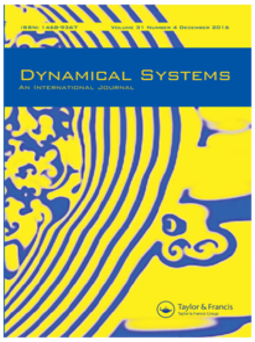

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Tempered exponential dichotomies: admissibility and stability under perturbations

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ABSTRACT
We give a characterization of the notion of a tempered exponential dichotomy on a Banach space in terms of an admissibility property. We note that for a linear cocycle over a measure-preserving transformation satisfying a certain integrability assumption, it follows from the multiplicative ergodic theorem that the dynamics admits a tempered exponential dichotomy if and only if all Lyapunov exponents are non-zero almost everywhere. As a consequence of our approach, we give a new proof of the robustness property of the notion of a tempered exponential dichotomy under sufficiently small linear perturbations and we establish a version of the Grobman–Hartman theorem yielding the existence of topological conjugacies between a linear dynamics with a tempered exponential dichotomy and any sufficiently small nonlinear perturbation. In addition, we show that the conjugacy maps vary continuously with the perturbation.

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1. Introduction

1.1. Tempered exponential dichotomies

In this paper, we consider the notion of a tempered exponential dichotomy (see Section 2 for the definition). For a linear cocycle over a measure-preserving transformation satisfying a certain integrability assumption, it follows from the multiplicative ergodic theorem that the dynamics admits a tempered exponential dichotomy if and only if all Lyapunov exponents are non-zero almost everywhere. Hence, a principal motivation to consider this notion of exponential dichotomy is its ubiquity in the context of smooth ergodic theory and of the non-uniform hyperbolicity theory. We refer to [1] for detailed expositions of the theory, which goes back to the landmark works of Oseledets [2] and Pesin.[3] More recently, Lian and Lu [4] showed that for a strongly measurable cocycle with values in the set of bounded linear operators acting on a separable Banach space, it is also true that if the usual integrability assumption holds and all the Lyapunov exponents are non-zero almost everywhere, then the dynamics admits a tempered exponential dichotomy.

Our main objective is twofold

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(1) to give a characterization of the notion of a tempered exponential dichotomy in terms of an admissibility property;
(2) to establish a Grobman–Hartman theorem, thus yielding a topological conjugacy between a tempered exponential dichotomy and any sufficiently small nonlinear perturbation.

As a consequence of our approach, we also give a new and much simpler proof of the robustness of the notion of a tempered exponential dichotomy established by Zhou *et al.*[5] More precisely, we use Lyapunov norms and the complete characterization of the notion of a tempered exponential dichotomy in terms of an admissibility property established in our paper.

1.2. Robustness and admissibility

Having in mind the central role of the notion of an exponential dichotomy in a large of part of the stability theory of differential equations and dynamical systems, it is not surprising that the study of its robustness has a long history. For some of the most relevant early contributions in the area we refer to the books by Massera and Schäffer [6] and Dalec'kiĭ and Kreĭn.[7] Related results for discrete time were obtained by Coffman and Schäffer.[8] We refer to the book [9] for some early results in infinite-dimensional spaces. For further references, see [10,11].

The robustness property of a tempered exponential dichotomy was studied in [12,13] and, more recently, in [5] with the most general result. In the last paper (in Remark 1), the authors note that it is unclear whether the existence of a tempered exponential dichotomy can be characterized in terms of an admissibility property. The study of admissibility properties goes back to Perron [14] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t) \tag{1}$$

in $\mathbb{R}^n$ for any bounded continuous perturbation $f : \mathbb{R}_0^+ \to \mathbb{R}^n$. The remark made in [5] is quite relevant since often the study of the robustness of an exponential dichotomy uses a characterization in terms of an admissibility property, and one might expect that this could again be the case with the tempered exponential dichotomies. Because of the lack of a characterization, they use other methods, partly based on techniques developed by Barreira and Valls [15] for non-uniform exponential dichotomies (some of which are inspired by the work of Popescu [16] for uniform exponential dichotomies).

We show in this paper that indeed the existence of a tempered exponential dichotomy can be completely characterized in terms of an admissibility property (see Theorems 3.1 and 3.2). Roughly speaking, the notion of a tempered exponential dichotomy can be expressed in terms of the invertibility of certain operators associated to single trajectories, which essentially corresponds to a discrete time version of (1) for cocycles (and thus with one equation for each trajectory $\theta^n(\omega)$ with $n \in \mathbb{Z}$, although satisfying a certain joint measurability). Our approach is partly inspired on the characterization of hyperbolic sets in [17] and on the characterization of non-uniform exponential dichotomies in terms of an admissibility property in [18]. Moreover, in contrast to many of the existing approaches, we are able to obtain bounds along the stable and unstable directions in a single step.

As already noted above, and as a consequence of our approach, we also give a new and much simpler proof of the main result in [5], thus showing that admissibility does play a role in the study of the robustness of the tempered exponential dichotomies.

1.3. A Grobman–Hartman theorem

Going back to pioneering work of Poincaré, a fundamental problem in the study of the local behaviour of a map or a flow is whether the linearization of the system along a given solution approximates well the solution itself in some open neighbourhood. That is, we look for an appropriate change of variables, called a conjugacy, that can take the system to a linear one. This can be done, for example, when the linear dynamics admits an exponential dichotomy: by the Grobman–Hartman theorem, under mild additional assumptions on the perturbation the two dynamics are topologically conjugate. The original references for the Grobman–Hartman theorem are Grobman [19,20] and Hartman.[21,22] Using the ideas in Moser's proof [23] of the structural stability of Anosov diffeomorphisms, the Grobman–Hartman theorem was extended to Banach spaces independently by Palis [24] and Pugh.[25]

We establish a version of the Grobman–Hartman theorem for the nonlinear perturbations of a tempered exponential dichotomy thus yielding the existence of topological conjugacies between a linear dynamics with a tempered exponential dichotomy and any sufficiently small nonlinear perturbation. In addition, we show that the conjugacy maps vary continuously with the perturbation.

2. Preliminaries

We first introduce some basic notions. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $\theta\colon \Omega \to \Omega$ be a measurable map with measurable inverse. We always assume that θ preserves the measure μ. Since θ has a measurable inverse, this is equivalent to require that

$$\mu(\theta(A)) = \mu(A) \quad \text{for } A \in \mathcal{F}.$$

Now let X be a Banach space and let $B(X)$ be the set of all bounded linear operators acting on X. Moreover, let $\Phi : \mathbb{N}_0 \times \Omega \to B(X)$ be a strongly measurable map. In the present work, this means that the map $\omega \mapsto \Phi(n, \omega)x$ is measurable for each $n \in \mathbb{N}_0$ and $x \in X$ (in the sense that the preimage of a measurable set is measurable). When some integrability conditions are also necessary (although they are not needed in our work), one must make the stronger assumption that for each n and x, there exists a sequence $F_m\colon \Omega \to X$ of simple (measurable) functions such that

$$\|F_m(\omega) - \Phi(n, \omega)x\| \to 0 \quad \text{when } m \to \infty \tag{2}$$

for μ-almost every $\omega \in \Omega$ (since the measure μ is finite, there is no need to consider countably-valued functions F_m).

The map Φ is called a *(measurable) cocycle* over θ, if

(1) $\Phi(0, \omega) = \mathrm{Id}$ for $\omega \in \Omega$;

(2) $\Phi(n + m, \omega) = \Phi(n, \theta^m(\omega))\Phi(m, \omega)$ for $m, n \geq 0$ and $\omega \in \Omega$.

The strongly measurable map $A_\Phi : \Omega \to B(X)$ defined by $A_\Phi(\omega) = \Phi(1, \omega)$ is called the *generator* of Φ. Conversely, each strongly measurable map $A\colon \Omega \to B(X)$ induces a strongly measurable cocycle by letting

$$\Phi(n, \omega) = \begin{cases} A(\theta^{n-1}(\omega)) \dots A(\omega), & n > 0, \\ \mathrm{Id}, & n = 0, \end{cases}$$

for $n \geq 0$ and $\omega \in \Omega$. When $A(\omega)$ is invertible for all $\omega \in \Omega$ and the map $\omega \mapsto A(\omega)^{-1}$ is strongly measurable, we obtain a $\mathbb{Z}$-cocycle by also defining

$$\Phi(n, \omega) = \Phi(-n, \theta^n(\omega)) \quad \text{for } n < 0.$$

It should be noted that for the stronger notion of measurability in (2), the product of strongly measurable bounded linear operator a priori may not be strongly measurable in that sense, although this is true, for example, when X is a separable Hilbert space (see [26]).

We say that a strongly measurable cocycle Φ admits a *tempered exponential dichotomy* if there exist a strongly measurable map $\Pi^s\colon \Omega \to B(X)$ and measurable functions $\alpha, K\colon \Omega \to (0, +\infty)$ with $K \geq 1$ such that for μ-almost every $\omega \in \Omega$:

(1) $\Pi^s(\omega)$ is a projection, $\alpha(\theta(\omega)) = \alpha(\omega)$ and

$$\limsup_{n \to \pm\infty} \frac{1}{|n|} \log K(\theta^n(\omega)) = 0; \tag{3}$$

(2) $\Pi^s(\theta^n(\omega))\Phi(n, \omega) = \Phi(n, \omega)\Pi^s(\omega)$ for $n \geq 0$;

(3) for each $n > 0$ the map

$$\Phi(n, \omega)|\ker \Pi^s(\omega) : \ker \Pi^s(\omega) \to \ker \Pi(\theta^s(\omega))$$

is invertible and $\omega \mapsto (\Phi(n, \omega)|\ker \Pi^s(\omega))^{-1}$ is strongly measurable;

(4)

$$\|\Phi(n, \omega)\Pi^s(\omega)\| \leq K(\omega)e^{-\alpha(\omega)n}, \quad n \geq 0 \tag{4}$$

and

$$\|\Phi(n, \omega)\Pi^u(\omega)\| \leq K(\omega)e^{\alpha(\omega)n}, \quad n \leq 0, \tag{5}$$

where $\Pi^u(\omega) = \mathrm{Id} - \Pi^s(\omega)$ and

$$\Phi(n, \omega) = \left(\Phi(-n, \theta^n(\omega))|\,\mathrm{Im}\,\Pi^u(\theta^n(\omega))\right)^{-1} \quad \text{for } n \leq 0.$$

3. Characterization of tempered exponential dichotomies

In this section, we give a complete characterization of the notion of a tempered exponential dichotomy in terms of an admissibility property. As a consequence of our approach, we also give a new proof of the robustness of a tempered exponential dichotomy.

Consider a measurable family of norms on X, that is, a family of norms $\|\cdot\|_\omega$ on X for $\omega \in \Omega$ such that the map $(x, \omega) \mapsto \|x\|_\omega$ is measurable. For each $\omega \in \Omega$, we define

$$Y^\omega = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sup_{n\in\mathbb{Z}} \|x_n\|_{\theta^n(\omega)} < +\infty \right\}.$$

It is easy to verify that Y^ω is a Banach space when equipped with the norm

$$\|\mathbf{x}\|_{\omega,\infty} = \sup_{n\in\mathbb{Z}} \|x_n\|_{\theta^n(\omega)}.$$

We also consider the set Y of all measurable functions $\mathbf{y} : \mathbb{Z} \times \Omega \to X$ such that $\mathbf{y}_\omega = (\mathbf{y}(n, \omega))_{n\in\mathbb{Z}} \in Y^\omega$ for each $\omega \in \Omega$, identified if they coincide $(\nu \times \mu)$-almost everywhere on $\mathbb{Z} \times \Omega$, where ν is the counting measure on $\mathbb{Z}$. Up to the measurability requirement, one can think of the set Y simply as $(Y^\omega)_{\omega\in\mathbb{Z}}$. We shall also write $\mathbf{y}(n, \omega) = y_n(\omega)$ for each $\mathbf{y} \in Y$.

Theorem 3.1: *Let Φ be a strongly measurable cocycle admitting a tempered exponential dichotomy. Then there exist a θ-invariant measurable set $\tilde{\Omega} \subset \Omega$ of full μ-measure, a measurable family of norms $\|\cdot\|_\omega$ on X for $\omega \in \Omega$, and measurable functions $K, \rho: \Omega \to (0, +\infty)$ with $K \geq 1$ and ρ θ-invariant such that*

(1) *property (3) holds and*

$$\frac{1}{2}\|x\| \leq \|x\|_\omega \leq K(\omega)\|x\| \quad \text{for } \omega \in \tilde{\Omega} \text{ and } x \in X; \tag{6}$$

(2) *given $\mathbf{y} \in Y$, there exists a unique $\mathbf{x}_\omega = (x_n(\omega))_{n\in Z} \in Y^\omega$ satisfying*

$$x_n(\omega) - A_\Phi(\theta^{n-1}(\omega))x_{n-1}(\omega) = y_n(\omega) \quad \text{for } n \in \mathbb{Z} \tag{7}$$

and

$$\|\mathbf{x}_\omega\|_{\omega,\infty} \leq \rho(\omega)\|\mathbf{y}\|_{\omega,\infty}; \tag{8}$$

moreover, the function $(n, \omega) \mapsto x_n(\omega)$ is measurable.

Proof: We first construct appropriate Lyapunov norms $\|\cdot\|_\omega$. For each $\omega \in \Omega$ and $x \in X$, let

$$\|x\|_\omega = \max\left\{\|x_1\|_\omega, \|x_2\|_\omega\right\}, \tag{9}$$

where

$$\|x_1\|_\omega = \sup_{m\geq 0}\left(\|\Phi(m, \omega)\Pi^s(\omega)x\|e^{\alpha(\omega)m}\right)$$

and

$$\|x_2\|_\omega = \sup_{m\leq 0}\left(\|\Phi(m, \omega)\Pi^u(\omega)x\|e^{-\alpha(\omega)m}\right).$$

By construction, the family of norms $\|\cdot\|_\omega$ is measurable. Moreover, it follows from (4) and (5) that property (6) holds for some θ-invariant measurable set $\tilde{\Omega} \subset \Omega$ of full μ-measure. $\square$

Lemma 3.1: *For each $\omega \in \tilde{\Omega}$, we have*

$$\|\Phi(m,\omega)\Pi^s(\omega)x\|_{\theta^m(\omega)} \le e^{-\alpha(\omega)m}\|x\|_\omega, \quad m \ge 0 \tag{10}$$

and

$$\|\Phi(m,\omega)\Pi^u(\omega)x\|_{\theta^m(\omega)} \le e^{\alpha(\omega)m}\|x\|_\omega, \quad m \le 0. \tag{11}$$

Proof of the lemma: We have

$$\begin{aligned}
&\|\Phi(m,\omega)\Pi^s(\omega)x\|_{\theta^m(\omega)}\\
&= \sup_{n\ge 0}\big(\|\Phi(n,\theta^m(\omega))\Pi^s(\theta^m(\omega))\Phi(m,\omega)\Pi^s(\omega)x\|e^{\alpha(\omega)n}\big)\\
&= e^{-\alpha(\omega)m}\sup_{n\ge 0}\big(\|\Phi(n+m,\omega)\Pi^s(\omega)x\|e^{\alpha(\omega)(n+m)}\big)\\
&\le e^{-\alpha(\omega)m}\|x\|_\omega
\end{aligned}$$

for $x \in X$ and so (10) holds. One can establish (11) in a similar manner. $\square$

Now take $\mathbf{y} \in Y$. For each $n \in \mathbb{Z}$ and $\omega \in \tilde{\Omega}$, let

$$x_n^1(\omega) = \sum_{m\ge 0}\Phi(m,\theta^{n-m}(\omega))\Pi^s(\theta^{n-m}(\omega))y_{n-m}(\omega)$$

and

$$x_n^2(\omega) = -\sum_{m\ge 1}\Phi(-m,\theta^{n+m}(\omega))\Pi^u(\theta^{n+m}(\omega))y_{n+m}(\omega).$$

It follows from (10) and (11) that

$$\|x_n^1(\omega)\|_{\theta^n(\omega)} \le \frac{1}{1-e^{-\alpha(\omega)}}\|\mathbf{y}_\omega\|_{\omega,\infty} \tag{12}$$

and

$$\|x_n^2(\omega)\|_{\theta^n(\omega)} \le \frac{e^{-\alpha(\omega)}}{1-e^{-\alpha(\omega)}}\|\mathbf{y}_\omega\|_{\omega,\infty}. \tag{13}$$

Let

$$x_n(\omega) = x_n^1(\omega) + x_n^2(\omega) \quad \text{for } n \in \mathbb{Z} \text{ and } \omega \in \tilde{\Omega}.$$

Moreover, define $\mathbf{x} \in Y$ by $\mathbf{x}(n,\omega) = x_n(\omega)$ (the measurability of $\mathbf{x}$ follows readily from the measurability assumptions in the theorem). It follows from (12) and (13) that $\mathbf{x} \in Y$ and that (8) holds with

$$\rho(\omega) = (1 + e^{-\alpha(\omega)})/(1 - e^{-\alpha(\omega)}).$$

Moreover, it is easy to verify that (7) holds.

In order to establish the uniqueness of $\mathbf{x}_\omega = (x_n(\omega))_{n\in\mathbb{Z}}$, it is sufficient to take $\mathbf{y} = 0$, in which case

$$x_n(\omega) = A_\Phi(\theta^{n-1}(\omega))x_{n-1}(\omega) \quad \text{for } n \in \mathbb{Z} \text{ and } x \in \tilde{\Omega}.$$

Let

$$x_n^s(\omega) = \Pi^s(\theta^n(\omega))x_n(\omega) \quad \text{and} \quad x_n^u(\omega) = \Pi^u(\theta^n(\omega))x_n(\omega).$$

Clearly, $x_n(\omega) = x_n^s(\omega) + x_n^u(\omega)$,

$$x_n^s(\omega) = A_\Phi(\theta^{n-1}(\omega))x_{n-1}^s(\omega) \quad \text{and} \quad x_n^u(\omega) = A_\Phi(\theta^{n-1}(\omega))x_{n-1}^u(\omega).$$

Since $x_k^s(\omega) = \Phi(m, \theta^{k-m}(\omega))x_{k-m}^s(\omega)$ for $m \geq 0$, we have

$$\begin{aligned}\|x_k^s(\omega)\|_{\theta^k(\omega)} &= \|\Phi(m, \theta^{k-m}(\omega))x_{k-m}^s(\omega)\|_{\theta^k(\omega)} \\ &= \|\Phi(m, \theta^{k-m}(\omega))\Pi^s(\theta^{k-m}(\omega))x_{k-m}(\omega)\|_{\theta^k(\omega)} \\ &\leq e^{-\alpha(\omega)m}\|x_{k-m}(\omega)\|_{\theta^{k-m}(\omega)} \\ &\leq e^{-\alpha(\omega)m}\|\mathbf{x}\|_{\omega,\infty}.\end{aligned}$$

Letting $m \to +\infty$ yields that $x_k^s(\omega) = 0$ for $k \in \mathbb{Z}$. One can show in a similar manner that $x_k^u(\omega) = 0$ for $k \in \mathbb{Z}$ and thus $\mathbf{x}_\omega = 0$ for each $\omega \in \tilde{\Omega}$. This completes the proof of the theorem.

Now we establish the converse of Theorem 3.1. □

Theorem 3.2: *Let Φ be a strongly measurable cocycle and assume that there exist a θ-invariant measurable set $\tilde{\Omega} \subset \Omega$ of full μ-measure, a measurable family of norms $\|\cdot\|_\omega$ on X for $\omega \in \Omega$, and measurable functions K, ρ: $\Omega \to (0, +\infty)$ with $K \geq 1$ and ρ θ-invariant satisfying properties* 1 *and* 2 *in Theorem* 3.1*. Then the cocycle Φ admits a tempered exponential dichotomy.*

Proof: We first introduce a family of linear operators related to condition (7). Namely, for each $\omega \in \tilde{\Omega}$, we define a linear operator $T^\omega : \mathcal{D}(T^\omega) \to Y^\omega$ by

$$(T^\omega \mathbf{x})_n = x_n - A_\Phi(\theta^{n-1}(\omega))x_{n-1}, \quad n \in \mathbb{Z}, \tag{14}$$

on the domain $\mathcal{D}(T^\omega)$ composed of those $\mathbf{x} \in Y^\omega$ such that $T^\omega \mathbf{x} \in Y^\omega$. □

Lemma 3.2: *The linear operator $T^\omega : \mathcal{D}(T^\omega) \to Y^\omega$ is closed.*

Proof of the lemma: Let $(\mathbf{x}^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(T^\omega)$ converging to $\mathbf{x} \in Y^\omega$ and assume that $T^\omega \mathbf{x}^k$ converges to $\mathbf{y} \in Y^\omega$. Then,

$$\begin{aligned} x_n - A_\Phi(\theta^{n-1}(\omega))x_{n-1} &= \lim_{k\to\infty} \left(x_n^k - A_\Phi(\theta^{n-1}(\omega))x_{n-1}^k\right) \\ &= \lim_{k\to\infty} (T^\omega \mathbf{x}^k)_n = y_n \end{aligned}$$

for $n \in \mathbb{Z}$, since the linear operator $A_\Phi(\theta^{n-1}(\omega))$ is continuous. Therefore, $\mathbf{x} \in \mathcal{D}(T^\omega)$ and $T^\omega \mathbf{x} = \mathbf{y}$. This shows that T^ω is closed. □

For $\mathbf{x} \in \mathcal{D}(T^\omega) \subset Y^\omega$, we consider the graph norm

$$\|\mathbf{x}\|'_{\omega,\infty} = \|\mathbf{x}\|_{\omega,\infty} + \|T^\omega \mathbf{x}\|_{\omega,\infty}.$$

Clearly, the operator

$$T^\omega : (\mathcal{D}(T^\omega), \|\cdot\|'_{\omega,\infty}) \to (Y^\omega, \|\cdot\|_{\omega,\infty})$$

is bounded. From now on we denote it simply by T^ω. It follows from Lemma 3.2 that $(\mathcal{D}(T^\omega), \|\cdot\|'_{\omega,\infty})$ is a Banach space for each $\omega \in \tilde{\Omega}$. Moreover, in view of the assumptions in the theorem, the operator T^ω is invertible and

$$\|(T^\omega)^{-1}\mathbf{y}\|'_{\omega,\infty} \le (\rho(\omega) + 1)\|\mathbf{y}\|_{\omega,\infty}$$

for $\mathbf{y} \in Y^\omega$ (see (8)).

For each $\omega \in \tilde{\Omega}$, let

$$F^s(\omega) = \left\{x \in X : \sup_{m\ge 0} \|\Phi(m,\omega)x\|_{\theta^m(\omega)} < +\infty\right\}.$$

Moreover, let $F^u(\omega)$ be the set of all vectors $x \in X$ for which there exists a sequence $(x_m)_{m\le 0} \subset X$ such that $x_0 = x$, $\sup_{m\le 0} \|x_m\|_{\theta^m(\omega)} < +\infty$ and

$$x_m = A_\Phi(\theta^{m-1}(\omega))x_{m-1} \quad \text{for } m \le 0.$$

It is easy to verify that $F^s(\omega)$ and $F^u(\omega)$ are subspaces of X.

Lemma 3.3: *For each $\omega \in \tilde{\Omega}$, we have*

$$X = F^s(\omega) \oplus F^u(\omega). \tag{15}$$

Proof of the lemma: Take $v \in X$ and consider the function $\mathbf{y} \in Y$ defined by

$$y(0,\omega) = v \quad \text{and} \quad y(n,\omega) = 0 \text{ for } n \ne 0, \tag{16}$$

for each $\omega \in \Omega$ (since $\mathbf{y}$ is independent of ω it is clearly measurable). Now take $\mathbf{x} \in Y$ such that $T^\omega \mathbf{x}_\omega = \mathbf{y}_\omega$ for $\omega \in \tilde{\Omega}$. We have

$$x_0(\omega) - A_\Phi(\theta^{-1}(\omega))x_{-1}(\omega) = v$$

and

$$x_n(\omega) = A_\Phi(\theta^{n-1}(\omega))x_{n-1}(\omega) \quad \text{for } n \neq 0.$$

Therefore, $x_n = \Phi(n, \omega)x_0(\omega)$ for $n \geq 0$ and so $x_0(\omega) \in F^s(\omega)$. One can show in a similar manner that $A_\Phi(\theta^{-1}(\omega))x_{-1}(\omega) \in F^u(\omega)$, and so

$$v = x_0(\omega) + \big(-A_\Phi(\theta^{-1}(\omega))x_{-1}(\omega)\big) \in F^s(\omega) + F^u(\omega).$$

Now take $v \in F^s(\omega) \cap F^u(\omega)$ and let $(x_n)_{n\leq 0} \subset X$ be a sequence such that $x_0 = v$, $\sup_{n\leq 0} \|x_n\|_{\theta^n(\omega)} < +\infty$ and

$$x_n = A_\Phi(\theta^{n-1}(\omega))x_{n-1} \quad \text{for } n \leq 0.$$

We define

$$z_n = \begin{cases} \Phi(n, \omega)v, & n > 0, \\ x_n, & n \leq 0. \end{cases}$$

Then $\mathbf{z} = (z_n)_{n\in\mathbb{Z}} \in Y^\omega$. It is easy to verify that $T^\omega \mathbf{z} = 0$ and thus $\mathbf{z} = 0$. Therefore, $x_0 = v = 0$. □

Let

$$\pi^s(\omega) : X \to F^s(\omega) \quad \text{and} \quad \pi^u(\omega) : X \to F^u(\omega)$$

be the projections associated to the decomposition in (15). According to the proof of Lemma 3.3, we have

$$\Pi^s(\omega)v = x_0(\omega), \qquad \Pi^u(\omega)v = -A_\Phi(\theta^{-1}(\omega))x_{-1}(\omega), \tag{17}$$

where $\mathbf{x}_\omega = (x_n(\omega))_{n\in\mathbb{Z}}$ are the unique sequences such that $T^\omega \mathbf{x}_\omega = \mathbf{y}_\omega$ for each $\omega \in \tilde{\Omega}$, with $\mathbf{y}$ as in (16). Since $\mathbf{y} \in Y$, it follows from the assumptions in the theorem that also $\mathbf{x} = (\mathbf{x}_\omega)_{\omega\in\mathbb{Z}} \in Y$, which ensures that the functions $\omega \mapsto x_n(\omega)$ are measurable. Hence, it follows from (17) that the maps $\Pi^s: \Omega \to B(X)$ and $\Pi^u: \Omega \to B(X)$ defined by

$$\Pi^s(\omega)v = \pi^s(\omega)v \quad \text{and} \quad \Pi^u(\omega)v = \pi^u(\omega)v$$

are strongly measurable. One can easily verify that

$$\Pi^s(\theta^n(\omega))\Phi(n, \omega) = \Phi(n, \omega)\Pi^s(\omega)$$

for $n \geq 0$ and $\omega \in \tilde{\Omega}$.

Lemma 3.4: *For each $n \geq 0$, the map*

$$\Phi(n, \omega)|F^u(\omega) : F^u(\omega) \to F^u(\theta^n(\omega)) \tag{18}$$

is invertible.

Proof of the lemma: Take a vector $v \in F^u(\omega)$ such that $\Phi(n, \omega)v = 0$ and let $(x_m)_{m \le 0} \subset X$ be a sequence such that $x_0 = v$, $\sup_{m \le 0} \|x_m\|_{\theta^m(\omega)} < +\infty$ and

$$x_m = A_\Phi(\theta^{m-1}(\omega))x_{m-1} \quad \text{for } m \le 0.$$

We define

$$y_m = \begin{cases} x_{m+n}, & m \le -n, \\ \Phi(m+n, \omega)v, & m > -n. \end{cases}$$

It is easy to verify that $\mathbf{y} = (y_m)_{m \in \mathbb{Z}} \in Y^{\theta^n(\omega)}$ and $T^{\theta^n(\omega)}\mathbf{y} = 0$. Hence, $\mathbf{y} = 0$ and so $v = y_{-n} = 0$. This shows that the map in (18) is one-to-one.

Now take $v \in F^u(\theta^n(\omega))$ and let $(x_m)_{m \le 0} \subset X$ be a sequence such that $x_0 = v$, $\sup_{m \le 0} \|x_m\|_{\theta^{m+n}(\omega)} < +\infty$ and

$$x_m = A_\Phi(\theta^{m+n-1}(\omega))x_{m-1} \quad \text{for } m \le 0.$$

We define $y_m = x_{m-n}$, $m \le 0$. Since

$$\|y_m\|_{\theta^m(\omega)} = \|x_{n-m}\|_{\theta^{m-n}(\theta^n(\omega))}$$

and

$$y_m = A_\Phi(\theta^{m-1}(\omega))y_{m-1} \quad \text{for } m \le 0,$$

we obtain $y_0 = x_{-n} \in F^u(\omega)$. Finally, we note that $v = \Phi(n, \omega)y_0$, and so the map is onto. □

In order to show that the map $\omega \mapsto (\Phi(n, \omega)|F^u(\omega))^{-1}$ is strongly measurable we proceed as follows. Given $w \in F^u(\theta^n(\omega))$, consider the map $\mathbf{y} \in Y$ defined by

$$y(n, \omega) = -w \quad \text{and} \quad y(m, \omega) = 0 \text{ for } m \ne n.$$

Then, $T^\omega \mathbf{x}_\omega = \mathbf{y}_\omega$, where

$$x(m, \omega) = \begin{cases} z_n(\omega), & m < n, \\ 0, & m \ge n, \end{cases}$$

with $z_n(\omega) = w$, $\sup_{m \le 0} \|z_{n+m}\|_{\theta^{n+m}(\omega)} < \infty$ and

$$z_{n+m}(\omega) = A_\Phi(\theta^{n+m-1}(\omega))z_{n+m-1} \quad \text{for } m \le 0.$$

Since $w \in F^u(\theta^n(\omega))$, by Lemma 3.4, the sequence $(z_{n+m}(\omega))_{m \le 0}$ is uniquely defined. Moreover, since $\mathbf{y} \in Y$, it follows from the hypotheses in the theorem that $\mathbf{x} \in Y$ and that $\mathbf{x}_\omega$ is unique for each $\omega \in \tilde{\Omega}$. In particular, this implies that

$$\omega \mapsto (\Phi(n, \omega)|F^u(\omega))^{-1}w = z_0(\omega) = x(0, \omega)$$

is measurable, since $\mathbf{x} \in Y$.

It remains to establish the exponential bounds along the stable and unstable directions. Take $\omega \in \tilde{\Omega}$ and $v \in X$. Moreover, let $\mathbf{y}$ and $\mathbf{x}$ be as in the proof of Lemma 3.3. For each $z \geq 1$, we define an operator

$$R(z) : (\mathcal{D}(T^\omega), \|\cdot\|'_{\omega,\infty}) \to (Y^\omega, \|\cdot\|_{\omega,\infty})$$

by

$$(R(z)\boldsymbol{v})_m = \begin{cases} zv_m - A_\Phi(\theta^{m-1}(\omega))v_{m-1}, & m \leq 0, \\ \frac{1}{z}v_m - A_\Phi(\theta^{m-1}(\omega))v_{m-1}, & m > 0. \end{cases}$$

Clearly,

$$\|(R(z) - T^\omega)\boldsymbol{v}\|_{\omega,\infty} \leq (z-1)\|\boldsymbol{v}\|'_{\omega,\infty}$$

for $\boldsymbol{v} \in \mathcal{D}(L^\omega)$ and $z \geq 1$. This implies that $R(z)$ is invertible whenever $1 \leq z < 1 + 1/(1 + \rho(\omega))$, in which case

$$\|R(z)^{-1}\| \leq \frac{1}{(1+\rho(\omega))^{-1} - (z-1)}.$$

Now let $\gamma \colon \Omega \to (0, 1)$ be a θ-invariant measurable function such that

$$\gamma(\omega)^{-1} < 1 + 1/(1+\rho(\omega)) \quad \text{for } \omega \in \Omega.$$

Moreover, take $\mathbf{z} \in Y^\omega$ such that $R(\gamma(\omega)^{-1})\mathbf{z} = \mathbf{y}_\omega$. Writing

$$D(\omega) = \frac{1}{(1+\rho(\omega))^{-1} - (\gamma(\omega)^{-1} - 1)},$$

we obtain

$$\|\mathbf{z}\|_{\omega,\infty} \leq \|R(\gamma(\omega)^{-1})^{-1}\mathbf{y}_\omega\|'_{\omega,\infty} \leq D(\omega)\|\mathbf{y}\|_{\omega,\infty} = D(\omega)\|v\|_\omega.$$

For each $m \in \mathbb{Z}$, let $w_m = \gamma(\omega)^{|m|-1}z_m$ and define $\mathbf{w} = (w_m)_{m\in\mathbb{Z}}$. Clearly, $\mathbf{w} \in Y^\omega$. Moreover, one can easily verify that $T^\omega\mathbf{w} = \mathbf{y}_\omega$ and hence $\mathbf{w} = \mathbf{x}_\omega$. Therefore,

$$\|x_m(\omega)\|_{\theta^m(\omega)} = \|w_m\|_{\theta^m(\omega)} = \gamma(\omega)^{|m|-1}\|z_m\|_{\theta^m(\omega)} \leq D(\omega)\gamma(\omega)^{|m|-1}\|v\|_\omega$$

for $m \in \mathbb{Z}$. By (6) and (17), we obtain

$$\|\Phi(m,\omega)\Pi^s(\omega)v\| \leq 2\frac{D(\omega)}{\gamma(\omega)}K(\omega)\gamma(\omega)^m\|v\| \quad \text{for } m \geq 0$$

and

$$\|\Phi(m,\omega)\Pi^u(\omega)v\| \leq 2\frac{D(\omega)}{\gamma(\omega)}K(\omega)\gamma(\omega)^{-m}\|v\| \quad \text{for } m \leq 0.$$

This shows that the cocycle Φ admits a tempered exponential dichotomy. □

The following result was established in [5]. We give an alternative proof using the complete characterization of tempered exponential dichotomies provided by Theorems 3.1 and 3.2.

Theorem 3.3: *Let Φ be a strongly measurable cocycle admitting a tempered exponential dichotomy. Then there exists a θ-invariant measurable function $\delta\colon \Omega \to (0, +\infty)$ such that any strongly measurable cocycle Ψ satisfying*

$$\|A_\Psi(\omega) - A_\Phi(\omega)\| \le \delta(\omega)/K(\theta(\omega)), \quad \omega \in \Omega, \tag{19}$$

also admits a tempered exponential dichotomy.

Proof: Let $\tilde{\Omega}$, $\|\cdot\|_\omega$ and ρ be as in Theorem 3.1. Moreover, let $\delta\colon \Omega \to (0, +\infty)$ be a measurable function such that

$$0 < 2\delta(\omega) < 1/(\rho(\omega) + 1) \quad \text{for } \omega \in \Omega.$$

Finally, let T^ω be the linear operators defined by (14).

We construct analogous linear operators associated to A_Ψ. Namely, for each $\omega \in \tilde{\Omega}$, let $L^\omega : \mathcal{D}(L^\omega) \to Y^\omega$ be the linear operator defined by

$$(L^\omega \mathbf{x})_n = x_n - A_\Psi(\theta^{n-1}(\omega))x_{n-1}, \quad n \in \mathbb{Z},$$

on the domain $\mathcal{D}(L^\omega)$ composed of those $\mathbf{x} \in Y^\omega$ such that $L^\omega \mathbf{x} \in Y^\omega$. It follows from (6) and (19) that

$$\begin{aligned}
\|(T^\omega - L^\omega)\mathbf{x}\|_{\omega,\infty} &= \sup_{n\in\mathbb{Z}} \left\| [A_\Phi(\theta^{n-1}(\omega)) - A_\Psi(\theta^{n-1}(\omega))]x_{n-1} \right\|_{\theta^n(\omega)} \\
&\le \sup_{n\in\mathbb{Z}} \Big(K(\theta^n(\omega)) \left\| [A_\Phi(\theta^{n-1}(\omega)) - A_\Psi(\theta^{n-1}(\omega))]x_{n-1} \right\| \Big) \\
&\le 2\delta(\omega) \sup_{n\in\mathbb{Z}} \|x_{n-1}\|_{\theta^{n-1}(\omega)} = 2\delta(\omega)\|\mathbf{x}\|_{\omega,\infty}
\end{aligned}$$

for $\mathbf{x} \in Y^\omega$. Therefore, $\mathcal{D}(T^\omega) = \mathcal{D}(L^\omega)$ and since $\|\mathbf{x}\|_{\omega,\infty} \le \|\mathbf{x}\|'_{\omega,\infty}$, we obtain

$$\|T^\omega - L^\omega\| \le 2\delta(\omega).$$

Hence,

$$\|T^\omega - L^\omega\| < \|(T^\omega)^{-1}\|^{-1},$$

which shows that L^ω is invertible for $\omega \in \tilde{\Omega}$, with inverse given by

$$(L^\omega)^{-1} = \sum_{n=0}^{\infty} \big(\mathrm{Id} - (T^\omega)^{-1}L^\omega\big)^n (T^\omega)^{-1}.$$

It follows from this identity that property 2 in Theorem 3.1 holds with the operator T^ω replaced by L^ω (the strong measurability condition follows from the explicit formula for

$(L^\omega)^{-1}$). Moreover,

$$\|(L^\omega)^{-1}\| \le \frac{1}{(\rho(\omega)+1)^{-1} - 2\delta(\omega)}.$$

Since the function on the right-hand side is measurable and θ-invariant, it follows from Theorem 3.2 that the cocycle Ψ admits a tempered exponential dichotomy. □

4. A Grobman–Hartman theorem

The goal of this section is to obtain a Grobman–Hartman theorem for the notion of tempered exponential dichotomy. In the remainder of the paper we always consider generators A such that $A(\omega)$ is invertible for all $\omega \in \Omega$ and $\omega \mapsto A(\omega)^{-1}$ is strongly measurable.

Let Φ be a strongly measurable cocycle admitting a tempered exponential dichotomy and let $\|\cdot\|_\omega$ be the measurable family of norms constructed in (9). We denote by Z the set of all measurable maps $u\colon \Omega \times X \to X$ such that the function $u_\omega\colon X \to X$ defined by $u_\omega(x) = u(\omega, x)$ is continuous for μ-almost every $\omega \in \Omega$ and

$$\|u\|' := \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in X} \|u_\omega(x)\|_\omega < +\infty.$$

One can easily verify that $(Z, \|\cdot\|')$ is a Banach space.

Although the measure μ need not be ergodic, one can always consider an ergodic decomposition and obtain results for each measure in the ergodic decomposition. Thus, without loss of generality one can and will assume in what follows that μ is ergodic.

Theorem 4.1: *Let Φ be a strongly measurable cocycle admitting a tempered exponential dichotomy and let $f\colon \Omega \times X \to X$ be a measurable map such that, for μ-almost every $\omega \in \Omega$,*

(1) *$f_\omega = f(\omega, \cdot)$ is continuous and $F_\omega = A_\Phi(\omega) + f_\omega$ is a homeomorphism;*
(2) *there exists a constant $\delta > 0$ such that, for $x, y \in X$, we have*

$$\|f_\omega(x)\| \le \delta K(\theta(\omega))^{-2} \tag{20}$$

and

$$\|f_\omega(x) - f_\omega(y)\| \le \delta K(\theta(\omega))^{-2}\|x-y\|. \tag{21}$$

Then provided that δ is sufficiently small,

(1) *there is a unique $u \in Z$ such that, for μ-almost every $\omega \in \Omega$, we have*

$$A_\Phi(\omega) \circ \hat{u}_\omega = \hat{u}_{\theta(\omega)} \circ (A_\Phi(\omega) + f_\omega), \quad \text{where } \hat{u}_\omega = \mathrm{Id} + u_\omega; \tag{22}$$

(2) *there is a unique $v \in Z$ such that, for μ-almost every $\omega \in \Omega$, we have*

$$\hat{v}_{\theta(\omega)} \circ A_\Phi(\omega) = (A_\Phi(\omega) + f_\omega) \circ \hat{v}_\omega, \quad \text{where } \hat{v}_\omega = \mathrm{Id} + v_\omega; \tag{23}$$

(3) *for μ-almost every $\omega \in \Omega$, the maps $\hat{u}_\omega$ and $\hat{v}_\omega$ are homeomorphism and*

$$\hat{u}_\omega \circ \hat{v}_\omega = \hat{v}_\omega \circ \hat{u}_\omega = \mathrm{Id}. \tag{24}$$

Proof:

(1) We note that the first identity in (22) is equivalent to

$$A_\Phi(\omega) \circ u_\omega - u_{\theta(\omega)} \circ F_\omega = f_\omega. \tag{25}$$

For each $\omega \in \Omega$, let

$$B(\omega) = A_\Phi(\omega)|\operatorname{Im}\Pi^s(\omega) \quad \text{and} \quad C(\omega) = A_\Phi(\omega)|\operatorname{Im}\Pi^u(\omega).$$

Clearly, the operators

$$B(\omega) : \operatorname{Im}\Pi^s(\omega) \to \operatorname{Im}\Pi^s(\theta(\omega)) \quad \text{and} \quad C(\omega) : \operatorname{Im}\Pi^u(\omega) \to \operatorname{Im}\Pi^u(\theta(\omega))$$

are invertible. Write $u_\omega = (b_\omega, c_\omega)$ and $f_\omega = (g_\omega, h_\omega)$, where

$$b_\omega = \Pi^s(\omega)u_\omega, \quad c_\omega = \Pi^u(\omega)u_\omega, \quad g_\omega = \Pi^s(\theta(\omega))f_\omega, \quad h_\omega = \Pi^u(\theta(\omega))f_\omega.$$

Then, (25) holds for μ-almost every $\omega \in \Omega$ if and only if $(b_\omega, c_\omega) = (\bar{b}_\omega, \bar{c}_\omega)$ for μ-almost every $\omega \in \Omega$, where

$$\bar{b}_\omega = (B(\eta) \circ b_\eta - g_\eta) \circ F_\eta^{-1}, \tag{26}$$

with $\eta = \theta^{-1}(\omega)$, and

$$\bar{c}_\omega = C(\omega)^{-1} \circ (c_{\theta(\omega)} \circ F_\omega + h_\omega). \tag{27}$$

Given $u = (b_\omega, c_\omega)_{\omega \in \Omega} \in Z$, we define $S(u) = (\bar{b}_\omega, \bar{c}_\omega)_{\omega \in \Omega}$. For each $z \in X$, we have

$$\begin{aligned}
\|\bar{b}_\omega(z)\|_\omega &\le \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)B(\eta)b_\eta(F_\eta^{-1}(z))\|e^{\alpha(\omega)n}\big)\\
&\quad + \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)g_\eta(F_\eta^{-1}(z))\|e^{\alpha(\omega)n}\big)\\
&= \sup_{n\ge 0}\big(\|\mathcal{B}(n+1,\eta)b_\eta(F_\eta^{-1}(z))\|e^{\alpha(\omega)n}\big)\\
&\quad + \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)g_\eta(F_\eta^{-1}(z))\|e^{\alpha(\omega)n}\big)\\
&\le e^{-\alpha(\omega)}\sup_{n\ge 0}\big(\|\mathcal{B}(n+1,\eta)b_\eta(F_\eta^{-1}(z))\|e^{\alpha(\omega)(n+1)}\big)\\
&\quad + \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)\|\cdot\|g_\eta\|_\infty e^{\alpha(\omega)n}\big),
\end{aligned}$$

where $\mathcal{B}(n,\omega) = \Phi(n,\omega)\Pi^s(\omega)$ and where $\|\cdot\|_\infty$ is the usual supremum norm. Using (4) and (20), we conclude that

$$\|\bar{b}_\omega(z)\|_\omega \le e^{-\alpha(\omega)}\|b_\eta(F_\eta^{-1}(z))\|_\eta + \delta$$

for μ-almost every $\omega \in \Omega$ and thus,

$$\|\bar{b}\|' \leq \|b\|' + \delta < +\infty. \tag{28}$$

Similarly, for each $z \in X$, we have

$$\begin{aligned}
\|\bar{c}_\omega(z)\|_\omega &\leq \sup_{n\leq 0}\big(\|\mathcal{C}(n,\omega)C(\omega)^{-1}c_{\theta(\omega)}(F_\omega(z))\|e^{-\alpha(\omega)n}\big) \\
&\quad + \sup_{n\leq 0}\big(\|\mathcal{C}(n,\omega)C(\omega)^{-1}h_\omega(z)\|e^{-\alpha(\omega)n}\big) \\
&= \sup_{n\leq 0}\big(\|\mathcal{C}(n-1,\theta(\omega))c_{\theta(\omega)}(F_\omega(z))\|e^{-\alpha(\omega)n}\big) \\
&\quad + \sup_{n\leq 0}\big(\|\mathcal{C}(n-1,\theta(\omega))h_\omega(z)\|e^{-\alpha(\omega)n}\big) \\
&\leq e^{-\alpha(\omega)}\sup_{n\leq 0}\big(\|\mathcal{C}(n-1,\theta(\omega))c_{\theta(\omega)}(F_\omega(z))\|e^{-\alpha(\omega)(n-1)}\big) \\
&\quad + \sup_{n\leq 0}\big(\|\mathcal{C}(n-1,\theta(\omega))\|\cdot\|h_\omega\|_\infty e^{-\alpha(\omega)n}\big),
\end{aligned}$$

where $\mathcal{C}(n,\omega) = \Phi(n,\omega)\Pi^u(\omega)$. Using (5) and (20), we obtain

$$\|\bar{c}_\omega(z)\|_\omega \leq e^{-\alpha(\omega)}\|c_{\theta(\omega)}(F_\omega(z))\|_{\theta(\omega)} + \delta e^{-\alpha(\omega)}$$

for μ-almost every $\omega \in \Omega$ and thus,

$$\|\bar{c}\|' \leq \|c\|' + \delta < +\infty. \tag{29}$$

It follows from (28) and (29) that $S(u) \in Z$ for every $u \in Z$ (the measurability of $S(u)$ follows readily from (26) and (27)).

Now we show that S is a contraction. Take $u_1 = (b_{1,\omega}, c_{1,\omega})_{\omega\in\Omega}$ and $u_2 = (b_{2,\omega}, c_{2,\omega})_{\omega\in\Omega}$ in Z. For each $z \in X$, we have

$$\begin{aligned}
\|\bar{b}_{1,\omega}(z) - \bar{b}_{2,\omega}(z)\|_\omega &= \sup_{n\geq 0}\big(\|\mathcal{B}(n+1,\eta)(b_{1,\eta} - b_{2,\eta})(F_\eta^{-1}(z))\|e^{\alpha(\omega)n}\big) \\
&\leq e^{-\alpha(\omega)}\|(b_{1,\eta} - b_{2,\eta})(F_\eta^{-1}(z))\|_\eta \\
&\leq e^{-a}\|(b_{1,\eta} - b_{2,\eta})(F_\eta^{-1}(z))\|_\eta,
\end{aligned}$$

for μ-almost every $\omega \in \Omega$, where a is the constant value of the θ-invariant function α on Ω up to a set of measure zero (recall that the measure μ is assumed to be ergodic). Therefore,

$$\|\bar{b}_1 - \bar{b}_2\|' \leq e^{-a}\|b_1 - b_2\|'. \tag{30}$$

Similarly,

$$\begin{aligned}
&\|\bar{c}_{1,\omega}(z) - \bar{c}_{2,\omega}(z)\|_\omega \\
&= \sup_{n\le 0}\big(\|\mathcal{C}(n-1,\theta(\omega))(c_{1,\theta(\omega)} - c_{2,\theta(\omega)})(F_\omega(z))\|e^{-\alpha(\omega)n}\big) \\
&\le e^{-\alpha(\omega)}\|(c_{1,\theta(\omega)} - c_{2,\theta(\omega)})(F_\omega(z))\|_{\theta(\omega)} \\
&\le e^{-a}\|(c_{1,\theta(\omega)} - c_{2,\theta(\omega)})(F_\omega(z))\|_{\theta(\omega)},
\end{aligned}$$

for $z \in X$ and μ-almost every $\omega \in \Omega$. Hence,

$$\|\bar{c}_1 - \bar{c}_2\|' \le e^{-a}\|c_1 - c_2\|'. \tag{31}$$

By (30) and (31), we have

$$\|S(u_1) - S(u_2)\|' \le e^{-a}\|u_1 - u_2\|'$$

and so S is a contraction. We conclude that S has a unique fixed point in Z which yields the first statement in the theorem.

(2) We note that (23) is equivalent to

$$v_{\theta(\omega)} \circ A_\Phi(\omega) - A_\Phi(\omega) \circ v_\omega = f_\omega \circ \hat{v}_\omega. \tag{32}$$

Write $v_\omega = (d_\omega, e_\omega)$, where $d_\omega = \Pi^s(\omega)v_\omega$ and $e_\omega = \Pi^u(\omega)v_\omega$. Then, (32) holds for μ-almost every $\omega \in \Omega$ if and only if $(d_\omega, e_\omega) = (\bar{d}_\omega, \bar{e}_\omega)$ for μ-almost every $\omega \in \Omega$, where

$$\bar{d}_\omega = (B(\eta) \circ d_\eta + g_\eta \circ \hat{v}_\eta) \circ A_\Phi(\eta)^{-1}, \tag{33}$$

with $\eta = \theta^{-1}(\omega)$, and

$$\bar{e}_\omega = C(\omega)^{-1} \circ (e_{\theta(\omega)} \circ A_\Phi(\omega) - h_\omega \circ \hat{v}_\omega). \tag{34}$$

Given $v = (d_\omega, e_\omega)_{\omega \in \Omega} \in Z$, we define $T(v) = (\bar{d}_\omega, \bar{e}_\omega)_{\omega\in\Omega}$. For each $z \in X$, we have

$$\begin{aligned}
\|\bar{d}_\omega(z)\|_\omega &\le \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)B(\eta)d_\eta(A_\Phi(\eta)^{-1}z)\|e^{\alpha(\omega)n}\big) \\
&\quad + \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)g_\eta(\hat{v}_\eta(A_\Phi(\eta)^{-1}z))\|e^{\alpha(\omega)n}\big) \\
&\le e^{-\alpha(\omega)}\sup_{n\ge 0}\big(\|\mathcal{B}(n+1,\eta)d_\eta(A_\Phi(\eta)^{-1}z)\|e^{\alpha(\omega)(n+1)}\big) \\
&\quad + \sup_{n\ge 0}\big(\|\mathcal{B}(n,\omega)\| \cdot \|g_\eta\|_\infty e^{\alpha(\omega)n}\big).
\end{aligned}$$

Using (10) and (20), we conclude that

$$\|\bar{d}_\omega(z)\|_\omega \le e^{-\alpha(\omega)}\|d_\eta(A_\Phi(\eta)^{-1}z)\|_\eta + \delta$$

for μ-almost every $\omega \in \Omega$ and thus,

$$\|\bar{d}\|' \leq \|d\|' + \delta < +\infty. \tag{35}$$

Similarly, for each $z \in X$, we have

$$\begin{aligned}
\|\bar{e}_\omega(z)\|_\omega &\leq \sup_{n\leq 0}\big(\|\mathcal{C}(n,\omega)C(\omega)^{-1}e_{\theta(\omega)}(A_\Phi(\omega)z)\|e^{-\alpha(\omega)n}\big) \\
&\quad + \sup_{n\leq 0}\big(\|\mathcal{C}(n,\omega)C(\omega)^{-1}h_\omega(\bar{v}_\omega(z))\|e^{-\alpha(\omega)n}\big) \\
&\leq e^{-\alpha(\omega)}\sup_{n\leq 0}\big(\|\mathcal{C}(n-1,\theta(\omega))e_\omega(A_\Phi(\omega)z)\|e^{-\alpha(\omega)(n-1)}\big) \\
&\quad + \sup_{n\leq 0}\big(\|\mathcal{C}(n-1,\theta(\omega))\|\cdot\|h_\omega\|_\infty e^{-\alpha(\omega)n}\big).
\end{aligned}$$

By (11) and (20),

$$\|\bar{e}_\omega(z)\|_\omega \leq e^{-\alpha(\omega)}\|e_{\theta(\omega)}(A_\Phi(\omega)z)\|_{\theta(\omega)} + \delta e^{-\alpha(\omega)}$$

for μ-almost every $\omega \in \Omega$ and thus,

$$\|\bar{e}\|' \leq \|e\|' + \delta < +\infty. \tag{36}$$

It follows from (35) and (36) that $T(v) \in Z$ for every $v \in Z$ (the measurability of $T(v)$ follows readily from (33) and (34)).

Now we show that T is a contraction. Take $v_1 = (d_{1,\omega}, e_{1,\omega})_{\omega\in\Omega}$ and $v_2 = (d_{2,\omega}, e_{2,\omega})_{\omega\in\Omega}$ in Z. Let $G_{i,\omega} = \hat{v}_{i,\omega} \circ A_\Phi(\omega)^{-1}$ for $i = 1, 2$. For each $z \in X$, using (10) and (21) we obtain

$$\begin{aligned}
&\|\bar{d}_{1,\omega}(z) - \bar{d}_{2,\omega}(z)\|_\omega \\
&\leq \sup_{n\geq 0}\big(\|\mathcal{B}(n+1,\eta)(d_{1,\eta}-d_{2,\eta})(A_\Phi(\eta)^{-1}z)\|e^{\alpha(\omega)n}\big) \\
&\quad + \sup_{n\geq 0}\big(\|\mathcal{B}(n,\omega)(g_\eta(G_{1,\eta}(z)) - g_\eta(G_{2,\eta}(z)))\|e^{\alpha(\omega)n}\big) \\
&\leq e^{-\alpha(\omega)}\|(d_{1,\eta}-d_{2,\eta})(A_\Phi(\eta)^{-1}z)\|_\eta \\
&\quad + K(\omega)\sup_{n\geq 0}\big(\|\mathcal{B}(n,\omega)\|\cdot\|f_\eta(G_{1,\eta}(z)) - f_\eta(G_{2,\eta}(z))\|e^{\alpha(\omega)n}\big) \\
&\leq e^{-\alpha(\omega)}\|(d_{1,\eta}-d_{2,\eta})(A_\Phi(\eta)^{-1}z)\|_\eta \\
&\quad + \delta\|v_{1,\eta}(A_\Phi(\eta)^{-1}(z)) - v_{2,\eta}(A_\Phi(\eta)^{-1}(z))\|_\eta
\end{aligned}$$

for μ-almost every $\omega \in \Omega$ and thus,

$$\|\bar{d}_1 - \bar{d}_2\|' \leq e^{-a}\|d_1 - d_2\|' + \delta\|v_1 - v_2\|'. \tag{37}$$

Similarly, using (11) and (21) we obtain

$$\begin{aligned}
&\|\bar{e}_{1,\omega}(z)-\bar{e}_{2,\omega}(z)\|_\omega[0]\\
&\le \sup_{n\le 0}\big(\|\mathcal{C}(n-1,\theta(\omega))[e_{1,\theta(\omega)}(A_\Phi(\omega)z)-e_{2,\theta(\omega)}(A_\Phi(\omega)z)]\|e^{-\alpha(\omega)n}\big)\\
&\quad+\sup_{n\le 0}\big(\|\mathcal{C}(n-1,\theta(\omega))(h_\omega(\hat{v}_{1,\omega}(z))-h_\omega(\hat{v}_{2,\omega}(z)))\|e^{-\alpha(\omega)n}\big)\\
&\le e^{-\alpha(\omega)}\|e_{1,\theta(\omega)}(A_\Phi(\omega)z)-e_{2,\theta(\omega)}(A_\Phi(\omega)z)\|_{\theta(\omega)}\\
&\quad+\delta e^{-\alpha(\omega)}\|v_{1,\omega}(z)-v_{2,\omega}(z)\|_\omega
\end{aligned}$$

for $z\in X$ and μ-almost every $\omega\in\Omega$. Hence,

$$\|\bar{e}_1-\bar{e}_2\|'\le e^{-a}\|e_1-e_2\|'+\delta\|v_1-v_2\|', \tag{38}$$

It follows from (37) and (38) that for any sufficiently small δ, the operator T is a contraction on Z which yields the second statement in the theorem.

(3) In order to complete the proof, it is sufficient to establish (24). By (22) and (23), we have

$$\hat{u}_{\theta(\omega)}\circ\hat{v}_{\theta(\omega)}\circ A_\Phi(\omega)=\hat{u}_{\theta(\omega)}\circ(A_\Phi(\omega)+f_{\theta(\omega)})\circ\hat{v}_\omega=A_\Phi(\omega)\circ\hat{u}_\omega\circ\hat{v}_\omega,$$

for μ-almost every $\omega\in\Omega$. Furthermore,

$$\hat{u}_\omega\circ\hat{v}_\omega-\mathrm{Id}=v_\omega+u_\omega\circ\hat{v}_\omega$$

for μ-almost every $\omega\in\Omega$ and thus $(\hat{u}_\omega\circ\hat{v}_\omega-\mathrm{Id})_{\omega\in\Omega}\in Z$. So it follows from the uniqueness statement in Theorem 4.1 that $\hat{u}_\omega\circ\hat{v}_\omega=\mathrm{Id}$ for μ-almost every $\omega\in\Omega$. Similarly, one can establish the second equality in (24). □

Finally, we describe how the conjugacies v_ω in Theorem 4.1 vary with the perturbations f_ω. A related study can be effected for the functions u_ω. We denote each function v_ω in (23) by

$$v_f(\omega,\cdot)=v_{\omega,f}=(d_{\omega,f},e_{\omega,f})=(d_f(\omega,\cdot),e_f(\omega,\cdot)).$$

We continue to assume that the measure μ is ergodic.

Theorem 4.2: *Let Φ be a strongly measurable cocycle. Then there exists $C>0$ such that for any sufficiently small δ as in (20) and (21) we have*

$$\|v_f-v_{\bar{f}}\|'\le C\operatorname*{ess\,sup}_{\omega\in\Omega}\Big(\max\big\{K(\omega),K(\theta(\omega))\big\}\sup_{x\in X}\|f_\omega(x)-\bar{f}_\omega(x)\|_\omega\Big)$$

for any maps f and $\bar{f}$ satisfying the hypotheses of Theorem 4.1.

Proof: Write

$$\widehat{v}_{\omega,f}=\mathrm{Id}+v_{\omega,f}\quad\text{and}\quad G_{\omega,f}=\widehat{v}_{\omega,f}\circ A_\Phi(\omega)^{-1}.$$

For μ-almost every $\omega \in \Omega$ and each $x \in X$, it follows from (33) that writing $\eta = \theta^{-1}(\omega)$, we have

$$\begin{aligned}
&\|d_{\omega,f}(x) - d_{\omega,\bar{f}}(x)\|_\omega \\
&\le \sup_{n\ge 0}\left(\left\|\mathcal{B}(n,\omega)B(\eta)[d_{\eta,f}(A_\Phi(\eta)^{-1}x) - d_{\eta,\bar{f}}(A_\Phi(\eta)^{-1}x)]\right\| e^{\alpha(\omega)n}\right) \\
&\quad + \sup_{n\ge 0}\left(\left\|\mathcal{B}(n,\omega)[g_\eta(G_{\eta,f}(x)) - \bar{g}_\eta(G_{\eta,\bar{f}}(x))]\right\| e^{\alpha(\omega)n}\right) \\
&\le e^{-\alpha(\omega)}\sup_{n\ge 0}\left(\left\|\mathcal{B}(n+1,\eta)[d_{\eta,f}(A_\Phi(\eta)^{-1}x) - d_{\eta,\bar{f}}(A_\Phi(\eta)^{-1}x)]\right\| e^{\alpha(\omega)(n+1)}\right) \\
&\quad + \sup_{n\ge 0}\left(\left\|\mathcal{B}(n,\omega)[g_\eta(G_{\eta,f}(x)) - \bar{g}_\eta(G_{\eta,f}(x))]\right\| e^{\alpha(\omega)n}\right) \\
&\quad + \sup_{n\ge 0}\left(\left\|\mathcal{B}(n,\omega)[\bar{g}_\eta(G_{\eta,f}(x)) - \bar{g}_\eta(G_{\eta,\bar{f}}(x))]\right\| e^{\alpha(\omega)n}\right) \\
&\le e^{-\alpha(\omega)}\|(d_{\eta,f} - d_{\eta,\bar{f}})(A_\Phi(\eta)^{-1}x)\|_\eta + K(\omega)\|(g_\eta - \bar{g}_\eta)(G_{\eta,f}(x))\|_\eta \\
&\quad + \delta\|(\widehat{v}_{\eta,f} - \widehat{v}_{\eta,\bar{f}})(A_\Phi(\eta)^{-1}(x))\|_\eta.
\end{aligned} \tag{39}$$

Thus,

$$\begin{aligned}
\sup_{x\in X}\|d_{\omega,f}(x) - d_{\omega,\bar{f}}(x)\|_\omega &\le e^{-a}\sup_{x\in X}\|d_{\eta,f}(x) - d_{\eta,\bar{f}}(x)\|_\eta \\
&\quad + K(\omega)\sup_{x\in X}\|g_\eta(x) - \bar{g}_\eta(x)\|_\eta \\
&\quad + \delta\sup_{x\in X}\|v_{\eta,f}(x) - v_{\eta,\bar{f}}(x)\|_\eta
\end{aligned}$$

and

$$(1-e^{-a})\|d_f - d_{\bar{f}}\|' \le \operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{x\in X}\left(K(\omega)\|g_\eta(x) - \bar{g}_\eta(x)\|_\eta\right) + \delta\|v_f - v_{\bar{f}}\|'. \tag{40}$$

Proceeding in a similar manner to that in (39), we obtain

$$\begin{aligned}
\sup_{x\in X}\|e_{\omega,f}(x) - e_{\omega,\bar{f}}(x)\|_\omega &\le e^{-a}\sup_{x\in X}\|e_{\theta(\omega),f}(x) - e_{\theta(\omega),\bar{f}}(x)\|_{\theta(\omega)} \\
&\quad + K(\omega)\sup_{x\in X}\|h_\omega(x) - \bar{h}_\omega(x)\|_\omega \\
&\quad + \delta\sup_{x\in X}\|v_{\omega,f}(x) - v_{\omega,\bar{f}}(x)\|_\omega
\end{aligned}$$

and

$$(1-e^{-a})\|e_f - e_{\bar{f}}\|' \le \operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{x\in X}\left(K(\omega)\|h_\omega(x) - \bar{h}_\omega(x)\|_\omega\right) + \delta\|v_f - v_{\bar{f}}\|'. \tag{41}$$

The desired statement follows now readily from (40) and (41). □

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