ALMOST SURE INVARIANCE PRINCIPLE FOR RANDOM PIECEWISE EXPANDING MAPS

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Abstract. We prove a fiberwise almost sure invariance principle for random piecewise expanding transformations in one and higher dimensions using recent developments on martingale techniques.

1. Introduction

The objective of this note is to prove the almost sure invariance principle (ASIP) for a large class of random dynamical systems. The random dynamics is driven by an invertible-measure preserving transformation σ of $(\Omega, \mathcal{F}, \mathbb{P})$ called the base transformation. Trajectories in the phase space X are formed by concatenations $f_\omega^n := f_{\sigma^{n-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega$ of maps from a family of maps $f_\omega : X \to X$, $\omega \in \Omega$. For a systematic treatment of these systems we refer to [2]. For sufficiently regular bounded observables $\psi_\omega : X \to \mathbb{R}$, $\omega \in \Omega$, an almost sure invariance principle guarantees that the random variables $\psi_{\sigma^n\omega} \circ f_\omega^n$ can be matched with trajectories of a Brownian motion, with the error negligible compared to the length of the trajectory. In the present paper, we consider observables defined on some measure space (X, m) which is endowed with a notion of variation. In particular, we consider examples where the observables are functions of bounded variation or quasi-Hölder functions on a compact subset X of $\mathbb{R}^n$. We emphasize that our setting is quite similar to that in [3], where the maps f_ω are called random Lasota-Yorke maps.

In a more general setting and under suitable assumptions, Kifer proved in [11] central limit theorems (CLT) and laws of iterated logarithm; we will briefly compare Kifer's assumptions with ours in Remark 2 below. In [11, Remark 2.7], Kifer claimed without proof (see [11, Remark 4.1]) a random functional CLT, i.e. the weak invariance principle (WIP), and also a strong version of the WIP with almost sure convergence, namely the almost sure invariance principle (ASIP), referring to techniques of Philip and Stout [13].

Here we present a proof of the ASIP for our class of random transformations, following a method recently proposed by Cuny and Merlèvede [6]. This method is particularly powerful when applied to non-stationary dynamical systems; it was successfully used in [9] for a large class of *sequential* systems with some expanding features and for which only the CLT was previously known [5]. We stress that ω-fibered random dynamical systems discussed above are also non-stationary since we use ω-dependent sample measures (see below) on the underlying probability space.

Date: November 15, 2016.

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The technique of Cuny and Merlèvede is based on martingale approximation; it was shown in [9] how to satisfy one of the main assumptions in [6] by using a result of Sprindzuk [15], which basically consists of getting an almost sure bound when the latter is known to hold in the L^1 norm. To prove such a result we also need two other ingredients: (i) the *error* in the martingale approximation must be bounded in a suitable Banach space; (ii) the *quenched* correlations with respect to the sample measures must decay with a summable rate.

We now compare our assumptions and results with those in Kifer's paper [11]. Kifer used a martingale approximation, but the martingale approximation error in [11] is given in terms of an infinite series (see the error g_ω in equation (4.18) in [11]), which appears difficult to estimate under general assumptions. Instead, our martingale approximation error term is explicitly given in terms of a finite sum (see (14)), and as mentioned above we can bound it easily. Furthermore, Kifer invoked a rate of mixing, but to deal with it he assumed strong conditions (ϕ-mixing and α-mixing), which are very hard to check on concrete examples. We use instead quenched decay of correlations on a space of regular observables, for example, bounded variation or quasi-Hölder and L^∞ functions (exponential decay was shown by Buzzi [3]), with an addition: the constant that scales the norm of the observable in the decay rate is independent of the noise ω; we can then satisfy the hypotheses of Sprindzuk's result. Further comparisons are deferred to Remark 2.

The rate which we obtain by approximating our process with a sum of i.i.d Gaussian variables (the content of the ASIP) is of order $n^{1/4}$, which is, up to logarithmic corrections, a rate previously obtained for deterministic uniformly expanding systems [8].

2. Preliminaries and statement of the main results

2.1. **Preliminaries.** We introduce in this section the fiber maps and the associated function spaces which we will use to form the random concatenations. We will call them *random expanding transformations*, or *random Lasota-Yorke maps.* We will refer to and use the general assumptions for these maps as proposed by Buzzi [3] in order to use his results on quenched decay of correlations. However, we will strengthen a few of those assumptions with the aim of obtaining limit theorems. Our additional conditions are similar to those called *Dec* and *Min* in the paper [5], and which were used to establish and recover a property akin to quasi-compactness for the composition of transfer operators.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\sigma : \Omega \to \Omega$ be an invertible $\mathbb{P}$-preserving transformation. We will assume that $\mathbb{P}$ is ergodic. Moreover, let $(X, \mathcal{B})$ be a measurable space endowed with a probability measure m and a notion of a variation $\mathrm{var}\colon L^1(X, m) \to [0, \infty]$ which satisfies the following conditions:

(V1) $\mathrm{var}(th) = |t| \,\mathrm{var}(h)$;
(V2) $\mathrm{var}(g + h) \le \mathrm{var}(g) + \mathrm{var}(h)$;
(V3) $\|h\|_\infty \le C_{\mathrm{var}}(\|h\|_1 + \mathrm{var}(h))$ for some constant $1 \le C_{\mathrm{var}} < \infty$;
(V4) for any $C > 0$, the set $\{h\colon X \to \mathbb{R} : \|h\|_1 + \mathrm{var}(h) \le C\}$ is $L^1(m)$-compact;
(V5) $\mathrm{var}(1_X) < \infty$, where 1_X denotes the function equal to 1 on X;
(V6) $\{h\colon X \to \mathbb{R}_+ : \|h\|_1 = 1 \text{ and } \mathrm{var}(h) < \infty\}$ is $L^1(m)$-dense in $\{h\colon X \to \mathbb{R}_+ : \|h\|_1 = 1\}$.
(V7) there exists $K_{\mathrm{var}} < \infty$ such that

$$\mathrm{var}(fg) + \|fg\|_1 \le K_{\mathrm{var}}(\mathrm{var}(f) + \|f\|_1)(\mathrm{var}(g) + \|g\|_1), \quad \text{for every } f, g \in BV. \tag{1}$$

(V8) for any $f \in L^1(X, m)$ such that $\operatorname{essinf} f > 0$, we have $\mathrm{var}(1/f) \le \frac{\mathrm{var}(f)}{(\operatorname{essinf} f)^2}$.

We denote by $BV = BV(X,m)$ the space of all $h \in L^1(X,m)$ such that $\text{var}(h) < \infty$. It is well known that BV is a Banach space with respect to the norm

$$\|h\|_{BV} = \text{var}(h) + \|h\|_1.$$

On several occasions we will also consider the following norm

$$\|h\|_{var} = \text{var}(h) + \|h\|_\infty,$$

on BV which (although different) is equivalent to $\|\cdot\|_{BV}$.

Let $f_\omega \colon X \to X$, $\omega \in \Omega$ be a collection of mappings on X. The associated skew product transformation $\tau \colon \Omega \times X \to \Omega \times X$ is defined by

$$\tau(\omega, x) = (\sigma(\omega), f_\omega(x)). \tag{2}$$

Each transformation f_ω induces the corresponding transfer operator $\mathcal{L}_\omega$ acting on $L^1(X,m)$ and defined by the following duality relation

$$\int_X (\mathcal{L}_\omega \phi)\psi \, dm = \int_X \phi(\psi \circ f_\omega)\, dm, \quad \phi \in L^1(X,m),\ \psi \in L^\infty(X,m).$$

For each $n \in \mathbb{N}$ and $\omega \in \Omega$, set

$$f_\omega^n = f_{\sigma^{n-1}(\omega)} \circ \cdots \circ f_\omega \quad \text{and} \quad \mathcal{L}_\omega^n = \mathcal{L}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}_\omega.$$

We say that the family of maps f_ω, $\omega \in \Omega$ (or the associated family of transfer operators $\mathcal{L}_\omega$, $\omega \in \Omega$) is *uniformly good* if:

(H1) The map $(\omega, x) \mapsto (\mathcal{L}_\omega H(\omega, \cdot))(x)$ is $\mathbb{P} \times m$-measurable for every $\mathbb{P} \times m$-measurable function H such that $H(\omega, \cdot) \in L^1(m)$ for a.e. $\omega \in \Omega$;

(H2) There exists $C > 0$ such that

$$\|\mathcal{L}_\omega \phi\|_{BV} \le C\|\phi\|_{BV}$$

for $\phi \in BV$ and $\mathbb{P}$ a.e. $\omega \in \Omega$.

(H3) For $\mathbb{P}$ a.e. $\omega \in \Omega$,

$$\sup_{n\ge 0} \|\phi_{n+1} \circ f_{\sigma^n(\omega)}\|_{BV} < \infty,$$

whenever $\{(\phi_n)_{n\ge 0}\} \subset BV$ and $\sup_n \|\phi_n\|_{BV} < \infty$.

(H4) There exist $K, \lambda > 0$ such that

$$\|\mathcal{L}_\omega^n \phi\|_{BV} \le K e^{-\lambda n}\|\phi\|_{BV},$$

for $n \ge 0$, $\mathbb{P}$ a.e. $\omega \in \Omega$ and $\phi \in BV$ such that $\int \phi \, dm = 0$.

(H5) There exists $c > 0$ such that

$$\mathcal{L}_\omega^n 1_X \ge c, \quad \text{for } \mathbb{P} \text{ a.e. } \omega \in \Omega \text{ and } n \in \mathbb{N}.$$

Using (H1), (H2), and (H4) we can prove the existence of a unique random ACIM $h : \Omega \times X \to \mathbb{R}$, with uniformly bounded fibres h_ω.

Proposition 1. *Let f_ω, $\omega \in \Omega$ be a uniformly good family of maps on X. Then there exist a unique measurable and nonnegative function $h \colon \Omega \times X \to \mathbb{R}$ with the property that $h_\omega := h(\omega, \cdot) \in BV$, $\int h_\omega \, dm = 1$, $\mathcal{L}(h_\omega) = h_{\sigma(\omega)}$ for a.e. $\omega \in \Omega$*

$$\operatorname{esssup}_{\omega\in\Omega} \|h_\omega\|_{BV} < \infty. \tag{3}$$

Proof. Let

$$Y = \Big\{ v\colon \Omega \times X \to \mathbb{R} : v \text{ measurable, } v_\omega := v(\omega,\cdot) \in BV \text{ and } \operatorname{esssup}_{\omega\in\Omega} \|v_\omega\|_{BV} < \infty \Big\}.$$

Then, Y is a Banach space with respect to the norm

$$\|v\|_\infty := \operatorname{esssup}_{\omega\in\Omega} \|v_\omega\|_{BV}.$$

Moreover, let Y_1 be the set of all $v \in Y$ such that $\int v_\omega \, dm = 1$ and $v_\omega \geq 0$ for a.e. $\omega \in \Omega$. It is easy to verify that Y_1 is a closed subset of Y and thus a complete metric space. We define a map $\mathcal{L}\colon Y_1 \to Y_1$ by

$$(\mathcal{L}(v))_\omega = \mathcal{L}_{\sigma^{-1}(\omega)} v_{\sigma^{-1}(\omega)}, \quad \omega \in \Omega, \ v \in Y_1.$$

Note that it follows from (H2) that

$$\|\mathcal{L}(v)\|_\infty = \operatorname{esssup}_{\omega\in\Omega} \|(\mathcal{L}(v))_\omega\|_{BV} \leq C \operatorname{esssup}_{\omega\in\Omega} \|v_{\sigma^{-1}(\omega)}\|_{BV} = C\|v\|_\infty.$$

Furthermore,

$$\int (\mathcal{L}(v))_\omega \, dm = \int \mathcal{L}_{\sigma^{-1}(\omega)} v_{\sigma^{-1}(\omega)} \, dm = \int v_{\sigma^{-1}(\omega)} \, dm = 1,$$

for a.e. $\omega \in \Omega$. Hence, $\mathcal{L}$ is well-defined. Similarly,

$$\|\mathcal{L}(v) - \mathcal{L}(w)\|_\infty \leq C\|v - w\|_\infty, \quad \text{for } v, w \in Y_1$$

which shows that $\mathcal{L}$ is continuous. Choose $n_0 \in \mathbb{N}$ such that $Ke^{-\lambda n_0} < 1$. Take arbitrary $v, w \in Y_1$ and note that by (H4),

$$\begin{aligned} \|\mathcal{L}^{n_0}(v) - \mathcal{L}^{n_0}(w)\|_\infty &= \operatorname{esssup}_{\omega\in\Omega} \|\mathcal{L}^{n_0}_{\sigma^{-n_0}(\omega)}(v_{\sigma^{-n_0}(\omega)} - w_{\sigma^{-n_0}(\omega))}\|_{BV} \\ &\leq Ke^{-\lambda n_0} \operatorname{esssup}_{\omega\in\Omega} \|v_{\sigma^{-n_0}(\omega)} - w_{\sigma^{-n_0}(\omega)}\|_{BV} = Ke^{-\lambda n_0}\|v - w\|_\infty. \end{aligned}$$

Hence, $\mathcal{L}^{n_0}$ is a contraction on Y_1 and thus has a unique fixed point $\tilde{h} \in Y_1$. Set

$$h_\omega := \frac{1}{n_0}\tilde{h}_\omega + \frac{1}{n_0}\mathcal{L}_{\sigma^{-1}(\omega)}(\tilde{h}_{\sigma^{-1}(\omega)}) + \ldots + \frac{1}{n_0}\mathcal{L}^{n_0-1}_{\sigma^{-(n_0-1)}\omega}(\tilde{h}_{\sigma^{-(n_0-1)}\omega}), \quad \omega \in \Omega.$$

Then, h is measurable, nonnegative, $\int h_\omega \, dm = 1$ and a simple computation yields $\mathcal{L}(h_\omega) = h_{\sigma(\omega)}$. Finally, by (H2) we have that

$$\operatorname{esssup}_{\omega\in\Omega} \|h_\omega\|_{BV} \leq \frac{C^{n_0} - 1}{n_0(C-1)} \operatorname{esssup}_{\omega\in\Omega} \|\tilde{h}_\omega\|_{BV} < \infty.$$

Thus, we have established existence of h. The uniqueness is obvious since each h satisfying the assertion of the theorem is a fixed point of $\mathcal{L}$ and thus also of $\mathcal{L}^{n_0}$ which implies that it must be unique. □

We note that [4, 3] prove the above existence result with weaker control on the properties of f_ω, and obtain existence of a random ACIM $\{h_\omega\}_{\omega\in\Omega}$ under less restrictive conditions. Indeed, those results don't require (H4) and in addition (H2) is allowed to hold with $C = C(\omega)$ such that $\log C \in L^1(\mathbb{P})$.

We now describe a large class of examples of good families of maps f_ω, $\omega \in \Omega$. We first show that they satisfy properties (H1)–(H3); this will crucially depend on the choice of the function space. We then give additional requirements in order for those maps to satisfy condition (H4), also called *Dec* in [5] when applied to sequential systems, and condition (H5), named *Min* in [5].

2.2. **Examples.**

2.2.1. *Random Lasota-Yorke maps.* Take $X = [0,1]$, a Borel σ-algebra $\mathcal{B}$ on $[0,1]$ and the Lebesgue measure m on $[0,1]$. Furthermore, let

$$\operatorname{var}(g) = \inf_{h=g(mod\ m)} \sup_{0=s_0<s_1<\ldots<s_n=1} \sum_{k=1}^{n} |h(s_k) - h(s_{k-1})|.$$

It is well known that var satisfies properties (V1)–(V8) with $C_{\text{var}}, K_{\text{var}} = 1$. For a piecewise C^2 $f : [0,1] \to [0,1]$, set $\delta(f) = \operatorname{essinf}_{x\in[0,1]}|f'|$. Consider now a finitely-valued, measurable map $\omega \mapsto f_\omega$, $\omega \in \Omega$ of piecewise C^2 maps on $[0,1]$ satisfying (H1) such that

$$\sup_{\omega\in\Omega} N(f_\omega) =: N < \infty, \quad \inf_{\omega\in\Omega} \delta(f_\omega) =: \delta > 1, \quad \text{and} \ \sup_{\omega\in\Omega} |f''_\omega|_\infty =: D < \infty.$$

It is proved in [3] that the family f_ω, $\omega \in \Omega$ satisfies (H2) with

$$C = 4\Big(\frac{N}{\delta} \vee 1\Big)\Big(\frac{D}{\delta^2} \vee 1\Big)\Big(\frac{1}{\delta} \vee 1\Big), \tag{4}$$

where for any two real-valued functions g_1 and g_2, $g_1 \vee g_2 = \max\{g_1, g_2\}$, and (V8) has been used for the bound $\operatorname{var}(1/f') \leq \frac{D}{\delta^2}$. We note that since $N < \infty$, condition (H3) holds.

The uniform decay rate (H4) has been treated in Propositions 2.10 and 2.11 in [5]. There, sufficient conditions were stated for sequential dynamical systems, but these conditions can be can be easily adapted to our random setting. Conze and Raugi [5] propose two types of conditions, either of which yields (H4). The first *local* type requires the existence of a fiber map, say f_0, whose transfer operator $\mathcal{L}_0$ is quasi-compact and exact. Defining a distance between two transfer operators $\mathcal{L}_1, \mathcal{L}_2$, as $d(\mathcal{L}_1, \mathcal{L}_2) := \sup_{\phi\in BV, \|\phi\|_{BV}\leq 1} ||\mathcal{L}_1\phi - \mathcal{L}_2\phi||_1$, it was proved in [5] that there is a neighborhood U_0 of $\mathcal{L}_0$ such that all allowed concatenations of transfer operators drawn from U_0 verify (H4) as well.

To establish (H4) for the second, more general, *nonlocal* type of random dynamics we require two conditions:

- Random covering: Let A_ω denote the collection of intervals of monotonicity for the map f_ω and define $A^n_\omega = \bigvee_{j=0}^{n-1} (f^j_\omega)^{-1} A_{\sigma^j\omega}$. We will say that the random Lasota-Yorke maps $\{f_\omega\}_{\omega\in\Omega}$ are *covering* if for each $n \geq 0$, $\omega \in \Omega$, and $J \in A^n_\omega$ there is an n_0 such that $f^{n_0}_\omega(J) = [0,1]$.
- Uniform Lasota-Yorke inequality: There exist $n \in \mathbb{N}$, $0 < \rho < 1$ and $B > 0$ such that for a.e. $\omega \in \Omega$, $\|\mathcal{L}^n_\omega f\|_{BV} \leq \rho\|f\|_{BV} + B\|f\|_1$.

For a fixed covering Lasota-Yorke map, Liverani [12, Theorem 3.6] established exponential decay of correlations for observables of bounded variation, using cone techniques and the property that the unique random invariant density is uniformly bounded below [12, Lemma 4.2]. His results, in particular [12, Lemma 3.5], which determines the rate of correlation decay, are directly applicable in our setting of random composition of finitely many Lasota-Yorke maps. A direct consequence of this decay of correlations result is exactness of the sequences $(f_{\sigma^j\omega})_{j\geq 0}$ for every $\omega \in \Omega$. That is, for every $\phi \in BV$ with $\int \phi\, dm = 0$, $\lim_{n\to\infty} \|\mathcal{L}^n_\omega\phi\|_1 = 0$; see for example the proof of Proposition 3.6 in [7].

The proof of (H4) now follows as the proof of [5, Proposition 2.11]. Indeed exactness, together with the compactness condition (for every sequence $(\mathcal{L}_{\omega_j})_{j\in\mathbb{N}}$, there exists a convergent subsequence) ensures that for every $\varepsilon_0 > 0$ and every $\omega \in \Omega$ there exists $q \in \mathbb{N}$ such that for every $\phi \in BV$ with $\int \phi\, dm = 0$ and every $j \in \mathbb{N}$, $\|\mathcal{L}^q_{\sigma^j\omega}\phi\|_1 \leq \varepsilon_0\|\phi\|_{BV}$, via a diagonal argument. This property, combined with the uniform Lasota-Yorke inequality implies (H4) essentially as in the proof of [5, Proposition 2.7].

With the random covering and uniform Lasota-Yorke conditions introduced above, and assuming $\text{esssup}_{\omega\in\Omega} |f'_\omega|_\infty \le C'$, it follows from Proposition 2 in [1] that (H5) holds. We note that [5] demonstrated that (H5) holds for compositions of β-transformations with β selected from an appropriate interval of values, and that [9] stated similar sufficient conditions for (H5) for sequences of Lasota-Yorke maps that are translations of a fixed Lasota-Yorke map or small perturbations of a fixed Lasota-Yorke map.

2.2.2. *Random piecewise expanding maps.* In higher dimensions, the properties (V1)-(V8) can be checked for a so-called quasi-Hölder space, which in particular is injected in L^∞ (condition (V3)) and has the algebra property (V7). Originally developed by Keller [10] for one-dimensional dynamics, we refer the reader to [14] for a detailed presentation of that space in higher dimensions, as well as for the proof of its main properties. In particular, using the same notation as in [14], we use the following notion of variation:

$$\text{var}(f) = \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{R}^n} \text{osc}(f, B_\varepsilon(x))\, dx,$$

where

$$\text{osc}(f, B_\varepsilon(x)) = \text{esssup}_{B_\varepsilon(x)} f - \text{essinf}_{B_\varepsilon(x)} f.$$

In [14] it is proved that this notion of variation satisfies (V1)–(V3) and (V5)–(V7) and noted that (V4) is a consequence of a result in [10]. We prove here that (V8) also holds. Assume that $\text{essinf}\, f \ge c$ and observe that

$$\begin{aligned}
\text{osc}(1/f, B_\varepsilon(x)) &= \text{esssup}_{B_\varepsilon(x)}(1/f) - \text{essinf}_{B_\varepsilon(x)}(1/f) \\
&= 1/\,\text{essinf}_{B_\varepsilon(x)}(f) - 1/\,\text{esssup}_{B_\varepsilon(x)}(f) \\
&= \frac{\text{esssup}_{B_\varepsilon(x)}(f) - \text{essinf}_{B_\varepsilon(x)}(f)}{(\text{esssup}_{B_\varepsilon(x)}(f))(\text{essinf}_{B_\varepsilon(x)}(f))} \\
&\le \frac{\text{osc}(f, B_\varepsilon(x))}{(\text{essinf}_{B_\varepsilon(x)}(f))^2} \le \frac{1}{c^2}\,\text{osc}(f, B_\varepsilon(x)),
\end{aligned}$$

which readily implies that $\text{var}(1/f) \le \frac{\text{var}(f)}{(\text{essinf}\, f)^2}$.

One can consider piecewise C^2 expanding maps on compact subdomains of $\mathbb{R}^d$ with C^2 boundary. As in the one-dimensional situation, condition (H3) holds because N is finite. In the local setting the conditions (H4) and (H5) can be verified as in the one-dimensional case using the results of [5] (see also Th. 7.7 in [9]). In the nonlocal setting, for maps of the type considered in [3, Appendix B], it is likely that (H4) and (H5) can be obtained as in the one-dimensional case.

2.3. **Further properties of the random ACIM.** Let μ_ω be the measure on X given by $d\mu_\omega = h_\omega dm$ for $\omega \in \Omega$. We have the following important consequence of (H4), which establishes the appropriate decay of correlations result that will be used later on.

Lemma 1. *There exists $K > 0$ and $\rho \in (0,1)$ such that*

$$\left| \int \mathcal{L}_\omega^n(\phi h_\omega)\psi\, dm - \int \phi\, d\mu_\omega \cdot \int \psi\, d\mu_{\sigma^n(\omega)} \right| \le K\rho^n \|\psi\|_\infty \cdot \|\phi\|_{var}, \tag{5}$$

for $n \ge 0$, $\psi \in L^\infty(X, m)$ and $\phi \in BV(X, m)$.

Proof. We consider two cases. Assume first that $\int \phi \, d\mu_\omega = \int \phi h_\omega \, dm = 0$. Then, it follows from (H4) that

$$\left| \int \mathcal{L}_\omega^n(\phi h_\omega)\psi \, dm - \int \phi \, d\mu_\omega \cdot \int \psi \, d\mu_{\sigma^n \omega} \right| = \left| \int \mathcal{L}_\omega^n(\phi h_\omega)\psi \, dm \right|$$
$$\leq \|\psi\|_\infty \cdot \|\mathcal{L}_\omega^n(\phi h_\omega)\|_1 \leq \|\psi\|_\infty \cdot \|\mathcal{L}_\omega^n(\phi h_\omega)\|_{BV} \leq K e^{-\lambda n} \|\phi h_\omega\|_{BV} \cdot \|\psi\|_\infty,$$

and thus (5) follows from (1) and (3). Now we consider the case when $\int \phi \, d\mu_\omega \neq 0$. We have

$$\begin{aligned}
&\left| \int \mathcal{L}_\omega^n(\phi h_\omega)\psi \, dm - \int \phi \, d\mu_\omega \cdot \int \psi \, d\mu_{\sigma^n(\omega)} \right| \\
&= \left| \int \mathcal{L}_\omega^n(\phi h_\omega)\psi \, dm - \int \phi h_\omega \, dm \cdot \int \psi h_{\sigma^n(\omega)} \, dm \right| \\
&\leq \|\psi\|_\infty \cdot \int \left| \left(\mathcal{L}_\omega^n(\phi h_\omega) - \left(\int \phi h_\omega \, dm \right) h_{\sigma^n(\omega)} \right) \right| dm \\
&= \|\psi\|_\infty \cdot \left| \int \phi h_\omega \, dm \right| \cdot \int \left| \mathcal{L}_\omega^n(\Phi - h_\omega) \right| dm \\
&\leq \|\psi\|_\infty \cdot \left| \int \phi h_\omega \, dm \right| \cdot \|\mathcal{L}_\omega^n(\Phi - h_\omega)\|_{BV},
\end{aligned}$$

where

$$\phi h_\omega = \left(\int \phi h_\omega \, dm \right) \Phi.$$

Note that $\int (\Phi - h_\omega) \, dm = 0$ and thus using (H4),

$$\begin{aligned}
\|\psi\|_\infty \cdot \left| \int \phi h_\omega \, dm \right| \cdot \|\mathcal{L}_\omega^n(\Phi - h_\omega)\|_{BV} &\leq K e^{-\lambda n} \|\psi\|_\infty \cdot \left| \int \phi h_\omega \, dm \right| \cdot \|\Phi - h_\omega\|_{BV} \\
&\leq K e^{-\lambda n} \|\psi\|_\infty \cdot \left\| \left(\phi - \int \phi h_\omega \, dm \right) h_\omega \right\|_{BV}.
\end{aligned}$$

Hence, it follows from (1) and (3) that

$$\left| \int \mathcal{L}_\omega^n(\phi h_\omega)\psi \, dm - \int \phi \, d\mu_\omega \cdot \int \psi \, d\mu_{\sigma^n(\omega)} \right| \leq K' e^{-\lambda n} \|\psi\|_\infty \cdot \|\phi\|_{BV}$$

for some $K' > 0$ and thus (5) follows from the observation that $\|\cdot\|_{BV} \leq \|\cdot\|_{var}$.

□

Remark 1. *We would like to emphasize that* (5) *is a special case of a more general decay of correlation result obtained in* [3] *which does not require (H4) and yields* (5) *but with* $K = K(\omega)$.

Finally, we prove that condition (H5) implies that we have a uniform lower bound for h_ω.

Lemma 2. *We have that*

$$\operatorname{essinf} h_\omega \geq c/2, \quad \text{for a.e. } \omega \in \Omega. \tag{6}$$

Proof. We note that

$$h_\omega = \mathcal{L}_{\sigma^{-n}(\omega)}^n 1_X - (\mathcal{L}_{\sigma^{-n}(\omega)}^n 1_X - h_\omega)$$

and thus it follows from (H5) that

$$\operatorname{essinf} h_\omega \geq c - \|\mathcal{L}_{\sigma^{-n}(\omega)}^n 1 - h_\omega)\|_\infty \geq c - C_{\text{var}} \|\mathcal{L}_{\sigma^{-n}(\omega)}^n 1 - h_\omega\|_{BV}. \tag{7}$$

On the other hand, it follows from (H4) and (3) that

$$\|\mathcal{L}^n_{\sigma^{-n}(\omega)}1 - h_\omega\|_{BV} = \|\mathcal{L}^n_{\sigma^{-n}(\omega)}(1 - h_{\sigma^{-n}(\omega)})\|_{BV} \le Ke^{-\lambda n}\|1 - h_{\sigma^{-n}(\omega)}\|_{BV} \le \tilde{K}e^{-\lambda n},$$

for some $\tilde{K} > 0$. Choosing n such that $C_{\text{var}}\tilde{K}e^{-\lambda n} \le c/2$, it follows from (7) that (6) holds. □

Remark 2. *We now briefly compare our setting with that in* [11]*. In the latter, the space X is replaced by a foliation $\Xi_\omega := \{\xi \in \mathbf{\Xi} : (\xi, \omega) \in \Xi\}$, where $\mathbf{\Xi}$ is some measurable space and ω belongs to the* base *space Ω. On the fibered subset Ξ it is defined the skew map $\tau(\xi, \omega) = (f_\omega\xi, \sigma\omega)$, with the associated* fiber maps *$f_\omega : \Xi_\omega \to \Xi_{\sigma\omega}$. In our situation the Ξ_ω's for all ω coincide with the set X and all $f_\omega : X \to X$ are endomorphisms of X with some regularity property; moreover the previous skew transformation will still hold on the product space $X \times \Omega$, see (2). Consequently, the conformal measure m is also allowed to depend on ω. In principle, all the arguments in the present paper could also be extended to this more general setting. However, we refrain from doing so since it would require heavy notation and more importantly since it is hard to verify that conditions like (H4) and (H5) hold in this more general setting unless fibers Ξ_ω can be identified in some natural way (like for example tangent spaces at different points on a manifold).*

2.4. **Statement of the main results.** We are now ready to state our main result. We will consider an observable $\psi \colon \Omega \times X \to \mathbb{R}$. Let $\psi_\omega = \psi(\omega, \cdot)$, $\omega \in \Omega$ and assume that

$$\sup_{\omega\in\Omega}\|\psi_\omega\|_{BV} < \infty. \tag{8}$$

We also introduce centered observable

$$\tilde{\psi}_\omega = \psi_\omega - \int \psi_\omega \, d\mu_\omega, \quad \omega \in \Omega.$$

and we consider the associated Birkhoff sum $\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega} \circ f^k_\omega$, and the variance $\tau_n^2 = \mathbb{E}_\omega\left(\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega} \circ f^k_\omega\right)^2$. The Almost Sure Invariance Principle is a matching of the trajectories of the dynamical system with a Brownian motion in such a way that the error is negligible in comparison with the Birkhoff sum. Limit theorems such as the central limit theorem, the functional central limit theorem and the law of the iterated logarithm transfer from the Brownian motion to time-series generated by observations on the dynamical system: these last results will therefore be immediate consequences of our proof of the ASIP for random Lasota-Yorke maps :

Theorem 1. *Let us consider the family of uniformly good random Lasota-Yorke maps. Then the variance τ_n^2 will grow linearly as $\tau_n^2 \sim n\Sigma^2$ and two cases will present:*
(i) either $\Sigma = 0$, and this is equivalent to the existence of $\phi \in L^2(\Omega \times X)$ such that (co-boundary condition)

$$\tilde{\psi} = \phi - \phi \circ \tau. \tag{9}$$

(ii) or $\Sigma^2 > 0$ and in this case for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $d \in (0, 1/2)$, by enlarging probability space $(X, \mathcal{B}, \mu_\omega)$ if necessary, it is possible to find a sequence $(Z_k)_k$ of independent centered Gaussian random variables such that

$$\sup_{1\le k\le n}\left|\sum_{k=1}^{n}(\tilde{\psi}_{\sigma^k\omega} \circ f^k_\omega) - \sum_{k=1}^{n} Z_k\right| = o((n^{1/2+d}(|\log n^{1/2-d}| + \log\log n^{1/2+d}))^{1/2}), \quad a.s.$$

3. Reverse martingale construction

In this section we construct the reverse martingale (or the reverse martingale difference) and establish various useful estimates that will play an important role in the rest of the paper. Indeed, the proof of our main result (Theorem 1) will be obtained as a consequence of the recent result by Cuny and Merlevède [6] applied to our reverse martingale.

For $\omega \in \Omega$ and $k \in \mathbb{N}$, let

$$\mathcal{T}_\omega^k := (f_\omega^k)^{-1}(\mathcal{B}).$$

Furthermore, for a measurable map $\phi\colon X \to \mathbb{R}$ and a σ-algebra $\mathcal{H}$ on X, $\mathbb{E}_\omega(\phi|\mathcal{H})$ will denote the conditional expectation of ϕ with respect to $\mathcal{H}$ and the measure μ_ω. Moreover, from now on, for $k \in \mathbb{N}$ and $\omega \in \Omega$ we will write $\sigma^k\omega$ instead of $\sigma^k(\omega)$. We begin by the following technical lemma.

Lemma 3. *We have*

$$\mathbb{E}_\omega(\phi \circ f_\omega^l | \mathcal{T}_\omega^n) = \left(\frac{\mathcal{L}_{\sigma^l\omega}^{n-l}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}} \right) \circ f_\omega^n, \tag{10}$$

for each $\omega \in \Omega$ and $0 \le l \le n$.

Proof. We note that the right-hand side of (10) is measurable with respect to $\mathcal{T}_\omega^n$. Take now an arbitrary $A \in \mathcal{T}_\omega^n$ and write it in the form $A = (f_\omega^n)^{-1}(B)$ for some $B \in \mathcal{B}$. We have

$$\begin{aligned}
\int_A \phi \circ f_\omega^l \, d\mu_\omega &= \int_X (\phi \circ f_\omega^l) \mathbf{1}_A \, d\mu_\omega \\
&= \int_X (\phi \circ f_\omega^l) \cdot (\mathbf{1}_B \circ f_\omega^n) \, d\mu_\omega = \int_X \phi (\mathbf{1}_B \circ f_{\sigma^l\omega}^{n-l}) \, d\mu_{\sigma^l\omega} \\
&= \int_X h_{\sigma^l\omega}\phi(\mathbf{1}_B \circ f_{\sigma^l\omega}^{n-l}) \, dm = \int_X \mathcal{L}_{\sigma^l\omega}^{n-l}(h_{\sigma^l\omega}\phi)\mathbf{1}_B \, dm \\
&= \int_X \frac{\mathcal{L}_{\sigma^l\omega}^{n-l}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}} \mathbf{1}_B \, d\mu_{\sigma^n\omega} = \int_X \left[\left(\frac{\mathcal{L}_{\sigma^l\omega}^{n-l}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}} \right) \circ f_\omega^n \right] (\mathbf{1}_B \circ f_\omega^n) \, d\mu_\omega \\
&= \int_X \left[\left(\frac{\mathcal{L}_{\sigma^l\omega}^{n-l}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}} \right) \circ f_\omega^n \right] \mathbf{1}_A \, d\mu_\omega = \int_A \left(\frac{\mathcal{L}_{\sigma^l\omega}^{n-l}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}} \right) \circ f_\omega^n \, d\mu_\omega,
\end{aligned}$$

which proves (10). □

We now return to the observable ψ_ω introduced in (8) and its centered companion $\tilde{\psi}_\omega = \psi_\omega - \int \psi_\omega \, d\mu_\omega, \quad \omega \in \Omega$.

Set

$$M_n = \tilde{\psi}_{\sigma^n\omega} + G_n - G_{n+1} \circ f_{\sigma^n\omega}, \quad n \ge 0, \tag{11}$$

where $G_0 = 0$ and

$$G_{k+1} = \frac{\mathcal{L}_{\sigma^k\omega}(\tilde{\psi}_{\sigma^k\omega} h_{\sigma^k\omega} + G_k h_{\sigma^k\omega})}{h_{\sigma^{k+1}\omega}}, \quad k \ge 0. \tag{12}$$

We emphasize that M_n and G_n depend on ω. However, in order to avoid complicating the notation, we will not make this dependence explicit. In preparation for the next proposition we need the following elementary result.

Lemma 4. *We have*

$$\mathcal{L}_\omega((\psi \circ f_\omega)\phi) = \psi \mathcal{L}_\omega \phi, \quad \textit{for } \phi \in L^1(X, m) \textit{ and } \psi \in L^\infty(X, m).$$

Proof. We first note that $(\psi \circ f_\omega)\phi \in L^1(X,m)$. Moreover, for an arbitrary $g \in L^\infty(X,m)$ we have that

$$\int_X \mathcal{L}_\omega((\psi \circ f_\omega)\phi)g\,dm = \int_X (\psi \circ f_\omega)\phi(g \circ f_\omega)\,dm = \int_X ((\psi g)\circ f_\omega)\phi\,dm = \int_X (\psi \mathcal{L}_\omega \phi)g\,dm,$$

which immediately implies the conclusion of the lemma. □

Now we prove that the sequence $(M_n \circ f_\omega^n)_n$ is a reversed martingale (or the reversed martingale difference) with respect to the sequence of σ-algebras $(\mathcal{T}_\omega^n)_n$.

Proposition 2. *We have*

$$\mathbb{E}_\omega(M_n \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = 0.$$

Proof. It follows from Lemma 3 that

$$\mathbb{E}_\omega(M_n \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = \left(\frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}M_n)}{h_{\sigma^{n+1}\omega}}\right) \circ f_\omega^{n+1}. \tag{13}$$

Moreover, by (11) we have

$$\frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}M_n)}{h_{\sigma^{n+1}\omega}} = \frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}\tilde{\psi}_{\sigma^n\omega} + h_{\sigma^n\omega}G_n - h_{\sigma^n\omega}(G_{n+1} \circ f_{\sigma^n\omega}))}{h_{\sigma^{n+1}\omega}}.$$

By Lemma 4,

$$\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}(G_{n+1} \circ f_{\sigma^n\omega})) = G_{n+1}\mathcal{L}_{\sigma^n\omega}h_{\sigma^n\omega} = G_{n+1}h_{\sigma^{n+1}\omega},$$

and thus it follows from (12) that

$$\frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}M_n)}{h_{\sigma^{n+1}\omega}} = 0.$$

This conclusion of the lemma now follows readily from (13). □

We now establish several auxiliary results that will be used in the following section. These results estimate various norms of functions related to M_n and G_n, defined in (11) and (12), respectively.

Lemma 5. *We have that*

$$\sup_{n\geq 0}\|G_n\|_{BV} < \infty.$$

Proof. By iterating (12), we obtain

$$G_n = \frac{1}{h_{\sigma^n\omega}}\sum_{j=0}^{n-1}\mathcal{L}_{\sigma^j\omega}^{n-j}(\tilde{\psi}_{\sigma^j\omega}h_{\sigma^j\omega}), \quad n \in \mathbb{N}. \tag{14}$$

We note that

$$\int \tilde{\psi}_{\sigma^j\omega}h_{\sigma^j\omega}\,dm = \int \tilde{\psi}_{\sigma^j\omega}\,d\mu_{\sigma^j\omega} = 0, \tag{15}$$

and thus it follows from (H4) that

$$\left\|\sum_{j=0}^{n-1}\mathcal{L}_{\sigma^j\omega}^{n-j}(\tilde{\psi}_{\sigma^j\omega}h_{\sigma^j\omega})\right\|_{BV} \leq K\sum_{j=0}^{n-1}e^{-\lambda(n-j)}\|\tilde{\psi}_{\sigma^j\omega}h_{\sigma^j\omega}\|_{BV},$$

for each $n \in \mathbb{N}$ which together with (V8), (1), (3), (6) and (8) implies the conclusion of the lemma. □

Lemma 6. *We have that*

$$\sup_{n\geq 0}\|M_n^2\|_{BV} < \infty.$$

Proof. In view of (8), (11) and Lemma 5, it is sufficient to show that

$$\sup_{n\geq 0} \|G_{n+1} \circ f_{\sigma^n\omega}\|_{BV} < \infty.$$

However, this follows directly from (H3) and Lemma 5. □

Lemma 7. *We have that*

$$\sup_{n\geq 0} \|\mathbb{E}_\omega(M_n^2 \circ f_\omega^n | \mathcal{T}_\omega^{n+1})\|_\infty < \infty.$$

Proof. It follows from Lemma 3 that

$$\mathbb{E}_\omega(M_n^2 \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = \left(\frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n^2)}{h_{\sigma^{n+1}\omega}} \right) \circ f_\omega^{n+1},$$

and thus, recalling (6),

$$\sup_{n\geq 0} \|\mathbb{E}_\omega(M_n^2 \circ f_\omega^n | \mathcal{T}_\omega^{n+1})\|_\infty \leq \frac{1}{c} \|\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n^2)\|_\infty.$$

Taking into account (1), (H2), (3), Lemma 6 and the fact that $\|\cdot\|_\infty \leq C_{var}\|\cdot\|_{BV}$ (see (V3)) we obtain the conclusion of the lemma. □

4. Sprindzuk's Theorem and consequences

The main tool in establishing the almost sure invariance principle is the recent result by Cuny and Merlevède (quoted in our Theorem 3 in Section 5). However, in order to verify the assumptions of that theorem we will first need to apply the following classical result due to Sprindzuk [15].

Theorem 2. *Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $(f_k)_k$ be a sequence of nonnegative and measurable functions on Ω. Moreover, let $(g_k)_k$ and $(h_k)_k$ be bounded sequences of real numbers such that $0 \leq g_k \leq h_k$. Assume that there exists $C > 0$ such that*

$$\int \left(\sum_{m<k\leq n} (f_k(\omega) - g_k) \right)^2 d\mu \leq C \sum_{m<k\leq n} h_k \tag{16}$$

for $m, n \in \mathbb{N}$ such that $m < n$. Then, for every $\varepsilon > 0$

$$\sum_{1\leq k\leq n} f_k(\omega) = \sum_{1\leq k\leq n} g_k + O(\Theta^{1/2}(n) \log^{3/2+\varepsilon} \Theta(n)),$$

for μ-a.e. $\omega \in \Omega$, where $\Theta(n) = \sum_{1\leq k\leq n} h_k$.

Lemma 8. *For each $\varepsilon > 0$,*

$$\sum_{k=0}^{n-1} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) = \sum_{k=0}^{n-1} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) + O(\Theta^{1/2}(n) \log^{3/2+\varepsilon} \Theta(n)),$$

for μ-a.e. $\omega \in \Omega$, where

$$\Theta(n) = \sum_{k=0}^{n-1} (\mathbb{E}_\omega(M_k^2 \circ f_\omega^k) + \|M_k^2\|_{var}). \tag{17}$$

Proof. Fix $\omega \in \Omega$. We want to apply Theorem 2 to

$$f_k = \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \quad \text{and} \quad g_k = \mathbb{E}_\omega(M_k^2 \circ f_\omega^k).$$

We have that

$$\begin{aligned}
&\int \Big[\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) - \sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Big]^2 d\mu_\omega \\
&= \int \Big(\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \Big)^2 d\mu_\omega \\
&\quad - 2\Big(\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Big) \int \Big(\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \Big) d\mu_\omega \\
&\quad + \Big(\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Big)^2 \\
&= \int \Big(\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \Big)^2 d\mu_\omega - \Big(\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Big)^2 \\
&= \sum_{m<k\le n} \int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1})^2 \, d\mu_\omega \\
&\quad + 2 \sum_{m<i<j\le n} \int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j | \mathcal{T}_\omega^{j+1}) \, d\mu_\omega \\
&\quad - \sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k)^2 - 2 \sum_{m<i<j\le n} \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j) \\
&\le \sum_{m<k\le n} \int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1})^2 \, d\mu_\omega \\
&\quad + 2 \sum_{m<i<j\le n} \int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j | \mathcal{T}_\omega^{j+1}) \, d\mu_\omega \\
&\quad - 2 \sum_{m<i<j\le n} \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j).
\end{aligned} \tag{18}$$

On the other hand, it follows from Lemma 3 that for $i < j$ we have

$$\begin{aligned}
&\int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j | \mathcal{T}_\omega^{j+1}) \, d\mu_\omega \\
&= \int \Big[\Big(\frac{\mathcal{L}_{\sigma^i\omega}(h_{\sigma^i\omega} M_i^2)}{h_{\sigma^{i+1}\omega}} \Big) \circ f_\omega^{i+1} \Big] \cdot \Big[\Big(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega} M_j^2)}{h_{\sigma^{j+1}\omega}} \Big) \circ f_\omega^{j+1} \Big] d\mu_\omega \\
&= \int \Big(\frac{\mathcal{L}_{\sigma^i\omega}(h_{\sigma^i\omega} M_i^2)}{h_{\sigma^{i+1}\omega}} \Big) \cdot \Big[\Big(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega} M_j^2)}{h_{\sigma^{j+1}\omega}} \Big) \circ f_{\sigma^{i+1}\omega}^{j-i} \Big] d\mu_{\sigma^{i+1}\omega} \\
&= \int \mathcal{L}_{\sigma^i\omega}(h_{\sigma^i\omega} M_i^2) \cdot \Big[\Big(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega} M_j^2)}{h_{\sigma^{j+1}\omega}} \Big) \circ f_{\sigma^{i+1}\omega}^{j-i} \Big] dm \\
&= \int \mathcal{L}_{\sigma^i\omega}^{j-i+1}(h_{\sigma^i\omega} M_i^2) \cdot \Big(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega} M_j^2)}{h_{\sigma^{j+1}\omega}} \Big) dm.
\end{aligned}$$

Moreover,

$$\mathbb{E}_\omega(M_i^2 \circ f_\omega^i) = \int M_i^2 \, d\mu_{\sigma^i\omega}$$

and

$$\begin{aligned}\mathbb{E}_\omega(M_j^2 \circ f_\omega^j) &= \int (M_j^2 \circ f_\omega^j) \, d\mu_\omega = \int M_j^2 \, d\mu_{\sigma^j\omega} = \int M_j^2 h_{\sigma^j\omega} \, dm \\ &= \int \mathcal{L}_{\sigma^j\omega}(M_j^2 h_{\sigma^j\omega}) \, dm = \int \frac{\mathcal{L}_{\sigma^j\omega}(M_j^2 h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}} \, d\mu_{\sigma^{j+1}\omega}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i|\mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j|\mathcal{T}_\omega^{j+1}) \, d\mu_\omega - \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j) \\ &= \int \mathcal{L}_{\sigma^i\omega}^{j-i+1}(h_{\sigma^i\omega} M_i^2) \cdot \left(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega} M_j^2)}{h_{\sigma^{j+1}\omega}}\right) dm - \int M_i^2 \, d\mu_{\sigma^i\omega} \cdot \int \frac{\mathcal{L}_{\sigma^j\omega}(M_j^2 h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}} \, d\mu_{\sigma^{j+1}\omega}.\end{aligned}$$

Therefore, it follows from Lemma 1 that

$$\begin{aligned}&\int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i|\mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j|\mathcal{T}_\omega^{j+1}) \, d\mu_\omega - \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j) \\ &\le K\rho^{j-i+1} \left\| \frac{\mathcal{L}_{\sigma^j\omega}(M_j^2 h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}} \right\|_\infty \cdot \|M_i^2\|_{var}.\end{aligned}$$

Furthermore,

$$\int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k|\mathcal{T}_\omega^{k+1})^2 \, d\mu_\omega \le \|\mathbb{E}_\omega(M_k^2 \circ f_\omega^k|\mathcal{T}_\omega^{k+1})\|_\infty \cdot \mathbb{E}_\omega(M_k^2 \circ f_\omega^k).$$

Thus, the last two inequalities combined with (18) imply that

$$\begin{aligned}&\int \Big[\sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k|\mathcal{T}_\omega^{k+1}) - \sum_{m<k\le n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Big]^2 \, d\mu_\omega \\ &\le \sum_{m<k\le n} \int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k|\mathcal{T}_\omega^{k+1})^2 \, d\mu_\omega \\ &\quad + 2 \sum_{m<i<j\le n} \int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i|\mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j|\mathcal{T}_\omega^{j+1}) \, d\mu_\omega \\ &\quad - 2 \sum_{m<i<j\le n} \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j). \\ &\le \sum_{m<k\le n} \|\mathbb{E}_\omega(M_k^2 \circ f_\omega^k|\mathcal{T}_\omega^{k+1})\|_\infty \cdot \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \\ &\quad + 2K \sum_{m<i<j\le n} \rho^{j-i+1} \left\| \frac{\mathcal{L}_{\sigma^j\omega}(M_j^2 h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}} \right\|_\infty \cdot \|M_i^2\|_{var},\end{aligned}$$

which combined with (H2), (3), (6) and Lemmas 6 and 7 implies that (16) holds with

$$h_k = \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) + \|M_k^2\|_{var}.$$

The conclusion of the lemma now follows directly from Theorem 2. □

5. Proof of Theorem 1

The goal of this section is to establish the almost sure invariance principle by proving Theorem 1. It is based on the following theorem due to Cuny and Merlevède.

Theorem 3 ([6]). *Let $(X_n)_n$ be a sequence of square integrable random variables adapted to a non-increasing filtration $(\mathcal{G}_n)_n$. Assume that $\mathbb{E}(X_n|\mathcal{G}_{n+1}) = 0$ a.s., $\sigma_n^2 := \sum_{k=1}^n \mathbb{E}(X_k^2) \to \infty$ when $n \to \infty$ and that $\sup_n \mathbb{E}(X_n^2) < \infty$. Moreover, let $(a_n)_n$ be a non-decreasing sequence of positive numbers such that the sequence $(a_n/\sigma_n^2)_n$ is non-increasing, (a_n/σ_n) is non-decreasing and such that :*

1.
$$\sum_{k=1}^{n} (\mathbb{E}(X_k^2|\mathcal{G}_{k+1}) - \mathbb{E}(X_k^2)) = o(a_n) \quad a.s.; \tag{19}$$

2.
$$\sum_{n\geq 1} a_n^{-v} \mathbb{E}(|X_n|^{2v}) < \infty \quad \text{for some } 1 \leq v \leq 2. \tag{20}$$

Then, enlarging our probability space if necessary, it is possible to find a sequence $(Z_k)_k$ of independent centered Gaussian variables with $\mathbb{E}(X_k^2) = \mathbb{E}(Z_k^2)$ such that

$$\sup_{1\leq k\leq n} \left| \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} Z_k \right| = o((a_n(|\log(\sigma_n^2/a_n)| + \log\log a_n))^{1/2}), \quad a.s.$$

In order to obtain the almost sure invariance principle for the sequence $(\tilde{\psi}_{\theta^k\omega} \circ f_\omega^k)_k$, $k \in \mathbb{N}$ we will first apply Theorem 3 for

$$X_n = M_n \circ f_\omega^n \quad \text{and} \quad \mathcal{G}_n = \mathcal{T}_\omega^n.$$

We note that it follows from Lemma 8 that

$$\sum_{k=1}^{n} (\mathbb{E}(X_k^2|\mathcal{G}_{k+1}) - \mathbb{E}(X_k^2)) = O(b_n),$$

with

$$b_n = \Theta^{1/2}(n) \log^{3/2+\varepsilon} \Theta(n), \tag{21}$$

and where $\Theta(n)$ is given by (17). On the other hand, it follows from Lemma 6 that $\Theta(n) \leq Dn$ for some $D > 0$ and every $n \in \mathbb{N}$ and therefore (19) holds with

$$a_n = n^{1/2+d}, \tag{22}$$

for any $d > 0$. From now on, we take $d \in (0, 1/2)$.

Lemma 9. *There exists $\Sigma^2 \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_\omega \left(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k \right)^2 = \Sigma^2, \quad \text{for a.e. } \omega \in \Omega. \tag{23}$$

Proof. Note that

$$\begin{aligned}\mathbb{E}_\omega \left(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k \right)^2 &= \sum_{k=0}^{n-1} \mathbb{E}_\omega(\tilde{\psi}_{\sigma^k\omega}^2 \circ f_\omega^k) + 2 \sum_{0\leq i<j\leq n-1} \mathbb{E}_\omega((\tilde{\psi}_{\sigma^i\omega} \circ f_\omega^i)(\tilde{\psi}_{\sigma^j\omega} \circ f_\omega^j)) \\ &= \sum_{k=0}^{n-1} \mathbb{E}_\omega(\tilde{\psi}_{\sigma^k\omega}^2 \circ f_\omega^k) + 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega} \circ f_{\sigma^i\omega}^{j-i})).\end{aligned}$$

Using the skew product transformation τ from (2), it follows from Birkhoff's ergodic theorem that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}_\omega(\tilde{\psi}^2_{\sigma^k\omega}\circ f^k_\omega)&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_X\tilde{\psi}(\tau^k(\omega,x))^2\,d\mu_\omega(x)\\&=\int_\Omega\int_X\tilde{\psi}(\omega,x)^2\,d\mu_\omega(x)\,d\mathbb{P}(\omega)=\int_{\Omega\times X}\tilde{\psi}(\omega,x)^2\,d\mu(\omega,x),\end{aligned}$$

for a.e. $\omega\in\Omega$, where μ is an invariant measure for τ given by

$$\mu(A\times B)=\int_A\mu_\omega(B)\,d\mathbb{P}(\omega),\quad \text{for } A\in\mathcal{F} \text{ and } B\in\mathcal{B}.$$

Furthermore, set

$$\Psi(\omega)=\sum_{n=1}^\infty\int_X\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,d\mu_\omega(x)=\sum_{n=1}^\infty\int_X\mathcal{L}^n_\omega(\tilde{\psi}_\omega h_\omega)\tilde{\psi}_{\sigma^n\omega}\,dm.$$

By (5) and (8), we have

$$|\Psi(\omega)|\le\sum_{n=1}^\infty\left|\int_X\mathcal{L}^n_\omega(\tilde{\psi}_\omega h_\omega)\tilde{\psi}_{\sigma^n\omega}\,dm\right|\le\tilde{K}\sum_{n=1}^\infty\rho^n=\frac{\tilde{K}\rho}{1-\rho},$$

for some $\tilde{K}>0$ and a.e. $\omega\in\Omega$. In particular, $\Psi\in L^1(\Omega)$ and thus it follows again from Birkhoff's ergodic theorem that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\Psi(\sigma^i\omega)=\int_\Omega\Psi(\omega)\,d\mathbb{P}(\omega)=\sum_{n=1}^\infty\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,d\mu(\omega,x),\tag{24}$$

for a.e. $\omega\in\Omega$. In order to complete the proof of the lemma, we are going to show that

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f^{j-i}_{\sigma^i\omega}))-\sum_{i=0}^{n-1}\Psi(\sigma^i\omega)\right)=0,\tag{25}$$

for a.e. $\omega\in\Omega$. Using (5), we have that for a.e. $\omega\in\Omega$,

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f^{j-i}_{\sigma^i\omega}))-\sum_{i=0}^{n-1}\Psi(\sigma^i\omega)\right|\\&=\left|\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f^{j-i}_{\sigma^i\omega}))-\sum_{i=0}^{n-1}\sum_{k=1}^{\infty}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^{k+i}\omega}\circ f^{k}_{\sigma^i\omega}))\right|\\&\le\sum_{i=0}^{n-1}\sum_{k=n-i}^{\infty}\left|\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^{k+i}\omega}\circ f^{k}_{\sigma^i\omega}))\right|=\sum_{i=0}^{n-1}\sum_{k=n-i}^{\infty}\left|\int_X\mathcal{L}^k_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}h_{\sigma^i\omega})\tilde{\psi}_{\sigma^{k+i}\omega}\,dm\right|\\&\le\tilde{K}\sum_{i=0}^{n-1}\sum_{k=n-i}^{\infty}\rho^k=\tilde{K}\frac{\rho}{(1-\rho)^2},\end{aligned}$$

which readily implies (25). It follows from (24) and (25) that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f^{j-i}_{\sigma^i\omega}))=\sum_{n=1}^\infty\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,d\mu(\omega,x)$$

for a.e. $\omega \in \Omega$ and therefore (23) holds with

$$\Sigma^2 = \int_{\Omega\times X} \tilde{\psi}(\omega,x)^2\,d\mu(\omega,x) + 2\sum_{n=1}^{\infty}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,d\mu(\omega,x). \tag{26}$$

Finally, we note that it follows readily from (23) that $\Sigma^2 \geq 0$ and the proof of the lemma is completed. □

We now present necessary and sufficient conditions under which $\Sigma^2 = 0$. We note that a similar result is stated in [11, (2.10)] with $\tilde{\psi}\circ\tau$ instead of $\tilde{\psi}$ in (27).

Proposition 3. *We have that $\Sigma^2 = 0$ if and only if there exists $\phi \in L^2(\Omega\times X)$ such that*

$$\tilde{\psi} = \phi - \phi\circ\tau. \tag{27}$$

Proof. We first observe that

$$\begin{aligned}
&\int_{\Omega\times X}\Big(\sum_{k=0}^{n-1}\tilde{\psi}(\tau^k(\omega,x))\Big)^2\,d\mu(\omega,x)\\
&\quad= \sum_{k=0}^{n-1}\int_{\Omega\times X}\tilde{\psi}^2(\tau^k(\omega,x))\,d\mu(\omega,x) + 2\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}\int_{\Omega\times X}\tilde{\psi}(\tau^k(\omega,x)\tilde{\psi}(\tau^j(\omega,x))\,d\mu(\omega,x)\\
&\quad= n\int_{\Omega\times X}\tilde{\psi}^2(\omega,x)\,d\mu(\omega,x) + 2\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^{k-j}(\omega,x))\,d\mu(\omega,x)\\
&\quad= n\int_{\Omega\times X}\tilde{\psi}^2(\omega,x)\,d\mu(\omega,x) + 2\sum_{k=1}^{n-1}(n-k)\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^k(\omega,x))\,d\mu(\omega,x),
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\Omega\times X}\Big(\sum_{k=0}^{n-1}\tilde{\psi}(\tau^k(\omega,x))\Big)^2\,d\mu(\omega,x) =\\
&= n\Big(\int_{\Omega\times X}\tilde{\psi}^2(\omega,x)\,d\mu(\omega,x) + 2\sum_{k=1}^{n-1}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^k(\omega,x))\,d\mu(\omega,x)\Big)\\
&\quad - 2\sum_{k=1}^{n-1}k\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^k(\omega,x))\,d\mu(\omega,x).
\end{aligned}$$

Assume now that $\Sigma^2 = 0$. Then, it follows from the above equality and (26) that

$$\int_{\Omega\times X}\Big(\sum_{k=0}^{n-1}\tilde{\psi}\circ\tau^k\Big)^2\,d\mu = -2n\sum_{k=n}^{\infty}\int_{\Omega\times X}\tilde{\psi}(\tilde{\psi}\circ\tau^k)\,d\mu - 2\sum_{k=1}^{n-1}k\int_{\Omega\times X}\tilde{\psi}(\tilde{\psi}\circ\tau^k)\,d\mu. \tag{28}$$

On the other hand, by (5) we have that $\int_{\Omega\times X}\tilde{\psi}(\tilde{\psi}\circ\tau^k)\,d\mu \to 0$ exponentially fast when $k\to\infty$ and hence, it follows from (28) that the sequence $(X_n)_n$ defined by

$$X_n(\omega,x) = \sum_{k=0}^{n-1}\tilde{\psi}(\tau^k(\omega,x)), \quad \omega\in\Omega,\ x\in X$$

is bounded in $L^2(\Omega\times X)$. Thus, it has a subsequence $(X_{n_k})_k$ which converges weakly to some $\phi\in L^2(\Omega\times X)$. We claim that ϕ satisfies (27). Indeed, take an arbitrary $g = \mathbf{1}_{A\times B}$, where $A \in \mathcal{F}$ and $B \in \mathcal{B}$ and observe that $g \in L^2(\Omega \times X)$ and

$$\begin{aligned}\int_{\Omega\times X} g(\tilde{\psi} - \phi + \phi \circ \tau) &= \lim_{k\to\infty} \int_{\Omega\times X} g(\tilde{\psi} - X_{n_k} + X_{n_k} \circ \tau)\, d\mu \\ &= \lim_{k\to\infty} \int_{\Omega\times X} g(\tilde{\psi} \circ \tau^{n_k})\, d\mu = 0,\end{aligned}$$

where in the last equality we used (5) again. Therefore, $\tilde{\psi} - \phi + \phi \circ \tau = 0$ which readily implies (27).

Suppose now that there exists $\phi \in L^2(\Omega \times X)$ satisfying (27). Then,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{\psi} \circ \tau^k = \frac{1}{\sqrt{n}}(\phi - \phi \circ \tau^n),$$

and thus

$$\left\| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{\psi} \circ \tau^k \right\|_{L^2(\Omega\times X)} \le \frac{2}{\sqrt{n}} \|\phi\|_{L^2(\Omega\times X)} \to 0,$$

when $n \to \infty$. Therefore, it follows by integrating (23) over Ω that

$$\Sigma^2 = \lim_{n\to\infty} \left\| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{\psi} \circ \tau^k \right\|^2_{L^2(\Omega\times X)} = 0.$$

This concludes the proof of the proposition. □

In the rest of the paper we assume that $\Sigma^2 > 0$. We also need the following lemmas.

Lemma 10. *We have that*

$$\mathbb{E}_\omega(X_i X_j) = 0, \quad \text{for } i < j.$$

Proof. By Lemma 2, we conclude that $\mathbb{E}_\omega(M_i \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) = 0$. Moreover, we note that $M_j \circ f_\omega^j$ is measurable with respect to $\mathcal{T}_\omega^{i+1}$ and thus

$$\mathbb{E}_\omega((M_j \circ f_\omega^j)(M_i \circ f_\omega^i) | \mathcal{T}_\omega^{i+1}) = (M_j \circ f_\omega^j)\mathbb{E}_\omega(M_i \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) = 0.$$

The conclusion of the lemma now follows simply by integrating the above equality. □

In what follows, we write $a_n \sim b_n$ if there exists $c \in \mathbb{R} \setminus \{0\}$ such that $\lim_{n\to\infty} a_n/b_n = c$.

Lemma 11. *We have that $\sigma_n^2 \to \infty$ as $n \to \infty$.*

Proof. It follows from (11) that

$$\sum_{k=0}^{n-1} X_k = \sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k - G_n \circ f_\omega^n, \tag{29}$$

and thus,

$$\left(\sum_{k=0}^{n-1} X_k\right)^2 = \left(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k\right)^2 - 2(G_n \circ f_\omega^n)\left(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k\right) + (G_n^2 \circ f_\omega^n). \tag{30}$$

By Lemma 9 and the assumption $\Sigma^2 > 0$,

$$\tau_n^2 := \mathbb{E}_\omega\left(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k\right)^2 \to \infty. \tag{31}$$

On the other hand, it follows from (8), (30) and Lemma 5 that

$$\mathbb{E}_\omega\Big(\sum_{k=0}^{n-1} X_k\Big)^2 \sim \tau_n^2. \tag{32}$$

By Lemma 10 and (32), we have that

$$\sigma_n^2 = \sum_{k=0}^{n-1} \mathbb{E}_\omega(X_k^2) = \mathbb{E}_\omega\Big(\sum_{k=0}^{n-1} X_k\Big)^2 \sim \tau_n^2, \tag{33}$$

which together with (31) implies the desired conclusion of Lemma 11. □

Lemma 12. *There exists $n_0 \in \mathbb{N}$ such that the sequence $(a_n/\sigma_n^2)_{n\geq n_0}$ is non-increasing and that the sequence $(a_n/\sigma_n)_{n\geq n_0}$ is non-decreasing.*

Proof. It follows from Lemma 9 and (33) that

$$\sigma_n^2 = \sum_{k=0}^{n-1} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \sim n\Sigma^2.$$

Therefore (22) yields,

$$a_n/\sigma_n^2 \sim \frac{n^{1/2+d}}{n} \quad \text{and} \quad a_n/\sigma_n \sim \frac{n^{1/2+d}}{\sqrt{n}},$$

and the conclusion of the lemma follows readily from the assumption that $d \in (0, 1/2)$. □

Since the conclusion of Theorem 3 concerns the tails of $(a_n)_n$ and $(\sigma_n)_n$, it will remain valid if the monotonicity hypotheses for $(a_n/\sigma_n^2)_n$ and $(a_n/\sigma_n)_n$ hold for sufficiently large n, and those are verified in Lemma 12. Finally, we show that (20) holds with $v = 2$.

Lemma 13. *We have that*

$$\sum_{n\geq 1} a_n^{-2}\mathbb{E}_\omega(|X_n|^4) < \infty.$$

Proof. Since $\sup_n \|M_n\|_\infty < \infty$, we have that $\sup_n \|X_n\|_\infty < \infty$ and thus

$$\sum_{n\geq 1} a_n^{-2}\mathbb{E}_\omega(|X_n|^4) \leq C\sum_{n\geq 1} a_n^{-2} = C\sum_{n\geq 1} \frac{1}{n^{1+2d}} < \infty.$$

□

Now we can conclude the proof or our main result.

Proof of Theorem 1. Using Theorem 3, we obtain the almost sure invariance principle for the sequence $(X_k)_k = (M_k \circ f_\omega^k)_k$. The almost sure invariance principle for the sequence $(\tilde{\psi}_{\theta^k\omega} \circ f_\omega^k)_k$, stated in Theorem 1, now follows from (29) and Lemma 5. □

Acknowledgments

The research of DD and GF was supported by the Australian Research Council Discovery Project DP150100017 and, in part, by the Croatian Science Foundation under the project IP-2014-09-228. CGT acknowledges the support of Australian Research Council Discovery Early Career Researcher Award (DECRA) DE160100147. SV was supported by the ANR-Project *Perturbations* and by the Leverhulme Trust for support thorough the Network Grant IN-2014-021. SV and CGT warmly thank the School of Mathematics and Statistics of the University of New South Wales for the kind hospitality during the preparation of this work. SV acknowledges several discussions with M. Abdelkader and

R. Aimino on the topics covered by this paper. GF, CGT and SV thank the American Institute of Mathematics which gave them the opportunity to meet and begin this project during the Conference *Stochastic methods for non-equilibrium dynamical systems*, in San José (CA, USA), in 2015. The authors also acknowledge the South University of Science and Technology of China in Shenzhen where they met and completed this work during the Conference *Statistical Properties of Nonequilibrium Dynamical Systems, in 2016.*

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