t,\tau)P_\tau^2\| \le De^{-b(\tau-t)+\varepsilon|\tau|}, \quad \|T(t,\tau)P_\tau^3\| \le De^{c(\tau-t)+\varepsilon|\tau|} \tag{2}$$

for $t,\tau \in \mathbb{R}$ with $t \le \tau$.

Moreover, we say that an evolution family $T(t,\tau)$ admits a *nonuniform exponential dichotomy* if it admits a nonuniform exponential trichotomy with $P_t^3 = 0$ for $t \in \mathbb{R}$.

Now we consider a family of norms $\|\cdot\|_t$, $t \in \mathbb{R}$, such that $\|\cdot\|_t$ is equivalent to $\|\cdot\|$ for each t and such that the map

$$t \mapsto \|x\|_t \text{ is continuous for each } x \in X. \tag{3}$$

We say that an evolution family $T(t,\tau)$ admits an *exponential dichotomy* with respect to the family of norms $\|\cdot\|_t$, where each norm $\|\cdot\|_t$ is equivalent to $\|\cdot\|$ if there exist projections P_t for $t \in \mathbb{R}$ satisfying

$$P_t T(t,\tau) = T(t,\tau)P_\tau, \quad t,\tau \in \mathbb{R},$$

and there exist constants $\lambda, D > 0$ such that for each $x \in X$ and $t,\tau \in \mathbb{R}$ we have

$$\|T(t,\tau)P_\tau x\|_t \le De^{-\lambda(t-\tau)}\|x\|_\tau \quad \text{for} \quad t \ge \tau \tag{4}$$

and

$$\|T(t,\tau)Q_\tau x\|_t \le De^{-\lambda(\tau-t)}\|x\|_\tau \quad \text{for} \quad t \le \tau, \tag{5}$$

where $Q_t = \mathrm{Id} - P_t$.

We will need the following auxiliary result.

Proposition 1. *For each $t \in \mathbb{R}$ we have*

$$\mathrm{Im}\, P_t = \left\{x \in X : \sup_{s \ge t} \|T(s,t)x\|_s < +\infty\right\}$$

and

$$\mathrm{Im}\, Q_t = \left\{x \in X : \sup_{s \le t} \|T(s,t)x\|_s < +\infty\right\}.$$

Proof. It follows from (4) that

$$\sup_{s\geq t}\|T(s,t)x\|_s<+\infty \tag{6}$$

for $x\in \operatorname{Im} P_t$. Now take $x\in X$ satisfying (6). Since $x=P_tx+Q_tx$, it follows from (4) that

$$\sup_{s\geq t}\|T(s,t)Q_tx\|_s<+\infty.$$

On the other hand, by (5), we have

$$\|Q_tx\|_t=\|T(t,s)T(s,t)Q_tx\|_t\leq e^{-\lambda(s-t)}\|T(s,t)Q_tx\|_s$$

for $s\geq t$. Letting $s\to\infty$ we obtain $Q_tx=0$ and so $x=P_tx\in \operatorname{Im} P_t$. The proof of the second identity in the proposition is analogous. □

The following result establishes the connection between the notions of a nonuniform exponential dichotomy and of an exponential dichotomy with respect to a sequence of norms (see for example [4]).

Proposition 2. *The following properties are equivalent:*

1. $T(t,\tau)$ *admits a nonuniform exponential dichotomy;*
2. $T(t,\tau)$ *admits an exponential dichotomy with respect to a family of norms* $\|\cdot\|_t$ *satisfying* (3) *and*

$$\|x\|\leq\|x\|_t\leq De^{\varepsilon|t|},\quad t\in\mathbb{R},\ x\in X$$

for some constant $D>0$.

Now let Y_1 be the set of all continuous functions $x\colon\mathbb{R}\to X$ such that

$$\|x\|_\infty:=\sup_{t\in\mathbb{R}}\|x(t)\|_t<+\infty$$

and let Y_2 be the set of all locally integrable measurable functions $x\colon\mathbb{R}\to X$ such that

$$\|x\|_L:=\sup_{t\in\mathbb{R}}\int_t^{t+1}\|x(s)\|_s\,ds<+\infty,$$

identified if they are equal Lebesgue-almost everywhere. We note that Y_1 and Y_2 are Banach spaces when equipped, respectively, with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_L$ (see [2] for a proof in the case of Y_2). We say that the evolution family $T(t,\tau)$ has an *admissibility property* with respect to the family of norms $\|\cdot\|_t$ if for each $y\in Y_2$ there exists a unique $x\in Y_1$ such that

$$x(t)=T(t,\tau)x(\tau)+\int_\tau^t T(t,s)y(s)\,ds\quad\text{for}\quad t\geq\tau.$$

The following result was established in [2].

Proposition 3. *The following statements are equivalent:*

1. *$T(t,\tau)$ admits an exponential dichotomy with respect to the family of norms $\|\cdot\|_t$;*
2. *$T(t,\tau)$ has an admissibility property with respect to the family of norms $\|\cdot\|_t$.*

3. Nonuniform exponential trichotomies

In this section we give a characterization of the notion of a nonuniform exponential trichotomy in terms of an admissibility property.

Theorem 1. *Assume that an evolution family $T(t,\tau)$ admits a nonuniform exponential trichotomy with $\varepsilon < b+d$. Then there exist families of norms $\|\cdot\|_{1,t}$ and $\|\cdot\|_{2,t}$ for $t \in \mathbb{R}$ satisfying property (3) and there exist constants $D', \omega > 0$ and $\omega' < 0$ with $\varepsilon \le \omega - \omega'$ such that:*

1. *the evolution family $e^{\omega(t-\tau)}T(t,\tau)$ has an admissibility property with respect to the family of norms $\|\cdot\|_{1,t}$;*
2. *the evolution family $e^{\omega'(t-\tau)}T(t,\tau)$ has an admissibility property with respect to the family of norms $\|\cdot\|_{2,t}$;*
3. *for $t \in \mathbb{R}$, $i \in \{1,2\}$ and $x \in X$, we have*

$$\|x\| \le \|x\|_{i,t} \le D' e^{\varepsilon|t|}\|x\|. \tag{7}$$

Proof. Take $\omega \in (c,d)$ and consider the evolution family

$$U(t,\tau) = e^{\omega(t-\tau)}T(t,\tau). \tag{8}$$

It follows from (1) and (2) that

$$\|U(t,\tau)P^1_\tau\| \le De^{-(d-\omega)(t-\tau)+\varepsilon|\tau|} \tag{9}$$

for $t \ge \tau$ and that

$$\|U(t,\tau)P^2_\tau\| \le De^{-(b+\omega)(\tau-t)+\varepsilon|\tau|} \tag{10}$$

and

$$\|U(t,\tau)P^3_\tau\| \le De^{-(\omega-c)(\tau-t)+\varepsilon|\tau|} \tag{11}$$

for $t \le \tau$. By (9), (10) and (11), the evolution family $U(t,\tau)$ admits a nonuniform exponential dichotomy with projections $P_t = P^1_t$. Hence, by Proposition 2, the evolution family $U(t,\tau)$ admits an exponential dichotomy with respect to a family of norms $\|\cdot\|_{1,t}$ satisfying (3) and (7) with some constant $D' > 0$. Moreover, it follows from Proposition 3 that the evolution family $U(t,\tau)$ has an admissibility property with respect to the family of norms $\|\cdot\|_{1,t}$.

Now take $\omega' \in (-b,-a)$ and consider the evolution family

$$U'(t,\tau) = e^{\omega'(t-\tau)}T(t,\tau). \tag{12}$$

It follows from (1) and (2) that

$$\|U'(t,\tau)P^1_\tau\| \le De^{-(d-\omega')(t-\tau)+\varepsilon|\tau|} \tag{13}$$

and

$$\|U'(t,\tau)P^3_\tau\| \le De^{-(-a-\omega')(t-\tau)+\varepsilon|\tau|} \tag{14}$$

for $t \geq \tau$ and that

$$\|U'(t,\tau)P^2_\tau\| \leq De^{-(b+\omega')(\tau-t)+\varepsilon|\tau|} \tag{15}$$

for $t \leq \tau$. By (13), (14) and (15), the evolution family $U'(t,\tau)$ admits a nonuniform exponential dichotomy with projections $P_t = P^1_t + P^3_t$. Hence, it follows from Propositions 2 and 3 that the evolution family $U'(t,\tau)$ has an admissibility property with respect to a family of norms $\|\cdot\|_{2,t}$ satisfying (3) and (7) with some constant $D' > 0$ (that can be the same as before).

Finally, since $\varepsilon < b + d$, one can choose ω and ω' so that $\varepsilon \leq \omega - \omega'$. □

Now we establish the converse of Theorem 1.

Theorem 2. *Assume that there exist families of norms $\|\cdot\|_{1,t}$ and $\|\cdot\|_{2,t}$ for $t \in \mathbb{R}$ satisfying (3) and constants $D', \omega > 0$, $\varepsilon \geq 0$ and $\omega' < 0$ with $\varepsilon \leq \omega - \omega'$ satisfying properties 1–3 in Theorem 1. Then the evolution family $T(t,\tau)$ admits a nonuniform exponential trichotomy.*

Proof. It follows from Proposition 3 that the evolution families $e^{\omega(t-\tau)}T(t,\tau)$ and $e^{\omega'(t-\tau)}T(t,\tau)$ admit exponential dichotomies, respectively, with respect to the families of norms $\|\cdot\|_{1,t}$ and $\|\cdot\|_{2,t}$. Hence, there exist projections P^1_t and P^2_t satisfying

$$P^1_t e^{\omega(t-\tau)}T(t,\tau) = e^{\omega(t-\tau)}T(t,\tau)P^1_\tau,$$
$$P^2_t e^{\omega'(t-\tau)}T(t,\tau) = e^{\omega'(t-\tau)}T(t,\tau)P^2_\tau$$

for $t, \tau \in \mathbb{R}$ and there exist constants $\lambda, D > 0$ such that

$$\|U(t,\tau)P^1_\tau x\|_{1,t} \leq De^{-\lambda(t-\tau)}\|x\|_{1,\tau}, \tag{16}$$
$$\|U'(t,\tau)P^2_\tau x\|_{2,t} \leq De^{-\lambda(t-\tau)}\|x\|_{2,\tau} \tag{17}$$

for $t \geq \tau$ and

$$\|U(t,\tau)Q^1_\tau x\|_{1,t} \leq De^{-\lambda(\tau-t)}\|x\|_{1,\tau}, \tag{18}$$
$$\|U'(t,\tau)Q^2_\tau x\|_{2,t} \leq De^{-\lambda(\tau-t)}\|x\|_{2,\tau} \tag{19}$$

for $t \leq \tau$, with the operators $U(t,\tau)$ and $U'(t,\tau)$ as in (8) and (12), and where $Q^i_t = \mathrm{Id} - P^i_t$.

Lemma 1. *For each $\tau \in \mathbb{R}$, we have*

$$\operatorname{Im} P^1_\tau \subset \operatorname{Im} P^2_\tau \quad \text{and} \quad \operatorname{Im} Q^2_\tau \subset \operatorname{Im} Q^1_\tau.$$

Proof. Take $x \in \operatorname{Im} P^1_\tau$. By (7), we have

$$\begin{aligned}\|U'(t,\tau)x\|_{2,t} &= e^{\omega'(t-\tau)}\|T(t,\tau)x\|_{2,t} \\ &\leq D'e^{\omega'(t-\tau)}e^{\varepsilon|t|}\|T(t,\tau)x\| \\ &\leq D'e^{\omega'(t-\tau)}e^{\varepsilon|t|}\|T(t,\tau)x\|_{1,t} \\ &= D'e^{(\omega'-\omega)(t-\tau)}e^{\varepsilon|t|}\|U(t,\tau)x\|_{1,t}\end{aligned}$$

for $t \ge \tau$. Since $\varepsilon \le \omega - \omega'$, it follows from Proposition 1 that

$$\sup_{t \ge \tau} \|U'(t,\tau)x\|_{2,t} < +\infty$$

and thus $x \in \operatorname{Im} P^2_\tau$. The second inclusion can be obtained in a similar manner. □

Lemma 2. *The map* $\mathrm{Id} - P^1_\tau - Q^2_\tau$ *is a projection for each* $\tau \in \mathbb{R}$.

Proof. It follows from Lemma 1 that

$$P^1_\tau Q^2_\tau = Q^2_\tau P^1_\tau = 0$$

for $\tau \in \mathbb{R}$. Hence,

$$\begin{aligned}(\mathrm{Id} - P^1_\tau - Q^2_\tau)^2 &= \mathrm{Id} - 2P^1_\tau - 2Q^2_\tau + (P^1_\tau)^2 + (Q^2_\tau)^2 + P^1_\tau Q^2_\tau + Q^2_\tau P^1_\tau \\ &= \mathrm{Id} - P^1_\tau - Q^2_\tau\end{aligned}$$

and the conclusion of the lemma follows. □

Lemma 3. *For each* $\tau \in \mathbb{R}$*, we have*

$$\operatorname{Im}(\mathrm{Id} - P^1_\tau - Q^2_\tau) = \operatorname{Im} P^2_\tau \cap \operatorname{Im} Q^1_\tau.$$

Proof. Take $x \in \operatorname{Im} P^2_\tau \cap \operatorname{Im} Q^1_\tau$. We have $Q^2_\tau x = P^1_\tau x = 0$ and thus,

$$(\mathrm{Id} - P^1_\tau - Q^2_\tau)x = x.$$

This shows that $x \in \operatorname{Im}(\mathrm{Id} - P^1_\tau - Q^2_\tau)$. Now take $x \in \operatorname{Im}(\mathrm{Id} - P^1_\tau - Q^2_\tau)$. Then $P^1_\tau x = -Q^2_\tau x$. Applying P^1_τ, it follows from Lemma 1 that $P^1_\tau x = 0$ and thus $x \in \operatorname{Im} Q^1_\tau$. One can show in a similar manner that $x \in \operatorname{Im} P^2_\tau$ and so $x \in \operatorname{Im} P^2_\tau \cap \operatorname{Im} Q^1_\tau$. □

Now we complete the proof of the theorem. It follows from (7) and (16) that

$$\|T(t,\tau)P^1_\tau\| \le DD' e^{-(\lambda+\omega)(t-\tau)+\varepsilon|\tau|} \quad \text{for} \quad t \ge \tau. \tag{20}$$

Similarly, by (7) and (19) we have

$$\|T(t,\tau)Q^2_\tau\| \le DD' e^{-(\lambda-\omega')(\tau-t)+\varepsilon|\tau|} \quad \text{for} \quad t \le \tau. \tag{21}$$

Moreover, it follows from (7), (17), (18) and Lemma 3 that for each $x \in \operatorname{Im}(\mathrm{Id} - P^1_\tau - Q^2_\tau)$, we have

$$\|T(t,\tau)x\| \le DD' e^{-(\lambda+\omega')(t-\tau)+\varepsilon|\tau|}$$

for $t \ge \tau$ and

$$\|T(t,\tau)x\| \le DD' e^{-(\lambda-\omega)(\tau-t)+\varepsilon|\tau|}$$

for $t \le \tau$. In addition, by (16) and (19),

$$\|\mathrm{Id} - P^1_\tau - Q^2_\tau\| \le 3DD' e^{\varepsilon|\tau|}$$

for $\tau \in \mathbb{R}$. Hence,

$$\|T(t,\tau)(\mathrm{Id} - P^1_\tau - Q^2_\tau)\| \le 3(DD')^2 e^{-(\lambda+\omega')(t-\tau)+2\varepsilon|\tau|} \quad \text{for} \quad t \ge \tau \tag{22}$$

and

$$\|T(t,\tau)(\mathrm{Id} - P^1_\tau - Q^2_\tau)\| \le 3(DD')^2 e^{-(\lambda-\omega)(\tau-t)+2\varepsilon|\tau|} \quad \text{for} \quad t \le \tau. \tag{23}$$

It follows from (20), (21), (22) and (23) that the evolution family $T(t,\tau)$ admits a nonuniform exponential trichotomy. □

4. Strong nonuniform exponential trichotomies

In this section we obtain corresponding results to those in the former section for the stronger notion of a strong nonuniform exponential trichotomy.

We say that an evolution family $T(t,\tau)$ admits a *strong nonuniform exponential trichotomy* if it admits a nonuniform exponential trichotomy and if there exist constants $d' \ge d$ and $b' \ge b$ such that

$$\|T(t,\tau)P^1_\tau\| \le De^{d'(\tau-t)+\varepsilon|\tau|} \quad \text{for} \quad t \le \tau$$

and

$$\|T(t,\tau)P^2_\tau\| \le De^{b'(t-\tau)+\varepsilon|\tau|} \quad \text{for} \quad t \ge \tau.$$

Theorem 3. *Assume that the evolution family $T(t,\tau)$ admits a nonuniform exponential trichotomy with $\varepsilon < b + d$. Then there exist families of norms $\|\cdot\|_{1,t}$ and $\|\cdot\|_{2,t}$ for $t \in \mathbb{R}$ satisfying (3) and constants $D', \omega > 0$, $c_1, c_2, K_1, K_2 > 0$ and $\omega' < 0$ with $\varepsilon \le \omega - \omega'$ satisfying properties 1–3 in Theorem 1 as well as*

$$\frac{1}{K_1} e^{c_1(\tau-t)}\|x\|_{1,\tau} \le \|T(t,\tau)x\|_{1,t}, \quad \|T(t,\tau)x\|_{2,t} \le K_2 e^{c_2(t-\tau)}\|x\|_{2,\tau} \tag{24}$$

for $t \ge \tau$ and $x \in X$.

Proof. Take $\omega \in (c,d)$ and consider the evolution family $U(t,\tau)$ in (8). In addition to (9), (10) and (11), we have

$$\|U(t,\tau)P^1_\tau\| \le De^{(d'-\omega)(\tau-t)+\varepsilon|\tau|} \quad \text{for} \quad t \le \tau. \tag{25}$$

For $\tau \in \mathbb{R}$ and $x \in X$, let

$$\|x\|_{1,\tau} = \sup_{t\ge\tau}\left(\|U(t,\tau)P^1_\tau x\|e^{\lambda(t-\tau)}\right) + \sup_{t\le\tau}\left(\|U(t,\tau)(\mathrm{Id} - P^1_\tau)x\|e^{\lambda(\tau-t)}\right)$$
$$+ \sup_{t<\tau}\left(\|U(t,\tau)P^1_\tau x\|e^{-(d'-\omega)(\tau-t)}\right),$$

where

$$\lambda = \min\left\{d-\omega, b+\omega, \omega-c\right\} > 0.$$

It follows from (9), (10), (11) and (25) that the norms $\|\cdot\|_{1,t}$ satisfy (7) with $D' = 4D$.

Lemma 4. *The evolution family $U(t,\tau)$ admits an exponential dichotomy with respect to the family of norms $\|\cdot\|_{1,t}$. Moreover, there exists $c_1 > 0$ such that the first inequality in (24) holds for $t \in \mathbb{R}$ and $x \in X$.*

Proof. For $t \geq \tau$, since $\lambda < d' - \omega$ we have

$$\begin{aligned}
\|U(t,\tau)P^1_\tau x\|_{1,t} &= \sup_{s\geq t}\left(\|U(s,\tau)P^1_\tau x\|e^{\lambda(s-t)}\right) \\
&\quad + \sup_{s<t}\left(\|U(s,\tau)P^1_\tau x\|e^{-(d'-\omega)(t-s)}\right) \\
&\leq \sup_{s\geq t}\left(\|U(s,\tau)P^1_\tau x\|e^{\lambda(s-t)}\right) \\
&\quad + \sup_{\tau\leq s<t}\left(\|U(s,\tau)P^1_\tau x\|e^{-\lambda(t-s)}\right) \\
&\quad + \sup_{s<\tau}\left(\|U(s,\tau)P^1_\tau x\|e^{-(d'-\omega)(t-s)}\right) \\
&\leq 2e^{-\lambda(t-\tau)}\sup_{s\geq \tau}\left(\|U(s,\tau)P^1_\tau x\|e^{\lambda(s-\tau)}\right) \\
&\quad + e^{-(d'-\omega)(t-\tau)}\sup_{s<\tau}\left(\|U(s,\tau)P^1_\tau x\|e^{-(d'-\omega)(\tau-s)}\right) \\
&\leq 2e^{-\lambda(t-\tau)}\|x\|_{1,\tau}.
\end{aligned} \tag{26}$$

One can show in a similar manner that

$$\|U(t,\tau)P^1_\tau x\|_{1,t} \leq 2e^{(d'-\omega)(\tau-t)}\|x\|_{1,\tau} \quad \text{for} \quad t \leq \tau. \tag{27}$$

Moreover,

$$\begin{aligned}
\|U(t,\tau)(\mathrm{Id}-P^1_\tau)x\|_{1,t} &= \sup_{s\leq t}\left(\|U(s,t)U(t,\tau)(\mathrm{Id}-P^1_\tau)x\|e^{\lambda(t-s)}\right) \\
&= e^{-\lambda(\tau-t)}\sup_{k\leq m}\left(\|U(s,\tau)(\mathrm{Id}-P^1_\tau)x\|e^{\lambda(\tau-s)}\right) \\
&\leq e^{-\lambda(\tau-t)}\|x\|_{1,\tau}
\end{aligned} \tag{28}$$

for $t \leq \tau$. It follows from (26) and (28) that the evolution family $U(t,\tau)$ admits an exponential dichotomy with respect to the family of norms $\|\cdot\|_{1,t}$. By (27) and (28), we have

$$\begin{aligned}
e^{-\omega(\tau-t)}\|T(t,\tau)x\|_{1,t} &= \|U(t,\tau)x\|_{1,t} \\
&\leq \|U(t,\tau)P^1_\tau x\|_{1,t} + \|U(t,\tau)(\mathrm{Id}-P^1_\tau)x\|_{1,t} \\
&\leq 2e^{(d'-\omega)(\tau-t)}\|x\|_{1,\tau} + e^{-\lambda(\tau-t)}\|x\|_{1,\tau} \\
&\leq 3e^{(d'-\omega)(\tau-t)}\|x\|_{1,\tau}
\end{aligned}$$

for $x \in X$ and $t \leq \tau$. This shows that the first inequality in (24) holds with $c_1 = d'$ and $K_1 = 3$. □

By Lemma 4 and Proposition 3, the evolution family $U(t,\tau)$ has an admissibility property with respect to the family of norms $\|\cdot\|_{1,t}$.

Now take $\omega' \in (-b, -a)$ and consider the evolution family $U'(t, \tau)$ in (12). In addition to (13), (14) and (15), we have

$$\|U'(t,\tau)P_\tau^2\| \le De^{(b'+\omega')(t-\tau)+\varepsilon|\tau|} \quad \text{for} \quad t \ge \tau. \tag{29}$$

For $\tau \in \mathbb{R}$ and $x \in X$, let

$$\begin{aligned}\|x\|_{2,\tau} = &\sup_{t\ge\tau}\left(\|U'(t,\tau)(\mathrm{Id}-P_\tau^2)x\|e^{\lambda'(t-\tau)}\right) + \sup_{t\le\tau}\left(\|U'(t,\tau)P_\tau^2 x\|e^{\lambda'(\tau-t)}\right)\\ &+ \sup_{t>\tau}\left(\|U'(t,\tau)P_n^2 x\|e^{-(b'+\omega')(t-\tau)}\right),\end{aligned}$$

where

$$\lambda' = \min\left\{d-\omega', -a-\omega', b+\omega'\right\} > 0.$$

It follows from (13), (14) and (15) and (29) that the norms $\|\cdot\|_{2,t}$ satisfy (7) with $D' = 4D$.

Lemma 5. *The evolution family $U'(t,\tau)$ admits an exponential dichotomy with respect to the family of norms $\|\cdot\|_{2,t}$. Moreover, there exists $c_2 > 0$ such that the second inequality in (24) holds for $t \in \mathbb{R}$ and $x \in X$.*

Proof. Take $t \le \tau$. Since $\lambda' < b' + \omega'$, proceeding as in (26) we obtain

$$\begin{aligned}\|U'(t,\tau)P_\tau^2 x\|_{2,t} &= \sup_{s\le t}\left(\|U'(s,\tau)P_\tau^2 x\|e^{\lambda'(t-s)}\right)\\ &\quad+ \sup_{s>t}\left(\|U'(s,\tau)P_\tau^2 x\|e^{-(b'+\omega')(s-t)}\right)\\ &\le \sup_{s\le t}\left(\|U'(s,\tau)P_\tau^2 x\|e^{\lambda'(t-s)}\right)\\ &\quad+ \sup_{\tau\ge s>t}\left(\|U'(s,\tau)P_\tau^2 x\|e^{\lambda'(t-s)}\right)\\ &\quad+ \sup_{s>\tau}\left(\|U'(s,\tau)P_\tau^2 x\|e^{-(b'+\omega')(s-t)}\right)\\ &\le 2e^{-\lambda'(\tau-t)}\sup_{s\le\tau}\left(\|U'(s,\tau)P_\tau^2 x\|e^{\lambda'(\tau-s)}\right)\\ &\quad+ e^{-(b'+\omega')(\tau-t)}\sup_{s>\tau}\left(\|U'(s,\tau)P_\tau^2 x\|e^{-(b'+\omega')(s-\tau)}\right)\\ &\le 2e^{-\lambda'(\tau-t)}\|x\|_{2,\tau}.\end{aligned} \tag{30}$$

One can show in a similar manner that

$$\|U'(t,\tau)P_\tau^2 x\|_{2,t} \le 2e^{(b'+\omega')(t-\tau)}\|x\|_{2,\tau} \tag{31}$$

and

$$\|U'(t,\tau)(\mathrm{Id}-P_\tau^2)x\|_{2,t} \le e^{-\lambda'(t-\tau)}\|x\|_{2,\tau} \tag{32}$$

for $t \ge \tau$. It follows from (30) and (32) that the evolution family $U'(t,\tau)$ admits an exponential dichotomy with respect to the family of norms $\|\cdot\|_{2,t}$. Moreover, by (31) and (32), we have

$$\begin{aligned} e^{\omega'(t-\tau)}\|T(t,\tau)x\|_{2,t} &= \|U'(t,\tau)x\|_{1,t} \\ &\le \|U'(t,\tau)P^2_\tau x\|_{2,t} + \|U'(t,\tau)(\mathrm{Id}-P^2_\tau)x\|_{2,t} \\ &\le 2e^{(b'+\omega')(t-\tau)}\|x\|_{2,\tau} + e^{-\lambda(t-\tau)}\|x\|_{2,\tau} \\ &\le 3e^{(b'+\omega')(t-\tau)}\|x\|_{2,\tau} \end{aligned}$$

for $x \in X$ and $t \ge \tau$. This shows that the second inequality in (24) holds with $c_2 = b'$ and $K_2 = 3$. □

By Lemma 5 and Proposition 3, the evolution family $U'(t,\tau)$ has an admissibility property with respect to the family of norms $\|\cdot\|_{2,t}$.

Finally, since $\varepsilon < b + d$, one can choose ω and ω' so that $\varepsilon \le \omega - \omega'$. □

Now we establish the converse of Theorem 3.

Theorem 4. *Assume that there exist families of norms $\|\cdot\|_{1,t}$ and $\|\cdot\|_{2,t}$ for $t \in \mathbb{R}$ satisfying (3) and constants $D', \omega > 0$, $\varepsilon \ge 0$, $c_1, c_2, K_1, K_2 > 0$ and $\omega' < 0$ with $\varepsilon \le \omega - \omega'$ satisfying properties 1–3 in Theorem 1 and (24). Then the evolution family $T(t,\tau)$ admits a strong nonuniform exponential trichotomy.*

Proof. Using the same notation as in the proof of Theorem 2, it follows from (24) that

$$\begin{aligned} \|T(t,\tau)P^1_\tau x\| &\le \|T(t,\tau)P^1_\tau x\|_{1,t} \\ &\le K_1 e^{c_1(\tau-t)}\|x\|_{1,\tau} \\ &\le K_1 D' e^{c_1(\tau-t)+\varepsilon|\tau|}\|x\| \end{aligned}$$

for $t \le \tau$ and $x \in X$. Similarly,

$$\begin{aligned} \|T(t,\tau)Q^2_\tau x\| &\le \|T(t,\tau)Q^2_\tau x\|_{2,t} \\ &\le K_2 e^{c_2(t-\tau)}\|x\|_{1,\tau} \\ &= K_2 D' e^{c_2(t-\tau)+\varepsilon|\tau|}\|x\| \end{aligned}$$

for $t \ge \tau$ and $x \in X$. This shows that the nonuniform exponential trichotomy given by Theorem 2 is strong. □

5. Robustness

In this section we establish the robustness of the notions of a nonuniform exponential trichotomy and of a strong nonuniform exponential trichotomy under sufficiently small linear perturbations.

Theorem 5. *Let $T(t,\tau)$ be an evolution family and let $B\colon \mathbb{R} \to B(X)$ be a strongly continuous function such that:*

1. *$T(t,\tau)$ admits a nonuniform exponential trichotomy with $\varepsilon < b + d$;*
2. *there exists $\rho > 0$ such that*

$$\|B(t)\| \le \rho e^{-\varepsilon|t|}, \quad t \in \mathbb{R}. \tag{33}$$

If ρ is sufficiently small, then the evolution family $U(t,\tau)$ defined by

$$U(t,\tau) = T(t,\tau) + \int_\tau^t T(t,s)B(s)U(s,\tau)\,ds \quad \text{for} \quad t \geq \tau \tag{34}$$

admits a nonuniform exponential trichotomy.

Proof. We first recall a result established in [2].

Lemma 6. *Assume that the evolution family $T(t,\tau)$ admits an exponential dichotomy with respect to a family of norms $\|\cdot\|_t$ and that $B\colon \mathbb{R} \to B(X)$ is a strongly continuous function satisfying (33). If ρ is sufficiently small, then the evolution family $U(t,\tau)$ defined by (34) admits an exponential dichotomy respect to the same family of norms.*

Let $\omega' < 0 < \omega$ be the constants and let $\|\cdot\|_{i,t}$ be the norms given by Theorem 1. It follows from Lemma 6 that for any sufficiently small ρ the evolution families $e^{\omega(t-\tau)}U(t,\tau)$ and $e^{\omega'(t-\tau)}U(t,\tau)$ have admissibility properties, respectively, with respect to the families of norms $\|\cdot\|_{1,t}$ and $\|\cdot\|_{2,t}$. Hence, by Theorem 2, the evolution family $U(t,\tau)$ admits a nonuniform exponential trichotomy. □

Now we consider the case of strong nonuniform exponential trichotomies.

Theorem 6. *Let $T(t,\tau)$ be an evolution family and let $B\colon \mathbb{R} \to B(X)$ be a strongly continuous function such that:*

1. *$T(t,\tau)$ admits a strong nonuniform exponential trichotomy with $\varepsilon < b + d$;*
2. *there exists $\rho > 0$ such that (33) holds.*

If ρ is sufficiently small, then the evolution family $U(t,\tau)$ defined by (34) admits a strong nonuniform exponential trichotomy.

Proof. In view of Theorem 5 and the characterization of a strong nonuniform exponential trichotomy given by Theorems 3 and 4, it is sufficient to show that there exist constants $c_1', c_2' > 0$ and $K_1', K_2' > 0$ such that

$$\frac{1}{K_1'} e^{c_1'(\tau-t)}\|x\|_{1,\tau} \leq \|U(t,\tau)x\|_{1,t}, \quad \|U(t,\tau)x\|_{2,t} \leq K_2' e^{c_2'(t-\tau)}\|x\|_{2,\tau}$$

for $t \geq \tau$ and $x \in X$. These inequalities can be easily deduced from (24) and Gronwall's lemma (see [1] for a detailed argument). □

6. Characterization of nonuniformly hyperbolic sets

In this section we obtain corresponding results to those in the former sections for nonuniformly hyperbolic sets. These will be used in Section 7 to characterize the notion of a partially hyperbolic set.

Let M be a compact d-dimensional Riemannian manifold and let $\Phi = (\phi^t)_{t\in\mathbb{R}}$ be a smooth flow on M. Throughout this section we denote by $E^0(x)$ the 1-dimensional subspace of T_xM spanned by the direction of the flow at x, that is, by the vector $(d/dt)\phi^t(x)|_{t=0}$.

A Φ-invariant measurable set $\Lambda \subset M$ is said to be *nonuniformly hyperbolic* for Φ if there exist constants $0 < \lambda < 1 < \mu$ and splittings

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^0(x)$$

for $x \in \Lambda$ so that given $\varepsilon > 0$, there exist measurable functions $C, K: \Lambda \to \mathbb{R}^+$ such that for each $x \in \Lambda$, $v \in T_xM$ and $t \in \mathbb{R}$ we have:

1. $d_x\phi^t E^s(x) = E^s(\phi^t(x))$ and $d_x\phi^t E^u(x) = E^u(\phi^t(x))$;
2. for $t \geq 0$,

$$\|d_x\phi^t v\|_{\phi^t(x)} \leq C(x)\lambda^t e^{\varepsilon t}\|v\|_x, \quad v \in E^s(x) \tag{35}$$

   and

$$\|d_x\phi^{-t} v\|_{\phi^{-t}(x)} \leq C(x)\mu^{-t} e^{\varepsilon t}\|v\|_x, \quad v \in E^s(x); \tag{36}$$

3. $\angle(E^s(x), E^u(x)) \geq K(x)$;
4.

$$C(\phi^t(x)) \leq C(x)e^{\varepsilon|t|}, \quad K(\phi^t(x)) \geq K(x)e^{-\varepsilon|t|}. \tag{37}$$

Now let $\Lambda \subset M$ be a Φ-invariant set. We say that a collection E of subspaces $E(x) \subset T_xM$, $x \in \Lambda$ is a *k-dimensional invariant distribution* on Λ if:

1. the subspaces $E(x)$ depend measurably on x;
2. $d_x\phi^t E(x) = E(\phi^t(x))$ for $t \in \mathbb{R}$ and $x \in \Lambda$;
3. $\dim E(x) = k$ for $x \in \Lambda$.

Consider a k-dimensional invariant distribution E on Λ and a norm $\|\cdot\|'_x$ on $E(x)$ for each $x \in \Lambda$. Given $x \in \Lambda$, we denote by Y_x the set of all continuous functions $v: \mathbb{R} \to T_\Lambda M$ such that $v(t) \in E(\phi^t(x))$ for $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \|v(t)\|'_{\phi^t(x)} < +\infty.$$

One can easily verify that Y_x is a Banach space when equipped with the norm

$$\|v\| = \sup_{t\in\mathbb{R}} \|v(t)\|'_{\phi^t(x)}.$$

Moreover, let R_x be the linear map defined by $R_xu = v$ in the domain $\mathcal{D}(R_x)$ formed by all $u \in Y_x$ for which there exists $v \in Y_x$ such that

$$u(t) = d_{\phi^s(x)}\phi^{t-s}u(s) + \int_s^t d_{\phi^\tau(x)}\phi^{t-\tau}v(\tau)\,d\tau \quad \text{for} \quad t \geq s. \tag{38}$$

We note that R_x is well-defined. Indeed, assume on the contrary that there exist $u, v, \tilde{v} \in Y_x$ such that

$$u(t) = d_{\phi^s(x)}\phi^{t-s}u(s) + \int_s^t d_{\phi^\tau(x)}\phi^{t-\tau}v(\tau)\,d\tau$$

and

$$u(t) = d_{\phi^s(x)}\phi^{t-s}u(s) + \int_s^t d_{\phi^\tau(x)}\phi^{t-\tau}\tilde{v}(\tau)\,d\tau$$

for $t \geq s$. Then

$$0 = \int_s^t d_{\phi^\tau(x)}\phi^{t-\tau}(v(\tau) - \tilde{v}(\tau))\, d\tau = d_x\phi^t \int_s^t d_{\phi^\tau(x)}\phi^{-\tau}(v(\tau) - \tilde{v}(\tau))\, d\tau,$$

which implies that

$$\int_s^t d_{\phi^\tau}(x)\phi^{-\tau}(v(\tau) - \tilde{v}(\tau))\, d\tau = 0.$$

Therefore,

$$d_{\phi^t}(x)\phi^{-t}(v(t) - \tilde{v}(t)) = 0$$

and so $v(t) = \tilde{v}(t)$ for all t. This shows that R_x is well-defined.

The following two results give a characterization of the notion of a nonuniformly hyperbolic set in terms of an admissibility property.

Theorem 7. *Let $\Lambda \subset M$ be a nonuniformly hyperbolic set for Φ. Then there exist a $(d-1)$-dimensional invariant distribution E on Λ, a measurable function $G\colon \Lambda \to \mathbb{R}^+$, and constants $\varepsilon_0, D, A > 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exists a norm $\|\cdot\|' = \|\cdot\|^\varepsilon$ on $T_\Lambda M$ and for each $x \in \Lambda$, $v \in T_xM$ and $t \in \mathbb{R}$ we have:*

1.
$$T_xM = E(x) \oplus E^0(x); \tag{39}$$

2.
$$\frac{1}{3}\|v\|_x \leq \|v\|_x^\varepsilon \leq G(x)\|v\|_x, \quad v \in E(x), \tag{40}$$

and

$$G(\phi^t(x)) \leq e^{2\varepsilon|t|}G(x); \tag{41}$$

3. *$R_x\colon \mathcal{D}(R_x) \to Y_x$ is a invertible linear operator with*
$$\|R_x^{-1}\| \leq D; \tag{42}$$

4.
$$\|d_x\phi^t v\|_{\phi^t(x)}^\varepsilon \leq 2A^{|t|}(e^{\varepsilon|t|} + 1)\|v\|_x^\varepsilon, \quad v \in E(x). \tag{43}$$

Proof. Let $E(x) = E^s(x) \oplus E^u(x)$ for $x \in \Lambda$. Clearly, E is a $(d-1)$-dimensional invariant distribution and satisfies (39).

Since M is compact, there exists $A > 0$ such that $\|d_x\phi^t\| \leq A^{|t|}$ for $x \in M$ and $t \in \mathbb{R}$. Without loss of generality, one may assume that $1/A \leq \lambda$ and $\mu \leq A$ (since otherwise one can simply increase A). Take

$\varepsilon_0 > 0$ such that $\lambda e^{\varepsilon_0} < 1 < \mu e^{-\varepsilon_0}$. For each $\varepsilon \in (0, \varepsilon_0)$, we introduce a norm $\|\cdot\|^\varepsilon$ on $T_\Lambda M$. For $v \in E^s(x)$, let

$$\|v\|_x^\varepsilon = \sup_{t\ge 0}\left(\lambda^{-t}e^{-\varepsilon t}\|d_x\phi^t v\|_{\phi^t(x)}\right) + \sup_{t<0}\left(e^{\varepsilon t}A^t\|d_x\phi^t v\|_{\phi^t(x)}\right).$$

It follows from (35) that

$$\|v\|_x \le \|v\|_x^\varepsilon \le (C(x)+1)\|v\|_x \quad \text{for} \quad v \in E^s(x). \tag{44}$$

Moreover,

$$\begin{aligned}
\|d_x\phi^t v\|_{\phi^t(x)}^\varepsilon &= \sup_{s\ge 0}\left(\lambda^{-s}e^{-\varepsilon s}\|d_x\phi^{s+t}v\|_{\phi^{s+t}(x)}\right)\\
&\quad + \sup_{s<0}\left(e^{\varepsilon s}A^s\|d_x\phi^{s+t}v\|_{\phi^{s+t}(x)}\right)\\
&\le \sup_{s\ge 0}\left(\lambda^{-s}e^{-\varepsilon s}\|d_x\phi^{s+t}v\|_{\phi^{s+t}(x)}\right)\\
&\quad + \sup_{s<-t}\left(e^{\varepsilon s}A^s\|d_x\phi^{s+t}v\|_{\phi^{s+t}(x)}\right)\\
&\quad + \sup_{-t\le s\le 0}\left(e^{-\varepsilon s}\lambda^{-s}\|d_x\phi^{s+t}v\|_{\phi^{s+t}(x)}\right)\\
&\le 2\lambda^t e^{\varepsilon t}\|v\|_x^\varepsilon
\end{aligned} \tag{45}$$

for $v \in E^s(x)$ and $t \ge 0$ and, analogously,

$$\|d_x\phi^{-t}v\|_{\phi^{-t}(x)}^\varepsilon \le A^t(2e^{\varepsilon t}+1)\|v\|_x^\varepsilon \tag{46}$$

for $v \in E^s(x)$ and $t \ge 0$. Similarly, for $v \in E^u(x)$, let

$$\|v\|_x^\varepsilon = \sup_{t\le 0}\left(\mu^{-t}e^{\varepsilon t}\|d_x\phi^t v\|_{\phi^t(x)}\right) + \sup_{t>0}\left(A^{-t}e^{-\varepsilon t}\|d_x\phi^t v\|_{\phi^t(x)}\right).$$

It follows from (36) that

$$\|v\|_x \le \|v\|_x^\varepsilon \le (C(x)+1)\|v\|_x \quad \text{for} \quad v \in E^u(x). \tag{47}$$

Moreover, proceeding as in (45), we obtain

$$\|d_x\phi^{-t}v\|_{\phi^{-t}(x)}^\varepsilon \le 2\mu^{-t}e^{\varepsilon t}\|v\|_x^\varepsilon \tag{48}$$

for $v \in E^u(x)$ and $t \ge 0$, and

$$\|d_x\phi^t v\|_{\phi^t(x)}^\varepsilon \le A^t(2e^{\varepsilon t}+1)\|v\|_x^\varepsilon \tag{49}$$

for $v \in E^u(x)$ and $t \ge 0$. For an arbitrary $v \in T_xM$, let

$$\|v\|_x^\varepsilon = \max\left\{\|v^s\|_x^\varepsilon, \|v^u\|_x^\varepsilon, \|v^0\|_x\right\},$$

where $v = v^s + v^u + v^0$ with $v^s \in E^s(x)$, $v^u \in E^u(x)$ and $v^0 \in E^0(x)$. It follows from (44) and (47) that (40) holds taking

$$G(x) = c\max\left\{1, (C(x)+1)/K(x)\right\}$$

for some constant $c > 0$. Moreover, it follows from (37) that (41) holds. Finally, it follows from (45), (46), (48) and (49) that (43) holds.

Now let

$$P(x) \colon E(x) \to E^s(x) \quad \text{and} \quad Q(x) \colon E(x) \to E^u(x)$$

be the projections associated to the decomposition $E(x) = E^s(x) \oplus E^u(x)$.

Lemma 7. *There exists a constant $Z > 0$ (independent of ε and x) such that*

$$\|P(x)v\|_x^\varepsilon \le Z\|v\|_x^\varepsilon \quad \text{and} \quad \|Q(x)v\|_x^\varepsilon \le Z\|v\|_x^\varepsilon \tag{50}$$

for $x \in \Lambda$ and $v \in T_xM$.

Proof. For each $x \in \Lambda$, let

$$\gamma_x^\varepsilon = \inf\left\{\|v^s + v^u\|_x^\varepsilon : \|v^s\|_x^\varepsilon = \|v^u\|_x^\varepsilon = 1, v^s \in E^s(x), v^u \in E^u(x)\right\}.$$

Take a vector $v \in E(x)$ such that $Pv \ne 0$ and $Qv \ne 0$, where $P = P(x)$ and $Q = Q(x)$. Then

$$\begin{aligned}\gamma_x^\varepsilon \le \left\|\frac{Pv}{\|Pv\|_x^\varepsilon} + \frac{Qv}{\|Qv\|_x^\varepsilon}\right\|_x^\varepsilon &= \frac{1}{\|Pv\|_x^\varepsilon}\left\|Pv + \frac{\|Pv\|_x^\varepsilon}{\|Qv\|_x^\varepsilon}Qv\right\|_x^\varepsilon \\ &= \frac{1}{\|Pv\|_x^\varepsilon}\left\|v + \frac{\|Pv\|_x^\varepsilon - \|Qv\|_x^\varepsilon}{\|Qv\|_x^\varepsilon}Qv\right\|_x^\varepsilon \\ &\le \frac{2\|v\|_x^\varepsilon}{\|Pv\|_x^\varepsilon}\end{aligned}$$

and so,

$$\|Pv\|_x^\varepsilon \le \frac{2}{\gamma_x^\varepsilon}\|v\|_x^\varepsilon$$

for $v \in E(x)$. In order to estimate γ_x^ε, take $v^s \in E^s(x)$ and $v^u \in E^u(x)$ such that $\|v^s\|_x^\varepsilon = \|v^u\|_x^\varepsilon = 1$. It follows from (45), (48) and (43) (recall that $\varepsilon < \varepsilon_0$) that

$$\begin{aligned}\|v^s + v^u\|_x^\varepsilon &\ge \frac{1}{2A(e^{\varepsilon_0}+1)}\|d_x\phi^1(v^s + v^u)\|_{\phi^1(x)}^\varepsilon \\ &\ge \frac{1}{2A(e^{\varepsilon_0}+1)}\left(\|d_x\phi^1 v^u\|_{\phi^1(x)}^\varepsilon - \|d_x\phi^1 v^s\|_{\phi^1(x)}^\varepsilon\right) \\ &\ge \frac{1}{2A(e^{\varepsilon_0}+1)}(\mu e^{-\varepsilon_0} - \lambda e^{\varepsilon_0})\end{aligned}$$

and so,

$$\gamma_x^\varepsilon \ge \frac{1}{2A(e^{\varepsilon_0}+1)}(\mu e^{-\varepsilon_0} - \lambda e^{\varepsilon_0}).$$

Therefore, (50) holds taking

$$Z = \frac{4A(e^{\varepsilon_0}+1)}{\mu e^{-\varepsilon_0} - \lambda e^{\varepsilon_0}}.$$

This completes the proof of the lemma. □

Now take $x \in \Lambda$. We first show that the map R_x is onto. Take $v \in Y_x$. We define

$$u(t) = \int_{-\infty}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} v^s(\tau)\, d\tau - \int_{t}^{\infty} d_{\phi^\tau(x)}\phi^{t-\tau} v^u(\tau)\, d\tau$$

for $t \in \mathbb{R}$, where

$$v^s(\tau) = P(\phi^\tau(x))v(\tau) \quad \text{and} \quad v^u(\tau) = Q(\phi^\tau(x))v(\tau).$$

It follows from (45), (48) and (50) that

$$\begin{aligned}
&\|u(t)\|^\varepsilon_{\phi^t(x)} \\
&\le 2Z\bigg(\int_{-\infty}^{t} (\lambda e^{\varepsilon_0})^{t-\tau}\|v(\tau)\|^\varepsilon_{\phi^\tau(x)}\, d\tau + \int_{t}^{\infty} (\mu e^{-\varepsilon_0})^{t-\tau}\|v(\tau)\|^\varepsilon_{\phi^\tau(x)}\, d\tau\bigg) \\
&= 2Z\bigg(\frac{1}{-\log(\lambda e^{\varepsilon_0})} + \frac{1}{\log(\mu e^{-\varepsilon_0})}\bigg)\|v\|
\end{aligned} \tag{51}$$

for $t \in \mathbb{R}$ and so $u \in Y_x$. Moreover, it is easy to verify that $R_x u = v$. Indeed,

$$\begin{aligned}
&u(t) - d_{\phi^s(x)}\phi^{t-s}u(s) \\
&= \int_{-\infty}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} v^s(\tau)\, d\tau - d_{\phi^s(x)}\phi^{t-s}\int_{-\infty}^{s} d_{\phi^\tau(x)}\phi^{s-\tau} v^s(\tau)\, d\tau \\
&\quad - \int_{t}^{\infty} d_{\phi^\tau(x)}\phi^{t-\tau} v^u(\tau)\, d\tau + d_{\phi^s(x)}\phi^{t-s}\int_{s}^{\infty} d_{\phi^\tau(x)}\phi^{s-\tau} v^u(\tau)\, d\tau \\
&= \int_{s}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} v^s(\tau)\, d\tau + \int_{s}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} v^u(\tau)\, d\tau \\
&= \int_{s}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} v(\tau)\, d\tau
\end{aligned}$$

for $t \ge s$.

Now we show that R_x is one-to-one. Assume that $R_x v = 0$ for some $v \in Y_x$. Then

$$v^s(t) = d_x\phi^{t-\tau}(\phi^\tau(x))v^s(\tau) \quad \text{and} \quad v^u(t) = d_x\phi^{t-\tau}(\phi^\tau(x))v^u(\tau)$$

for $t \ge \tau$. For each $t \in \mathbb{R}$, it follows from (45) that

$$\|v^s(t)\|^\varepsilon_{\phi^t(x)} \le 2(\lambda e^{\varepsilon})^{t-\tau}\|v(\tau)\|^\varepsilon_{\phi^\tau(x)} \le 2(\lambda e^{\varepsilon_0})^{t-\tau}\|v\|$$

for $\tau \le t$. Letting $\tau \to -\infty$, we obtain $v^s(t) = 0$. Similarly, $v^u(t) = 0$ for $t \in \mathbb{R}$ and so $v = 0$. We conclude that the map R_x is one-to-one for each $x \in \Lambda$. Moreover, it follows from (51) that (42) holds with

$$D = 2Z\bigg(\frac{1}{-\log(\lambda e^{\varepsilon_0})} + \frac{1}{\log(\mu e^{-\varepsilon_0})}\bigg).$$

This completes the proof of the theorem. □

Now we establish the converse of Theorem 7.

Theorem 8. *Let $\Lambda \subset M$ be a Φ-invariant measurable set. Assume that there exist a $(d-1)$-dimensional invariant distribution E on Λ and constants $\varepsilon_0, D, A > 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist a norm $\|\cdot\|^\varepsilon$ on $T_\Lambda M$ and a measurable function $G: \Lambda \to \mathbb{R}^+$ satisfying properties 1–4 in Theorem 7. Then Λ is a nonuniformly hyperbolic set for Φ.*

Proof. Take $x \in \Lambda$ and $\varepsilon \in (0, \varepsilon_0)$. We define

$$E^s(x,\varepsilon) = \left\{ v \in E(x) : \sup_{t \geq 0} \left(\|d_x\phi^t v\|^\varepsilon_{\phi^t(x)} \right) < +\infty \right\}$$

and

$$E^u(x,\varepsilon) = \left\{ v \in E(x) : \sup_{t \leq 0} \left(\|d_x\phi^t v\|^\varepsilon_{\phi^t(x)} \right) < +\infty \right\}.$$

Lemma 8. *For each $x \in \Lambda$, we have*

$$E(x) = E^s(x,\varepsilon) \oplus E^u(x,\varepsilon). \tag{52}$$

Proof. Let $\psi: \mathbb{R} \to \mathbb{R}$ be a continuous function supported on $[0,1]$ such that $\int_0^1 \psi(s)\, ds = 1$. Given a vector $v \in E(x)$, we define a function $g: \mathbb{R} \to X$ by

$$g(t) = \psi(t) d_x\phi^t v.$$

Clearly, $g \in Y_x$. Since R_x is invertible, there exists $w \in Y_x$ such that $R_x w = g$. Moreover, it follows from (38) that

$$w(t) = d_x\phi^t(w(0) + v)$$

for $t \geq 1$ and thus $w(0) + v \in E^s(x,\varepsilon)$. Furthermore, again by (38), we have $w(t) = d_x\phi^t w(0)$ for $t \leq 0$ and thus $w(0) \in E^u(x,\varepsilon)$. This shows that $v \in E^s(x,\varepsilon) + E^u(x,\varepsilon)$.

Now take $v \in E^s(x,\varepsilon) \cap E^u(x,\varepsilon)$ and let $w(t) = d_x\phi^t v$. Clearly, $w \in Y_x$ and one can easily verify that $R_x w = 0$. Since R_x is invertible, we obtain $w = 0$ and so $v = 0$. □

It follows from (39) and (52) that

$$T_x M = E^s(x,\varepsilon) \oplus E^u(x,\varepsilon) \oplus E^0(x) \quad \text{for} \quad x \in \Lambda. \tag{53}$$

Now let

$$P(x): E(x) \to E^s(x,\varepsilon) \quad \text{and} \quad Q(x): E(x) \to E^u(x,\varepsilon)$$

be the projections associated to the decomposition (52).

Lemma 9. *There exists $M > 0$ (independent of x and ε) such that*

$$\|P(x)v\|^\varepsilon_x \leq M\|v\|^\varepsilon_x \tag{54}$$

for $x \in \Lambda$ and $v \in E(x)$.

Proof. Using the same notation as in the proof of Lemma 8, we obtain

$$\begin{aligned}\|P(x)v\|_x^\varepsilon &= \|w(0)+v\|_x^\varepsilon \\ &\le \|w(0)\|_x^\varepsilon + \|v\|_x^\varepsilon \le \|w\| + \|v\|_x^\varepsilon \\ &= \|R_x^{-1}g\| + \|v\|_x^\varepsilon \le D\|g\| + \|v\|_x^\varepsilon,\end{aligned}$$

using (42) in the last inequality. On the other hand, it follows from (43) that $\|g\| \le CA(e^{\varepsilon_0}+1)\|v\|_x^\varepsilon$, where

$$C = \sup\big\{|\phi(t)| : t \in [\tau, \tau+1]\big\}.$$

This shows that (54) holds taking $M = CA(e^{\varepsilon_0}+1)+1$. □

Lemma 10. *There exist constants $\lambda, C > 0$ (independent of x and ε) with $\lambda < 1$ such that*

$$\|d_x\phi^t v\|_{\phi^t(x)}^\varepsilon \le C\lambda^t \|v\|_x^\varepsilon, \tag{55}$$

for $v \in E^s(x,\varepsilon)$ and $t \ge 0$.

Proof. Take $v \in E^s(x,\varepsilon)$ and let $u(t) = d_x\phi^t v$. Moreover, let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[0,+\infty)$ such that $0 \le \psi \le 1$, $\psi = 1$ on $[1,+\infty)$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Clearly, $\psi u \in Y_x$ and one can easily verify that $R_x(\psi u) = \psi' u$. Moreover,

$$\begin{aligned}\sup\big\{\|u(t)\|_{\phi^t(x)}^\varepsilon : t \in [1,+\infty)\big\} &= \sup\big\{\|\psi(t)u(t)\|_{\phi^t(x)}^\varepsilon : t \in [1,+\infty)\big\} \\ &\le \|\psi u\| = \|R_x^{-1}(\psi' u)\| \\ &\le \|R_x^{-1}\| \cdot \|\psi' u\| \\ &= \|R_x^{-1}\| \sup\big\{\|(\psi' u)(t)\|_{\phi^t(x)}^\varepsilon : t \in [0,1]\big\} \\ &\le 2\|R_x^{-1}\| \sup\big\{\|u(t)\|_{\phi^t(x)}^\varepsilon : t \in [0,1]\big\} \\ &= 2\|R_x^{-1}\| \sup\big\{\|d_x\phi^t v\|_{\phi^t(x)}^\varepsilon : t \in [0,1]\big\} \\ &\le 2K\|R_x^{-1}\| \cdot \|v\|_x^\varepsilon,\end{aligned}$$

where $K = 2A(e^{\varepsilon_0}+1)$, using (43) in the last inequality. Hence, again using (43), we obtain

$$\|u(t)\|_{\phi^t(x)}^\varepsilon \le C\|v\|_x^\varepsilon \quad \text{for} \quad t \ge 0, \tag{56}$$

where $C = 2K\max\{1, \|R_x^{-1}\|\}$.

Now we show that there exists $N \in \mathbb{N}$ such that for every $x \in \Lambda$ and $v \in E^s(x,\varepsilon)$,

$$\|u(t)\|_{\phi^t(x)}^\varepsilon \le \frac{1}{2}\|v\|_x^\varepsilon \quad \text{for} \quad t \ge N. \tag{57}$$

Take t_0 such that $t_0 > 0$ and $\|u(t_0)\|_{\phi^{t_0}(x)}^\varepsilon > \|v\|_x^\varepsilon/2$. It follows from (56) that

$$\frac{1}{2C}\|v\|_x^\varepsilon < \|u(s)\|_{\phi^s(x)}^\varepsilon \le C\|v\|_x^\varepsilon, \quad 0 \le s \le t_0. \tag{58}$$

Now take $\delta > 0$ and let $\psi: \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[0, t_0]$ such that $0 \le \psi \le 1$ and $\psi = 1$ on $[\delta, t_0 - \delta]$. Moreover, let

$$y(t) = \psi(t)u(t) \quad \text{and} \quad w(t) = u(t) \int_{-\infty}^{t} \psi(s)\, ds \tag{59}$$

for $t \in \mathbb{R}$. Clearly, y and w belong to Y_x and one can easily verify that $R_x w = y$. Indeed,

$$\begin{aligned} w(t) - d_{\phi^s(x)}\phi^{t-s} w(s) &= u(t) \int_{-\infty}^{t} \psi(\tau)\, d\tau - d_{\phi^s(x)}\phi^{t-s} u(s) \int_{-\infty}^{s} \psi(\tau)\, d\tau \\ &= u(t) \int_{s}^{t} \psi(\tau)\, d\tau \\ &= \int_{s}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} \psi(\tau) u(\tau)\, d\tau = \int_{s}^{t} d_{\phi^\tau(x)}\phi^{t-\tau} y(\tau)\, d\tau \end{aligned}$$

for $t \ge s$. Therefore,

$$\|R_x^{-1}\| \sup \left\{ \|u(t)\|^\varepsilon_{\phi^t(x)} : t \in [0, t_0] \right\} \ge \|R_x^{-1}\| \cdot \|y\| \ge \|w\|.$$

Hence, it follows from (58) that

$$\begin{aligned} C\|R_x^{-1}\| \cdot \|v\|^\varepsilon_x &\ge \|w(t_0)\|^\varepsilon_{\phi^{t_0}(x)} \\ &\ge \|u(t_0)\|^\varepsilon_{\phi^{t_0}(x)} \int_{\delta}^{t_0-\delta} \psi(s)\, ds \\ &\ge \frac{1}{2C}(t_0 - 2\delta)\|v\|^\varepsilon_x. \end{aligned}$$

Letting $\delta \to 0$ we obtain

$$t_0 \le 2C^2 \|R_x^{-1}\|$$

and so property (57) holds taking $N > 2C^2\|R_x^{-1}\|$.

Now take $t \ge 0$ and write $t = kN + r$, with $k \in \mathbb{N}$ and $0 \le r < N$. By (56) and (57), we obtain

$$\begin{aligned} \|d_x\phi^t v\|^\varepsilon_{\phi^t(x)} &= \|d_x\phi^{kN+r}(x) v\|^\varepsilon_{\phi^{kN+r}(x)} \\ &\le \frac{1}{2^k} \|d_x\phi^r v\|^\varepsilon_{\phi^r(x)} \\ &\le \frac{C}{2^k} \|v\|^\varepsilon_x \\ &\le 2Ce^{-t\log 2/N} \|v\|^\varepsilon_x, \end{aligned}$$

for $v \in E^s(x, \varepsilon)$ and so property (55) holds taking $\lambda = e^{-\log 2/N}$. □

Lemma 11. *There exist constants $\lambda, C > 0$ (independent of x and ε) with $\mu > 1$ such that*

$$\|d_x\phi^{-t}v\|^{\varepsilon}_{\phi^{-t}(x)} \le C\mu^{-t}\|v\|^{\varepsilon}_x, \tag{60}$$

for $v \in E^u(x,\varepsilon)$ and $t \ge 0$.

Proof. Take $v \in E^u(x,\varepsilon)$ and let $u(t) = d_x\phi^t v$. Moreover, let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $(-\infty, 0]$ such that $0 \le \psi \le 1$, $\psi = 1$ on $(-\infty, -1]$ and $\sup_{t\in\mathbb{R}} |\psi'(t)| \le 2$. Clearly, $\psi u \in Y_x$ and one can easily verify that $R_x(\psi u) = \psi' u$. Moreover,

$$\begin{aligned}
\sup\left\{\|u(t)\|^{\varepsilon}_{\phi^t(x)} : t \in (-\infty, -1]\right\} &= \sup\left\{\|\psi(t)u(t)\|^{\varepsilon}_{\phi^t(x)} : t \in (-\infty, -1]\right\} \\
&\le \|\psi u\| = \|R_x^{-1}(\psi' u)\| \\
&\le \|R_x^{-1}\| \cdot \|\psi' u\| \\
&= \|R_x^{-1}\| \sup\left\{\|(\psi' u)(t)\|^{\varepsilon}_{\phi^t(x)} : t \in [-1, 0]\right\} \\
&\le 2\|R_x^{-1}\| \sup\left\{\|u(t)\|^{\varepsilon}_{\phi^t(x)} : t \in [-1, 0]\right\} \\
&= 2\|R_x^{-1}\| \sup\left\{\|d_x\phi^t v\|^{\varepsilon}_{\phi^t(x)} : t \in [-1, 0]\right\} \\
&\le 2K\|R_x^{-1}\| \cdot \|v\|^{\varepsilon}_x,
\end{aligned}$$

where $K = 2A(e^{\varepsilon_0} + 1)$, using (43) in the last inequality. Hence, again using (43), we obtain

$$\|u(t)\|^{\varepsilon}_{\phi^t(x)} \le C\|v\|^{\varepsilon}_x \quad \text{for} \quad t \le 0, \tag{61}$$

where $C = 2K \max\{1, \|R_x^{-1}\|\}$.

Now we show that there exists $N \in \mathbb{N}$ such that for every $x \in \Lambda$ and $v \in E^u(x,\varepsilon)$,

$$\|u(t)\|^{\varepsilon}_{\phi^t(x)} \le \frac{1}{2}\|v\|^{\varepsilon}_x \quad \text{for} \quad t \le -N. \tag{62}$$

Take t_0 such that $t_0 < 0$ and $\|u(t_0)\|^{\varepsilon}_{\phi^{t_0}(x)} > \|v\|^{\varepsilon}_x/2$. It follows from (61) that

$$\frac{1}{2C}\|v\|^{\varepsilon}_x < \|u(s)\|^{\varepsilon}_{\phi^s(x)} \le C\|v\|^{\varepsilon}_x, \quad t_0 \le s \le 0. \tag{63}$$

Now take $\delta > 0$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported on $[t_0, 0]$ such that $0 \le \psi \le 1$ and $\psi = 1$ on $[t_0 + \delta, -\delta]$. Moreover, let $y(t)$ and $w(t)$ be as in (59). Clearly, y and w belong to Y_x and one can easily verify that $R_x w = y$. Therefore,

$$\|R_x^{-1}\| \sup\left\{\|u(t)\|^{\varepsilon}_{\phi^t(x)} : t \in [t_0, 0]\right\} \ge \|R_x^{-1}\| \cdot \|y\| \ge \|w\|.$$

Hence, it follows from (63) that

$$\begin{aligned}
C\|R_x^{-1}\| \cdot \|v\|^{\varepsilon}_x &\ge \|w(t_0)\|^{\varepsilon}_{\phi^{t_0}(x)} \\
&\ge \|u(t_0)\|^{\varepsilon}_{\phi^{t_0}(x)} \int_{t_0+\delta}^{-\delta} \psi(s)\, ds \\
&\ge \frac{1}{2C}(-t_0 - 2\delta)\|v\|^{\varepsilon}_x.
\end{aligned}$$

Letting $\delta \to 0$ we obtain

$$-t_0 \le 2C^2 \|R_x^{-1}\|$$

and so property (62) holds taking $N > 2C^2\|R_x^{-1}\|$.

Now take $t \le 0$ and write $-t = kN + r$, with $k \in \mathbb{N}$ and $0 \le r < N$. By (61) and (62), we obtain

$$\begin{aligned}\|d_x\phi^t v\|^{\varepsilon}_{\phi^t(x)} &= \|d_x\phi^{-kN-r}(x)v\|^{\varepsilon}_{\phi^{-kN-r}(x)} \\ &\le \frac{1}{2^k}\|d_x\phi^{-r}v\|^{\varepsilon}_{\phi^{-r}(x)} \\ &\le \frac{C}{2^k}\|v\|^{\varepsilon}_x \\ &\le 2Ce^{t\log 2/N}\|v\|^{\varepsilon}_x,\end{aligned}$$

for $v \in E^u(x, \varepsilon)$ and so property (60) holds taking $\mu = e^{\log 2/N}$. □

Now we complete the proof of the theorem. It follows from (40) and (54) that

$$\|P(x)v\|_x \le \frac{1}{K(x)}\|v\|_x \quad \text{and} \quad \|Q(x)v\| \le \frac{1}{K(x)}\|v\|_x, \tag{64}$$

where $K(x) = 1/((2M + 1)G(x))$. Moreover, by (40), (55) and (60), we have

$$\|d_x\phi^t v\|_{\phi^t(x)} \le 3CG(x)\lambda^t\|v\|_x \tag{65}$$

for $v \in E^s(x, \varepsilon)$ and $t \ge 0$, and

$$\|d_x\phi^{-t} v\|_{\phi^{-t}(x)} \le 3CG(x)\mu^{-t}\|v\|_x \tag{66}$$

for $v \in E^u(x, \varepsilon)$ and $t \ge 0$. It follows from (53), (64), (65) and (66) that Λ is a nonuniformly hyperbolic set for Φ. □

7. Characterization of partially hyperbolic sets

Let $\Phi = (\phi^t)_{t\in\mathbb{R}}$ be a flow on a compact d-dimensional Riemannian manifold M. A Φ-invariant measurable set $\Lambda \subset M$ is said to be *nonuniformly partially hyperbolic* if there exist constants $0 \le a < b$, $0 \le c < d$ and splittings

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^c(x) \oplus E^0(x)$$

for $x \in \Lambda$, so that given $\varepsilon > 0$, there exist measurable functions $C, K\colon \Lambda \to \mathbb{R}^+$ such that for each $x \in \Lambda$, $v \in T_xM$ and $t \in \mathbb{R}$ we have:

1. $d_x\phi^t E^\alpha(x) = E^\alpha(\phi^t(x))$ for $\alpha \in \{s, u, c\}$;
2. for $t \ge 0$,

$$\|d_x\phi^t v\|_{\phi^t(x)} \le C(x)e^{-dt}e^{\varepsilon t}\|v\|_x, \quad v \in E^s(x) \tag{67}$$

and

$$\|d_x\phi^{-t} v\|_{\phi^{-t}(x)} \le C(x)e^{-bt}e^{\varepsilon t}\|v\|_x, \quad v \in E^u(x); \tag{68}$$

3. for $t \ge 0$ and $v \in E^c(x)$,

$$\|d_x\phi^t v\|_{\phi^t(x)} \le C(x)e^{at}e^{\varepsilon t}\|v\|_x$$

and

$$\|d_x\phi^{-t} v\|_{\phi^{-t}(x)} \le C(x)e^{ct}e^{\varepsilon t}\|v\|_x; \tag{69}$$

4. $\angle(E^\alpha(x), E^\beta(x)) \ge K(x)$ for $\alpha, \beta \in \{s, u, c\}$ with $\alpha \ne \beta$;
5.

$$C(\phi^t(x)) \le C(x)e^{\varepsilon|t|}, \quad K(\phi^t(x)) \ge K(x)e^{-\varepsilon|t|}. \tag{70}$$

The following two results give a characterization of the notion of a partially hyperbolic set in terms of an admissibility property.

Theorem 9. *Let $\Lambda \subset M$ be a nonuniformly partially hyperbolic set for Φ. Then there exist a $(d-1)$-dimensional invariant distribution E on Λ, a measurable function $G: \Lambda \to \mathbb{R}^+$, and constants $\varepsilon_0, \omega, D, A > 0$ and $\omega' < 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist norms $\|\cdot\|' = \|\cdot\|^{\varepsilon,1}$ and $\|\cdot\|'' = \|\cdot\|^{\varepsilon,2}$ on $T_\Lambda M$ and for each $x \in \Lambda$, $v \in T_x M$, $t \in \mathbb{R}$ and $i = 1, 2$ we have:*

1. $T_x M = E(x) \oplus E^0(x)$;
2.

$$\frac{1}{3}\|v\|_x \le \|v\|_x^{\varepsilon,i} \le G(x)\|v\|_x, \quad v \in E(x),$$

and

$$G(\phi^t(x)) \le e^{2\varepsilon|t|}G(x);$$

3. $R_x^1: \mathcal{D}(R_x^1) \to Y_x^1$ *defined with respect to the norm* $\|\cdot\|'$ *and the cocycle* $\mathcal{B}(x, t) = e^{\omega t}d_x\phi^t$ *is invertible and* $\|(R_x^1)^{-1}\| \le D$;
4. $R_x^2: \mathcal{D}(R_x^2) \to Y_x^2$ *defined with respect to the norm* $\|\cdot\|''$ *and the cocycle* $\mathcal{B}'(x, t) = e^{\omega' t}d_x\phi^t$ *is invertible and* $\|(R_x^2)^{-1}\| \le D$;
5. $\|d_x\phi^t v\|_{\phi^t(x)}^\varepsilon \le 3A^{|t|}(e^{\varepsilon|t|} + 1)\|v\|_x^\varepsilon$, $v \in E(x)$.

Proof. The proof is similar to the proof of Theorem 1. Let

$$E(x) = E^s(x) \oplus E^u(x) \oplus E^c(x)$$

for $x \in \Lambda$. Take $\omega \in (c, d)$. It follows from (67), (68) and (69) that for each $x \in \Lambda$, $v \in T_x M$ and $t \ge 0$, we have

$$\|\mathcal{B}(x, t)v\|_{\phi^t(x)} \le C(x)e^{(\omega-d)t}e^{\varepsilon t}\|v\|_x, \quad v \in E^s(x),$$
$$\|\mathcal{B}(x, -t)v\|_{\phi^{-t}(x)} \le C(x)e^{-(b+\omega)t}e^{\varepsilon t}\|v\|_x, \quad v \in E^u(x)$$

and

$$\|\mathcal{B}(x, -t)v\|_{\phi^{-t}(x)} \le C(x)e^{-(\omega-c)t}e^{\varepsilon t}\|v\|_x, \quad v \in E^c(x).$$

This implies that Λ is a nonuniformly hyperbolic set for Φ with respect to the cocycle $\mathcal{B}$. It follows from Theorem 7 that there exists $\varepsilon_0 > 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist a norm $\|\cdot\|' = \|\cdot\|^{\varepsilon,1}$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ satisfying properties 1 and 2 in the theorem.

Similarly, Λ is a nonuniformly hyperbolic set with respect to the cocycle $\mathcal{B}'$, where $\omega' \in (-b, -a)$. Using Theorem 7, we obtain norms $\|\cdot\|^{\varepsilon,2}$ satisfying properties 1 and 3 in the theorem. $\square$

Now we establish the converse of Theorem 9.

Theorem 10. *Let $\Lambda \subset M$ be a Φ-invariant measurable set. Assume that there exist a $(d-1)$-dimensional invariant distribution E on Λ and constants $\varepsilon_0, \omega, D, A > 0$ and $\omega' < 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist norms $\|\cdot\|^{\varepsilon,1}$ and $\|\cdot\|^{\varepsilon,2}$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ satisfying properties 1–5 in Theorem 9. Then Λ is a nonuniformly partially hyperbolic set for Φ.*

Proof. It follows from Theorem 8 that Λ is a nonuniformly hyperbolic set for Φ with respect to the cocycles $\mathcal{B}$ and $\mathcal{B}'$ in Theorem 9. Hence, there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a measurable function $C\colon \Lambda \to (0, +\infty)$ satisfying (70) such that for each $x \in \Lambda$, $v \in T_x M$ and $t \geq 0$:

$$\|\mathcal{B}(x,t)v\|_{\phi^t(x)} \leq C(x)e^{-\lambda t}e^{\varepsilon t}\|v\|_x, \quad v \in E^s_{\mathcal{B}}(x) \tag{71}$$

$$\|\mathcal{B}(x,-t)v\|_{\phi^{-t}(x)} \leq C(x)e^{-\lambda t}e^{\varepsilon t}\|v\|_x, \quad v \in E^u_{\mathcal{B}}(x) \tag{72}$$

and

$$\|\mathcal{B}'(x,t)v\|_{\phi^t(x)} \leq C(x)e^{-\lambda t}e^{\varepsilon t}\|v\|_x, \quad v \in E^s_{\mathcal{B}'}(x) \tag{73}$$

$$\|\mathcal{B}'(x,-t)v\|_{\phi^{-t}(x)} \leq C(x)e^{-\lambda t}e^{\varepsilon t}\|v\|_x, \quad v \in E^u_{\mathcal{B}'}(x). \tag{74}$$

One can now repeat the arguments in the proof of Lemma 1 to show that

$$E^s_{\mathcal{B}}(x) \subset E^s_{\mathcal{B}'}(x) \quad \text{and} \quad E^u_{\mathcal{B}'}(x) \subset E^u_{\mathcal{B}}(x)$$

for $x \in \Lambda$. Moreover, proceeding as in the proof of Lemmas 2 and 3 we find that the operator $\mathrm{Id} - P^s_{\mathcal{B}}(x) - P^u_{\mathcal{B}'}(x)$ is a projection on $T_x M$ (where $P^s_{\mathcal{B}}(x)$ is the projection on the stable space for the cocycle $\mathcal{B}$ and analogously for the other maps) with range $E^s_{\mathcal{B}'}(x) \cap E^u_{\mathcal{B}}(x)$, for $x \in \Lambda$. It follows directly from (71) and (74) that

$$\|d_x\phi^t v\|_{\phi^t(x)} \leq C(x)e^{-(\lambda+\omega)t}e^{\varepsilon t}\|v\|_x, \quad v \in E^s_{\mathcal{B}}(x) \tag{75}$$

and

$$\|d_x\phi^{-t} v\|_{\phi^{-t}(x)} \leq C(x)e^{-(\lambda-\omega')t}e^{\varepsilon t}\|v\|_x, \quad v \in E^u_{\mathcal{B}'}(x) \tag{76}$$

for $x \in \Lambda$, $v \in T_x M$ and $t \geq 0$. Similarly, it follows from (72) and (73) that

$$\|d_x\phi^{-t} v\|_{\phi^{-t}(x)} \leq C(x)e^{-(\lambda-\omega)t}e^{\varepsilon t}\|v\|_x$$

and

$$\|d_x\phi^t v\|_{\phi^t(x)} \leq C(x)e^{-(\lambda+\omega')t}e^{\varepsilon t}\|v\|_x$$

for $x \in \Lambda$, $v \in E^s_{\mathcal{B}'}(x) \cap E^u_{\mathcal{B}}(x)$ and $t \ge 0$. On the other hand, by (72) and (73),

$$\|\mathrm{Id} - P^s_{\mathcal{B}}(x) - P^u_{\mathcal{B}'}(x)\| \le 3C(x).$$

Hence,

$$\|d_x\phi^{-t}(\mathrm{Id} - P^s_{\mathcal{B}}(x) - P^u_{\mathcal{B}'}(x))v\|_{f^{-t}(x)} \le 3C(x)^2 e^{-(\lambda-\omega)t} e^{\varepsilon t} \|v\|_x \tag{77}$$

and

$$\|d_x\phi^{t}(\mathrm{Id} - P^s_{\mathcal{B}}(x) - P^u_{\mathcal{B}'}(x))v\|_{\phi^{t}(x)} \le 3C(x)^2 e^{-(\lambda+\omega')t} e^{\varepsilon t} \|v\|_x \tag{78}$$

for $x \in \Lambda$, $v \in T_xM$ and $t \ge 0$. It follows from (75), (76), (77) and (78) that Λ is a nonuniformly partially hyperbolic set for Φ. □

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