LYAPUNOV TYPE CHARACTERIZATION OF HYPERBOLIC BEHAVIOR

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Abstract. We give a complete characterization of the uniform hyperbolicity and nonuniform hyperbolicity of a cocycle with values in the space of bounded linear operators acting on a Hilbert space in terms of the existence of appropriate quadratic forms. Our work unifies and extends many results in the literature by considering the general case of not necessarily invertible cocycles acting on a Hilbert space over an arbitrary invertible dynamics. As a nontrivial application of, we study the persistence of hyperbolicity under small linear perturbations.

1. Introduction

Our main objective is to give a complete characterization of the uniform hyperbolicity and nonuniform hyperbolicity of a cocycle with values in the space of bounded linear operators acting on a Hilbert space in terms of the existence of appropriate quadratic forms. Our work unifies and extends many results in the literature by considering the general case of not necessarily invertible cocycles acting on a Hilbert space over an arbitrary invertible dynamics $f\colon M \to M$.

For example, when $M = \mathbb{Z}$ and $f(n) = n+1$, the concept of hyperbolicity introduced in Section 3 reduces to the notion of a uniform exponential dichotomy. This notion was essentially introduced by Perron in [17] and plays a central role in the qualitative theory of dynamical systems. We refer the reader to [6, 9, 10, 21] for details and further references. In the same setting, the concept of nonuniform hyperbolicity considered in Section 5 includes the notion of a nonuniform exponential dichotomy as a particular case. We refer to [4] for the discussion of many related developments.

Many works in the literature have been devoted to the characterization of an exponential dichotomy in terms of the existence of appropriate quadratic forms. For some early contributions, we refer to the work of Maizel [14], Coppel [6, 7] and Papaschinopoulos [16]. For more recent work dealing with nonuniform exponential dichotomies, see [5]. We emphasize that all these works consider only the particular case of an invertible finite-dimensional dynamics. Moreover, to the best of our understanding, the arguments in those works cannot be extended to our setting. This forced us to develop

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a different approach that relies on the spectral characterization of the hyperbolic behavior (see Section 2) and on the characterization of hyperbolic operators given in [8].

On the other hand, the generality of our setting enables to consider more complicated forms of hyperbolicity. For example, when f is differentiable, the notion of hyperbolicity considered in Section 3 reduces to the classical concept of uniform hyperbolicity introduced and studied by Smale [22] and Anosov [1]. The notion of uniform hyperbolicity was characterized in terms of the existence of quadratic forms by Lewowicz [12, 13] but again only in the finite-dimensional setting and in the particular case of derivative cocycles. In the present work, we extend the results in [12] to an arbitrary noninvertible cocycle acting on an infinite-dimensional Hilbert space.

Moreover, the notion of hyperbolicity considered in Section 5 includes the concept of nonuniform hyperbolicity in the sense of Pesin [3, 18] in the particular case of a finite-dimensional setting when the function K is tempered (see also [15, 20] for related work in the infinite-dimensional setting). See [11] and the references therein for generalizations of the work of Lewowicz in this context. However, these works give characterizations of systems with nonzero Lyapunov exponents rather then of nonuniformly hyperbolic cocycles.

As a nontrivial application of our characterizations of the hyperbolic behavior, we study the persistence of hyperbolicity under small linear perturbations. This problem has a long history, especially in relation to exponential dichotomies. We refer the reader to [4, 19] for details and further references.

2. Preliminaries

We first introduce some notions and results related to hyperbolicity that will be used throughout the paper.

2.1. **Hyperbolic operators.** Let $B(X)$ be the set of all bounded linear operators acting on a Hilbert space X. Given self-adjoint operators $A, B \in B(X)$, we write $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for all $x \in X$. Moreover, an operator $A \in B(X)$ is said to be *hyperbolic* if its spectrum does not intersect the unit circle $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

We also recall two important results about hyperbolicity taken from [8].

Theorem 1. *Assume that the spectrum of an operator $T \in B(X)$ does not cover the whole S^1. Then every self-adjoint operator $W \in B(X)$ with the property that there exists $\delta > 0$ satisfying*

$$T^*WT - W \le -\delta I \tag{1}$$

is invertible.

The following result will be of particular importance.

Theorem 2. *Let $T \in B(X)$ and assume that there exists an invertible self-adjoint operator $W \in B(X)$ satisfying (1) for some $\delta > 0$. Then T is hyperbolic if and only if there exists $\delta' > 0$ satisfying*

$$TW^{-1}T^* - W^{-1} \le -\delta' I.$$

2.2. **Hyperbolic sequences.** Now let $\|\cdot\|_n$, for $n \in \mathbb{Z}$, be a sequence of norms on X such that $\|\cdot\|_n$ is equivalent to the original norm $\|\cdot\|$ for each $n \in \mathbb{Z}$. Given a sequence $(A_n)_{n\in\mathbb{Z}} \subset B(X)$, we define

$$\mathcal{A}(n,m) = \begin{cases} A_{n-1}\cdots A_m, & n > m, \\ I, & n = m \end{cases}$$

for $n, m \in \mathbb{Z}$ with $n \ge m$. We say that the sequence $(A_n)_{n\in\mathbb{Z}}$ is *hyperbolic with respect to the norms* $\|\cdot\|_n$ if:

1. there exist projections $P_m \in B(X)$ for $m \in \mathbb{Z}$ satisfying
$$\mathcal{A}(n,m)P_m = P_n\mathcal{A}(n,m) \quad \text{for } n \ge m$$
such that each map
$$\mathcal{A}(n,m)|\operatorname{Ker} P_m \colon \operatorname{Ker} P_m \to \operatorname{Ker} P_n$$
is invertible;
2. there exist constants $\lambda, D > 0$ such that for $n, m \in \mathbb{Z}$ and $x \in X$ we have
$$\|\mathcal{A}(n,m)P_m x\|_n \le De^{-\lambda(n-m)}\|x\|_m \quad \text{for } n \ge m \tag{2}$$
and
$$\|\mathcal{A}(n,m)Q_m x\|_n \le De^{-\lambda(m-n)}\|x\|_m \quad \text{for } n \le m, \tag{3}$$
where $Q_m = I - P_m$ and
$$\mathcal{A}(n,m) = (\mathcal{A}(m,n)|\operatorname{Ker} P_n)^{-1} : \operatorname{Ker} P_m \to \operatorname{Ker} P_n$$
for $n < m$.

We also consider the space

$$Y = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sum_{n\in\mathbb{Z}}\|x_n\|_n^2 < \infty\right\}.$$

One can easily verify that Y is a Hilbert space with the scalar product

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{n\in\mathbb{Z}}\langle x_n, y_n\rangle_n \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}}, \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y.$$

Now consider a sequence $(A_n)_{n\in\mathbb{Z}} \subset B(X)$ and assume that there exists $C > 0$ such that

$$\|A_n x\|_{n+1} \le C\|x\|_n \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X. \tag{4}$$

Then the map $T \colon Y \to Y$ given by

$$(T\mathbf{x})_n = A_{n-1}x_{n-1} \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y,$$

is a well-defined bounded linear operator. The following two results are taken from [2].

Theorem 3. *Assume that $(A_n)_{n\in\mathbb{Z}}$ satisfies (4) and is hyperbolic with respect to a sequence of norms $\|\cdot\|_n$. Then $I - T$ is invertible and*

$$\|(I-T)^{-1}\| \le K \tag{5}$$

for some constant K depending only on D and λ.

Theorem 4. *Given a sequence $(A_n)_{n\in\mathbb{Z}}$ satisfying* (4), *if $I-T$ is invertible, then $(A_n)_{n\in\mathbb{Z}}$ is hyperbolic with respect to a sequence of norms $\|\cdot\|_n$. Moreover, the constants D,λ in* (2) *and* (3) *can be chosen so that they depend only on the constant K in* (5).

2.3. **Hyperbolic cocycles.** Given a map $f\colon M\to M$, we say that a function $\mathcal{A}\colon M\times\mathbb{N}_0\to B(X)$ is a *cocycle* (over f) if for every $q\in M$ and $n,m\in\mathbb{N}_0$ we have:

1. $\mathcal{A}(q,0)=I$;
2. $\mathcal{A}(q,n+m)=\mathcal{A}(f^n(q),m)\mathcal{A}(q,n)$.

The map $A\colon M\to B(X)$ given by $A(q)=\mathcal{A}(q,1)$ is called the *generator* of the cocycle $\mathcal{A}$.

Now assume that the map f is invertible and let $\|\cdot\|_q$, for $q\in M$, be a family of norms on X such that $\|\cdot\|_q$ is equivalent to $\|\cdot\|$ for each $q\in M$. We say that the cocycle $\mathcal{A}$ is *hyperbolic with respect to the norms* $\|\cdot\|_q$ if:

1. there exists projections $P(q)$, for $q\in M$, satisfying
$$A(q)P(q)=P(f(q))A(q)\quad\text{for } q\in M, \tag{6}$$
such that each map
$$A(q)|\operatorname{Ker}P(q)\colon\ \operatorname{Ker}P(q)\to\operatorname{Ker}P(f(q)) \tag{7}$$
is invertible;
2. there exist constants $\lambda,D>0$ such that for each $q\in M$, $x\in X$ and $n\ge 0$ we have
$$\|\mathcal{A}(q,n)P(q)x\|_{f^n(q)}\le De^{-\lambda n}\|x\|_q \tag{8}$$
and
$$\|\mathcal{A}(q,-n)Q(q)\|_{f^{-n}(q)}\le De^{-\lambda n}\|x\|_q, \tag{9}$$
where $Q(q)=I-P(p)$ and
$$\mathcal{A}(q,-n)=\big(\mathcal{A}(f^{-n}(q),n)|\operatorname{Ker}P(f^{-n}(q))\big)^{-1}. \tag{10}$$

For each $q\in M$, we define
$$Y_q=\bigg\{\mathbf{x}=(x_n)_{n\in\mathbb{Z}}:\sum_{n=-\infty}^{\infty}\|x_n\|^2_{f^n(q)}<+\infty\bigg\}. \tag{11}$$
One can easily verify that Y_q is a Hilbert space with the scalar product
$$\langle\mathbf{x},\mathbf{y}\rangle=\sum_{n\in\mathbb{Z}}\langle x_n,y_n\rangle_{f^n(q)}\quad\text{for } \mathbf{x}=(x_n)_{n\in\mathbb{Z}},\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_q.$$
Now, assume that for a given cocycle $\mathcal{A}$ there exists $C>0$ such that
$$\|A(q)x\|_{f(q)}\le C\|x\|_q\quad\text{for } q\in M \text{ and } x\in X. \tag{12}$$
Then the operator $T_q\colon Y_q\to Y_q$ given by
$$(T_q\mathbf{x})_n=A(f^{n-1}(q))x_{n-1}\quad\text{for } n\in\mathbb{Z} \text{ and } \mathbf{x}=(x_n)_{n\in\mathbb{Z}}\in Y_q \tag{13}$$
is well-defined.

Theorem 5. *Assume that the cocycle $\mathcal{A}$ satisfies* (12) *and is hyperbolic with respect to the norms* $\|\cdot\|_q$*. Then* $I-T_q$ *is invertible for each* $q \in M$ *and there exists* $K > 0$ *such that*

$$\|(I-T_q)^{-1}\| \le K \quad \text{for } q \in M. \tag{14}$$

Proof. It follows easily from (8) and (9) that for each $q \in M$ the sequence $(A_n)_{n\in\mathbb{Z}}$ defined by $A_n = A(f^n(q))$ is hyperbolic with respect to the norms $\|\cdot\|_{f^n(q)}$, for $n \in \mathbb{Z}$. Moreover, the constants D, λ in (2) and (3) are independent of q. The conclusion of the theorem follows now directly from Theorem 3. ☐

Theorem 6. *Given a cocycle $\mathcal{A}$ satisfying* (12)*, assume that the operator* $I - T_q$ *is invertible for each* $q \in M$ *and that there exists a constant* $K > 0$ *satisfying* (14)*. Then $\mathcal{A}$ is hyperbolic with respect to the norms* $\|\cdot\|_q$*.*

Proof. It follows from Theorem 4 that the sequence $(A_n)_{n\in\mathbb{Z}}$ in the proof of Theorem 5 is hyperbolic with respect to the norms $\|\cdot\|_{f^n(q)}$, for $n \in \mathbb{Z}$. Moreover, the constants D, λ in (2) and (3) can be chosen so that they depend only on K (and not on q). This establishes (8) and (9). ☐

3. Hyperbolic cocycles

This section contains our mains results, concerning the characterization of the hyperbolicity of a cocycle in terms of Lyapunov functions.

Let $\mathcal{A}$ be a cocycle. We say that $\mathcal{A}$ is *hyperbolic* if:

1. there exists a family of projections $P(q)$, for $q \in M$, satisfying (6) such that each map in (7) is invertible;
2. there exist constants $\lambda, D > 0$ such that for each $q \in M$ and $n \ge 0$ we have
$$\|\mathcal{A}(q,n)P(q)\| \le De^{-\lambda n} \tag{15}$$
and
$$\|\mathcal{A}(q,-n)Q(q)\| \le De^{-\lambda n}, \tag{16}$$
where $Q(q) = I - P(p)$, with $\mathcal{A}(q,-n)$ as in (10).

An easy consequence of the definition of hyperbolicity is the following.

Proposition 1. *Assume that the cocycle $\mathcal{A}$ is hyperbolic. Then*

$$\operatorname{Im} P(q) = \left\{ v \in X : \sup_{n\ge 0} \|\mathcal{A}(q,n)v\| < +\infty \right\}$$

and $\operatorname{Ker} P(q)$ *consists of all vectors* $v \in X$ *for which there exists a sequence* $(v_n)_{n\le 0}$ *satisfying* $v_0 = v$*,* $\sup_{n\le 0}\|v_n\| < +\infty$ *and* $v_n = \mathcal{A}(f^m(q), n-m)v_m$ *for* $0 \ge n \ge m$*.*

The following theorem is our first main result.

Theorem 7. *Assume that the cocycle $\mathcal{A}$ is hyperbolic and that there exists* $C > 0$ *such that*

$$\|A(q)\| \le C \quad \text{for } q \in M. \tag{17}$$

Then there exist invertible self-adjoint operators $S(q) \in B(X)$*, for* $q \in M$*, and constants* $K, \delta > 0$ *such that for each* $q \in M$*:*

1.
$$\|S(q)\| \le K \quad \textit{and} \quad \|S(q)^{-1}\| \le K; \tag{18}$$
2.
$$A(q)^* S(f(q)) A(q) - S(q) \le -\delta I; \tag{19}$$
3.
$$A(q) S(q)^{-1} A(q)^* - S(f(q))^{-1} \le -\delta I. \tag{20}$$

Proof. Take $\rho \in (0, \lambda)$ and let

$$\begin{aligned} S(q) = &\sum_{n \ge 0} \left(\mathcal{A}(q,n)P(q)\right)^* \mathcal{A}(q,n)P(q) e^{2(\lambda-\rho)n} \\ &- \sum_{n>0} \left(\mathcal{A}(q,-n)Q(q)\right)^* \mathcal{A}(q,-n)Q(q) e^{2(\lambda-\rho)n}. \end{aligned}$$

By (15) and (16), for each $x \in X$ we have

$$\begin{aligned} |\langle S(q)x, x\rangle| &\le \sum_{n\ge 0} \|\mathcal{A}(q,n)P(q)x\|^2 e^{2(\lambda-\rho)n} + \sum_{n>0} \|\mathcal{A}(q,-n)Q(q)x\|^2 e^{2(\lambda-\rho)n} \\ &\le \sum_{n\ge 0} D^2 e^{-2\lambda n} e^{2(\lambda-\rho)n} \|x\|^2 + \sum_{n>0} D^2 e^{-2\lambda n} e^{2(\lambda-\rho)n} \|x\|^2 \\ &= D^2 \left(\sum_{n\ge 0} e^{-2\rho n} + \sum_{n>0} e^{-2\rho n} \right) \|x\|^2 = K\|x\|^2, \end{aligned}$$

where

$$K = D^2 \frac{1+e^{-2\rho}}{1-e^{-2\rho}}.$$

Since $S(q)$ is self-adjoint, we conclude that

$$\|S(q)\| = \sup_{\|x\|=1} |\langle S(q)x, x\rangle| \le K.$$

This establishes the first inequality in (18).

On the other hand, using (6) we obtain

$$\begin{aligned} &A(q)^* S(f(q)) A(q) = \\ &= A(q)^* \sum_{n\ge 0} \left(\mathcal{A}(f(q),n)P(f(q))\right)^* \mathcal{A}(f(q),n)P(f(q)) e^{2(\lambda-\rho)n} A(q) \\ &\quad - A(q)^* \sum_{n>0} \left(\mathcal{A}(f(q),-n)Q(f(q))\right)^* \mathcal{A}(f(q),-n)Q(f(q)) e^{2(\lambda-\rho)n} A(q) \\ &= \sum_{n\ge 0} \left(\mathcal{A}(f(q),n)P(f(q))A(q)\right)^* \mathcal{A}(f(q),n)P(f(q))A(q) e^{2(\lambda-\rho)n} \\ &\quad - \sum_{n>0} \left(\mathcal{A}(f(q),-n)Q(f(q))A(q)\right)^* \mathcal{A}(f(q),-n)Q(f(q))A(q) e^{2(\lambda-\rho)n} \\ &= \sum_{n\ge 0} \left(\mathcal{A}(f(q),n)A(q)P(q)\right)^* \mathcal{A}(f(q),n)A(q)P(q) e^{2(\lambda-\rho)n} \\ &\quad - \sum_{n>0} \left(\mathcal{A}(f(q),-n)A(q)Q(q)\right)^* \mathcal{A}(f(q),-n)A(q)Q(q) e^{2(\lambda-\rho)n} \end{aligned}$$

$$
\begin{aligned}
&= e^{-2(\lambda-\rho)} \sum_{n\geq 0} \left(\mathcal{A}(q,n+1)P(q)\right)^* \mathcal{A}(q,n+1)P(q)e^{2(\lambda-\rho)(n+1)} \\
&\quad - e^{2(\lambda-\rho)} \sum_{n>0} \left(\mathcal{A}(q,-(n-1))Q(q)\right)^* \mathcal{A}(q,-(n-1))Q(q)e^{2(\lambda-\rho)(n-1)} \\
&= e^{-2(\lambda-\rho)} \sum_{n\geq 1} \left(\mathcal{A}(q,n)P(q)\right)^* \mathcal{A}(q,n)P(q)e^{2(\lambda-\rho)n} \\
&\quad - e^{2(\lambda-\rho)} \sum_{n\geq 0} \left(\mathcal{A}(q,-n)Q(q)\right)^* \mathcal{A}(q,-n)Q(q)e^{2(\lambda-\rho)n} \\
&= e^{-2(\lambda-\rho)} \sum_{n\geq 0} \left(\mathcal{A}(q,n)P(q)\right)^* \mathcal{A}(q,n)P(q)e^{2(\lambda-\rho)n} - e^{-2(\lambda-\rho)}P(q)^*P(q) \\
&\quad - e^{2(\lambda-\rho)} \sum_{n>0} \left(\mathcal{A}(q,-n)Q(q)\right)^* \mathcal{A}(q,-n)Q(q)e^{2(\lambda-\rho)n} - e^{2(\lambda-\rho)}Q(q)^*Q(q).
\end{aligned}
\tag{21}
$$

Thus,

$$
\begin{aligned}
&A(q)^*S(f(q))A(q) - S(q) = \\
&= (e^{-2(\lambda-\rho)} - 1) \sum_{n\geq 0} \left(\mathcal{A}(q,n)P(q)\right)^* \mathcal{A}(q,n)P(q)e^{2(\lambda-\rho)n} \\
&\quad + (1 - e^{2(\lambda-\rho)}) \sum_{n>0} \left(\mathcal{A}(q,-n)Q(q)\right)^* \mathcal{A}(q,-n)Q(q)e^{2(\lambda-\rho)n} \\
&\quad - e^{-2(\lambda-\rho)}P(q)^*P(q) - e^{2(\lambda-\rho)}Q(q)^*Q(q).
\end{aligned}
\tag{22}
$$

Since $e^{-2(\lambda-\rho)} - 1 < 0$ and $1 - e^{2(\lambda-\rho)} < 0$, we obtain

$$
\begin{aligned}
A(q)^*S(f(q))A(q) - S(q) &\leq -e^{-2(\lambda-\rho)}P(q)^*P(q) - e^{2(\lambda-\rho)}Q(q)^*Q(q) \\
&\leq -e^{-2(\lambda-\rho)}(P(q)^*P(q) + Q(q)^*Q(q)).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&2\langle (P(q)^*P(q) + Q(q)^*Q(q))x, x\rangle \\
&= 2\|P(q)x\|^2 + 2\|Q(q)x\|^2 \\
&\geq \|P(q)x\|^2 + 2\|P(q)x\| \cdot \|Q(q)x\| + \|Q(q)x\|^2 \\
&= (\|P(q)x\| + \|Q(q)x\|)^2 \geq \|x\|^2
\end{aligned}
\tag{23}
$$

for each $x \in X$, which implies that

$$
-e^{-2(\lambda-\rho)}(P(q)^*P(q) + Q(q)^*Q(q)) \leq -\frac{1}{2}e^{-2(\lambda-\rho)}I.
$$

Hence, (19) holds taking $\delta = \frac{1}{2}e^{-2(\lambda-\rho)} > 0$.

In order to prove (20) we consider the space $l^2 = Y$ introduced in Section 2.2 taking $\|\cdot\|_n = \|\cdot\|$ for all $n \in \mathbb{Z}$. For each $q \in M$, we define a map $T_q \colon l^2 \to l^2$ by

$$
(T_q\mathbf{x})_n = A(f^{n-1}(q))x_{n-1} \quad \text{for } n \in Z \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.
$$

It follows from (17) that T_q is a well-defined bounded linear operator.

Lemma 1. *The adjoint $T_q^* \colon l^2 \to l^2$ is given by*

$$
(T_q^*\mathbf{x})_n = A(f^n(q))^*x_{n+1} \quad \textit{for } n \in \mathbb{Z} \textit{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.
$$

Proof of the lemma. We define a linear operator $G\colon l^2 \to l^2$ by

$$(G\mathbf{x})_n = A(f^n(q))^* x_{n+1} \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.$$

It follows from (17) that G is well-defined. For each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in l^2$ we have

$$\begin{aligned}\langle G\mathbf{x}, \mathbf{y}\rangle &= \sum_{n\in\mathbb{Z}} \langle (G\mathbf{x})_n, y_n\rangle \\ &= \sum_{n\in\mathbb{Z}} \langle A(f^n(q))^* x_{n+1}, y_n\rangle = \sum_{n\in\mathbb{Z}} \langle x_{n+1}, A(f^n(q)) y_n\rangle \\ &= \sum_{n\in\mathbb{Z}} \langle x_{n+1}, (T_q\mathbf{y})_{n+1}\rangle = \langle \mathbf{x}, T_q\mathbf{y}\rangle,\end{aligned}$$

which implies that $G = T_q^*$. □

Finally, for each $q \in X$, we define a map $W_q\colon l^2 \to l^2$ by

$$(W_q\mathbf{x})_n = S(f^n(q)) x_n \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.$$

Lemma 2. *W_q is a well-defined invertible self-adjoint linear operator and satisfies*

$$T_q^* W_q T_q - W_q \le -\delta I \tag{24}$$

for each $q \in M$.

Proof of the lemma. It follows from the first inequality in (18) that W_q is well-defined. Moreover, since $S(f^n(q))$ is self-adjoint we have

$$\langle W_q\mathbf{x}, \mathbf{y}\rangle = \sum_{n\in\mathbb{Z}} \langle S(f^n(q)) x_n, y_n\rangle = \sum_{n\in\mathbb{Z}} \langle x_n, S(f^n(q)) y_n\rangle = \langle \mathbf{x}, W_q\mathbf{y}\rangle$$

and so W_q is also self-adjoint. Moreover, from (19) and Lemma 1, we conclude that (24) holds. Hence, applying Theorem 1 together with Theorems 5 and 6 in the particular case when $\|\cdot\|_q = \|\cdot\|$ for all $q \in M$, we conclude that W_q is invertible for each $q \in M$. □

Lemma 3. *The map $q \mapsto \|W_q^{-1}\|$ is bounded on M.*

Proof of the lemma. For each $q \in M$, let

$$H_q = -T_q^* W_q T_q + W_q.$$

By (24) we have $H_q \ge \delta I$. It is easy to verify that

$$(T_q^* - I) W_q (T_q + I) + (T_q^* + I) W_q (T_q - I) = 2T_q^* W_q T_q - 2W_q.$$

Hence,

$$(T_q^* - I) W_q (T_q + I) + (T_q^* + I) W_q (T_q - I) = -2H_q.$$

Multiplying this identity on the right by $(T_q - I)^{-1}$ and on the left by $(T_q^* - I)^{-1}$, we obtain

$$W_q (T_q + I)(T_q - I)^{-1} + (T_q^* - I)^{-1}(T_q^* + I) W_q = -2(T_q^* - I)^{-1} H_q (T_q - I)^{-1}. \tag{25}$$

Thus,

$$\langle (T_q^* - I)^{-1} H_q (T_q - I)^{-1}\mathbf{x}, \mathbf{x}\rangle \le \frac{1}{2}\|W_q\mathbf{x}\| \cdot \|T_q + I\| \cdot \|(I - T_q)^{-1}\| \cdot \|\mathbf{x}\|$$

for each $\mathbf{x} \in l^2$.

On the other hand, we have

$$\begin{aligned}2\langle (T_q^*-I)^{-1}H_q(T_q-I)^{-1}\mathbf{x},\mathbf{x}\rangle &= 2\langle H_q(T_q-I)^{-1}\mathbf{x},(T_q-I)^{-1}\mathbf{x}\rangle\\ &\geq 2\delta\langle (T_q-I)^{-1}\mathbf{x},(T_q-I)^{-1}\mathbf{x}\rangle\\ &\geq 2\delta\|(T_q-I)^{-1}\mathbf{x}\|^2\\ &\geq 2\delta\frac{\|\mathbf{x}\|^2}{\|I-T_q\|^2}\end{aligned}$$

Combining these estimates, we obtain

$$2\delta\frac{\|\mathbf{x}\|^2}{\|I-T_q\|^2}\leq \|W_q\mathbf{x}\|\cdot\|T_q+I\|\cdot\|(I-T_q)^{-1}\|\cdot\|\mathbf{x}\|$$

and thus,

$$\|\mathbf{x}\|\leq\frac{1}{2\delta}\|W_q\mathbf{x}\|\cdot\|T_q+I\|\cdot\|(I-T_q)^{-1}\|\cdot\|I-T_q\|^2,$$

for each $\mathbf{x}\in l^2$. It follows from (17) and Theorem 5, taking $\|\cdot\|_q=\|\cdot\|$ for all $q\in M$, that there exists $L>0$ such that

$$\|\mathbf{x}\|\leq L\|W_q\mathbf{x}\|\quad\text{for } \mathbf{x}\in l^2.$$

This implies that $\|W_q^{-1}\|\leq L$ for $q\in M$. ☐

Lemma 4. *For each $q\in M$ the operator $S(q)$ is invertible and the second inequality in* (18) *holds.*

Proof of the lemma. We first show that the operators $S(q)$ are invertible. Assume that $S(q)v=0$ for some $v\in X$. Define $\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\in l^2$ by $x_0=v$ and $x_n=0$ for $n\neq 0$. Then $W_q\mathbf{x}=0$ and since W_q is invertible, we conclude that $\mathbf{x}=0$. Hence, $v=0$ and $S(q)$ is injective for each $q\in M$.

Now take $v\in X$ and define $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in l^2$ by $y_0=v$ and $y_n=0$ for $n\neq 0$. Since W_q is invertible, there exists $\mathbf{x}\in l^2$ such that $W_q\mathbf{x}=\mathbf{y}$. Thus, $(W_q\mathbf{x})_0=y_0$, which shows that $S(q)x_0=y_0=v$ and so $S(q)$ is surjective. Hence $S(q)$ is invertible. Moreover, we have

$$(S(q))^{-1}v=(W_q^{-1}\mathbf{y})_0$$

and so,

$$\|(S(q))^{-1}v\|=\|(W_q^{-1}\mathbf{y})_0\|\leq\|W_q^{-1}\mathbf{y}\|\leq\|W_q^{-1}\|\cdot\|\mathbf{y}\|=\|W_q^{-1}\|\cdot\|v\|.$$

Therefore, $\|(S(q))^{-1}\|\leq\|W_q^{-1}\|$ for all $q\in M$ and the second inequality in (18) follows directly from Lemma 3. ☐

Now we establish (20). Let H_q be as in the proof of Lemma 3. Multiplying identity (25) on the left and on the right by W_q^{-1}, we obtain

$$\begin{aligned}-2W_q^{-1}(T_q^*-I)^{-1}H_q(T_q-I)^{-1}W_q^{-1} &= (T_q+I)(T_q-I)^{-1}W_q^{-1}\\ &\quad + W_q^{-1}(T_q^*-I)^{-1}(T_q^*+I).\end{aligned}$$

Moreover, multiplying this equality on the left by T_q-I and on the right by T_q^*-I yields that

$$\begin{aligned}&-2(T_q-I)W_q^{-1}(T_q^*-I)^{-1}H_q(T_q-I)^{-1}W_q^{-1}(T_q^*-I)\\ &=(T_q+I)W_q^{-1}(T_q^*-I)+(T_q-I)W_q^{-1}(T_q^*+I).\end{aligned}$$

Since

$$(T_q+I)W_q^{-1}(T_q^*-I)+(T_q-I)W_q^{-1}(T_q^*+I)=2T_qW_q^{-1}T_q^*-2W_q^{-1},$$

we obtain

$$T_qW_q^{-1}T_q^*-W_q^{-1}=-(T_q-I)W_q^{-1}(T_q^*-I)^{-1}H_q(T_q-I)^{-1}W_q^{-1}(T_q^*-I).$$

Finally, note that for every $\mathbf{x}\in l^2$ we have

$$\begin{aligned}&\langle (T_q-I)W_q^{-1}(T_q^*-I)^{-1}H_q(T_q-I)^{-1}W_q^{-1}(T_q^*-I)\mathbf{x},\mathbf{x}\rangle \ge\\ &\ge \delta\|(T_q-I)^{-1}W_q^{-1}(T_q^*-I)\mathbf{x}\|^2\\ &\ge \frac{\delta\|\mathbf{x}\|^2}{\|T_q-I\|\cdot\|W_q\|\cdot\|(T_q^*-I)^{-1}\|}.\end{aligned}$$

Since there exists $L>0$ such that

$$\|T_q-I\|\cdot\|W_q\|\cdot\|(T_q^*-I)^{-1}\|=\|T_q-I\|\cdot\|W_q\|\cdot\|(T_q-I)^{-1}\|\le L$$

for every $q\in X$, we conclude that

$$T_qW_q^{-1}T_q^*-W_q^{-1}\le -\frac{\delta}{L}I \tag{26}$$

for $q\in M$, which establishes inequality (20). □

Now we establish the converse of Theorem 7.

Theorem 8. *Assume that the cocycle $\mathcal{A}$ satisfies* (17) *for some $C>0$ and that there exists a family $S(q)$, for $q\in M$, of invertible self-adjoint operators in $B(X)$ and constants $K,\delta>0$ satisfying* (18)*,* (19) *and* (20) *for every $q\in X$. Then the cocycle $\mathcal{A}$ is hyperbolic.*

Proof. For each $q\in M$, let T_q and W_q be as in the proof of Theorem 1. It follows from (19) and (20) that (24) and (26) hold. Hence, by Theorem 2, the operator T_q is hyperbolic and $I-T_q$ is invertible for each $q\in M$.

Lemma 5. *The map $q\mapsto\|(I-T_q)^{-1}\|$ is bounded on M.*

Proof of the lemma. It follows from the proof of Lemma 3 that

$$\delta\|(T_q-I)^{-1}\mathbf{x}\|^2\le\|W_q\|\cdot\|T_q+I\|\cdot\|(I-T_q)^{-1}\mathbf{x}\|\cdot\|\mathbf{x}\| \tag{27}$$

for every $q\in X$ and $\mathbf{x}\in l^2$. Moreover, by (17) and (18), there exists $L>0$ such that

$$\|W_q\|\cdot\|T_q+I\|\le L\quad\text{for } q\in M. \tag{28}$$

Hence, by (27) and (28), we obtain

$$\|(I-T_q)^{-1}\|\le\frac{L}{\delta}\quad\text{for } q\in M.$$

This completes the proof of the lemma. □

It follows from Theorem 6 taking $\|\cdot\|_q=\|\cdot\|$ for all $q\in M$ and Lemma 5 that the cocycle $\mathcal{A}$ is hyperbolic. □

As an application of the previous results, we establish in a simple manner the robustness of hyperbolicity.

Theorem 9. *Let $\mathcal{A}$ be a hyperbolic cocycle satisfying* (17) *for some* $C > 0$, *and let* $\mathcal{B}$ *be a cocycle with generator* B *such that there exists* $c > 0$ *satisfying*

$$\|A(q) - B(q)\| \le c \quad \text{for } q \in M. \tag{29}$$

If $c > 0$ *is sufficiently small, then* $\mathcal{B}$ *is also uniformly hyperbolic.*

Proof. We first note that it follows from (17) and (29) that

$$\|B(q)\| \le c + C \quad \text{for } q \in M. \tag{30}$$

By Theorem 7, there exists a family $S(q)$, for $q \in M$, of invertible self-adjoint operators in $B(X)$ and constants $K, \delta > 0$ satisfying (18), (19) and (20). For each $q \in M$ and $x \in X$ we have

$$\begin{aligned}
&\langle B(q)^* S(f(q))B(q)x, x\rangle - \langle S(q)x, x\rangle \\
&= \langle (B(q) - A(q))^* S(f(q))(B(q) - A(q))x, x\rangle \\
&\quad + \langle (B(q) - A(q))^* S(f(q))A(q)x, x\rangle \\
&\quad + \langle A(q)^* S(f(q))(B(q) - A(q))x, x\rangle \\
&\quad + \langle A(q)^* S(f(q))A(q)x, x\rangle - \langle S(q)x, x\rangle.
\end{aligned}$$

Using (18), (19), (29) and (30), we obtain

$$\begin{aligned}
\langle B(q)^* S(f(q))B(q)x, x\rangle - \langle S(q)x, x\rangle &\le -\delta\langle x, x\rangle + c^2 K\langle x, x\rangle + 2KcC\langle x, x\rangle \\
&= -(\delta - c^2 K - 2KcC)\langle x, x\rangle
\end{aligned}$$

for $q \in M$ and $x \in X$. Thus,

$$B(q)^* S(f(q))B(q) - S(q) \le -\tilde{\delta} I, \tag{31}$$

where $\tilde{\delta} = \delta - c^2 K - 2KcC$. Note that for c sufficiently small, we have $\tilde{\delta} > 0$. Similarly,

$$B(q)S(q)^{-1}B(q)^* - S(f(q))^{-1} \le -\tilde{\delta}' I \tag{32}$$

for some $\tilde{\delta}' > 0$. It follows from (18), (30), (31), (32) and Theorem 8 that the cocycle $\mathcal{B}$ is hyperbolic. ☐

4. Partially hyperbolic cocycles

In this section we consider briefly the notion of a partially hyperbolic cocycle. Let $\mathcal{A}$ be a cocycle. We say that $\mathcal{A}$ is *partially hyperbolic* if:

1. there exist projections $P^i(q)$, for $q \in M$ and $i = 1, 2, 3$, satisfying
$$P^1(q) + P^2(q) + P^3(q) = I, \quad A(q)P^i(q) = P^i(f(q))A(q)$$
for $q \in M$ and $i = 1, 2, 3$ such that the operator
$$A(q)|\operatorname{Im} P^i(q) \colon \operatorname{Im} P^i(q) \to \operatorname{Im} P^i(f(q))$$
is invertible for each $q \in M$ and $i = 2, 3$;
2. there exist constants
$$D > 0, \quad 0 \le a < b \quad \text{and} \quad 0 \le c < d$$
such that for each $q \in M$ we have
$$\|\mathcal{A}(q, n)P^1(q)\| \le De^{-dn}, \quad \|\mathcal{A}(q, n)P^3(q)\| \le De^{an} \tag{33}$$
for $n \ge 0$ and
$$\|\mathcal{A}(q, -n)P^2(q)\| \le De^{-bn}, \quad \|\mathcal{A}(q, -n)P^3(q)\| \le De^{cn} \tag{34}$$

for $n \geq 0$, with $\mathcal{A}(q,-n)P^i(q)$ in (34) given by

$$\big(\mathcal{A}(f^{-n}(q),n)|\operatorname{Im} P^i(f^{-n}(q))\big)^{-1} \colon \operatorname{Im} P^i(q) \to \operatorname{Im} P^i(f^{-n}(q))$$

for $i = 2, 3$.

The following result is a version of Theorem 7 for partially hyperbolic cocycles.

Theorem 10. *Assume that the cocycle $\mathcal{A}$ is partially hyperbolic and that there exists $C > 0$ satisfying* (17)*. Then there exist families $S^i(q)$, for $q \in M$ and $i = 1, 2$, of invertible self-adjoint operators in $B(X)$ and constants $K, \delta, \omega_1 > 0$ and $\omega_2 < 0$ such that for each $q \in M$ and $i = 1, 2$:*

1. $$\|S^i(q)\| \leq K \quad \text{and} \quad \|S^i(q)^{-1}\| \leq K; \tag{35}$$
2. $$e^{2\omega_i} A(q)^* S(f(q)) A(q) - S(q) \leq -\delta I; \tag{36}$$
3. $$e^{2\omega_i} A(q) S(q)^{-1} A(q)^* - S(f(q))^{-1} \leq -\delta I. \tag{37}$$

Proof. Take $\omega_1 \in (c, d)$ and consider the cocycle $\mathcal{B}$ with generator $B(q) = e^{\omega_1} A(q)$. We note that

$$\mathcal{B}(q,n) = e^{\omega_1 n} \mathcal{A}(q,n) \quad \text{for } q \in M \text{ and } n \geq 0.$$

It follows from (33) and (34) that

$$\|\mathcal{B}(q,n)P^1(q)\| \leq De^{-(d-\omega_1)n}, \tag{38}$$

$$\|\mathcal{B}(q,-n)P^2(q)\| \leq De^{-(b+\omega_1)n} \tag{39}$$

and

$$\|\mathcal{B}(q,-n)P^3(q)\| \leq De^{-(\omega_1-c)n} \tag{40}$$

for $q \in M$ and $n \geq 0$. By (39) and (40), we obtain

$$\|\mathcal{B}(q,-n)(P^2(q) + P^3(q))\| \leq 2De^{-\min\{b+\omega_1, \omega_1-c\}n} \tag{41}$$

for $q \in M$ and $n \geq 0$. In view of (38) and (41), the cocycle $\mathcal{B}$ is hyperbolic. Hence, applying Theorem 7 we obtain a family $S^1(q)$, for $q \in M$, of invertible self-adjoint operators satisfying (35), (36) and (37) for $i = 1$.

Now take $\omega_2 \in (-b, -a)$ and consider the cocycle $\mathcal{C}$ with generator $C(q) = e^{\omega_2} A(q)$. Then

$$\mathcal{C}(q,n) = e^{\omega_2 n} \mathcal{A}(q,n) \quad \text{for } n \geq 0 \text{ and } q \in M.$$

It follows from (33) and (34) that

$$\|\mathcal{C}(q,n)P^1(q)\| \leq De^{-(d-\omega_2)n}, \tag{42}$$

$$\|\mathcal{C}(q,n)P^3(q)\| \leq De^{-(-a-\omega_2)n} \tag{43}$$

and

$$\|\mathcal{C}(q,-n)P^2(q)\| \leq De^{-(b+\omega_2)n} \tag{44}$$

for $q \in M$ and $n \geq 0$. By (42) and (43), we obtain

$$\|\mathcal{C}(q,n)(P^1(q) + P^3(q))\| \leq 2De^{-\min\{d-\omega_2, -a-\omega_2\}n} \tag{45}$$

for $q \in M$ and $n \geq 0$. In view of (44) and (45), the cocycle $\mathcal{C}$ is hyperbolic. Again applying Theorem 7 we obtain a family $S^2(q)$, for $q \in M$, of invertible self-adjoint operators satisfying (35), (36) and (37) for $i = 2$. This completes the proof of the theorem. ☐

The following result is a converse of Theorem 10.

Theorem 11. *Assume that the cocycle $\mathcal{A}$ satisfies* (17) *for some $C > 0$ and that there exists families $S^i(q)$, for $q \in M$ and $i = 1, 2$, of invertible self-adjoint operators in $B(X)$ and constants $K, \delta, \omega_1 > 0$ and $\omega_2 < 0$ satisfying* (35), (36) *and* (37) *for $q \in X$. Then the cocycle $\mathcal{A}$ is partially hyperbolic.*

Proof. Let $\mathcal{B}$ be the cocycle with $B(q) = e^{\omega_1} A(q)$ and let $\mathcal{C}$ be the cocycle with generator $C(q) = e^{\omega_2} A(q)$. It follows from Theorem 8 that the cocycles $\mathcal{B}$ and $\mathcal{C}$ are hyperbolic. Hence, there exist families of projections $R^i(q)$, for $q \in M$ and $i = 1, 2$, satisfying

$$A(q)R^i(q) = R^i(f(q))A(q) \quad \text{for } q \in M \text{ and } i = 1, 2,$$

such that each map

$$A(q)|\operatorname{Ker} R^i(q) \colon \operatorname{Ker} R^i(q) \to \operatorname{Ker} R^i(f(q))$$

is invertible, and there exists constants $D, \lambda > 0$ such that for each $q \in X$ and $n \geq 0$ we have

$$\|\mathcal{B}(q, n)R^1(q)\| \leq De^{-\lambda n}, \tag{46}$$

$$\|\mathcal{C}(q, n)R^2(q)\| \leq De^{-\lambda n}, \tag{47}$$

$$\|\mathcal{B}(q, -n)(I - R^1(q))\| \leq De^{-\lambda n} \tag{48}$$

and

$$\|\mathcal{C}(q, -n)(I - R^2(q))\| \leq De^{-\lambda n}. \tag{49}$$

Lemma 6. *For each $q \in M$, we have*

$$\operatorname{Im} R^1(q) \subset \operatorname{Im} R^2(q) \quad \text{and} \quad \operatorname{Ker} R^2(q) \subset \operatorname{Ker} R^1(q).$$

Proof of the lemma. Take $v \in \operatorname{Im} R^1(q)$. It follows from Proposition 1 that $\sup_{n\geq 0}\|\mathcal{B}(q, n)v\| < +\infty$. Since $\omega_2 < \omega_1$, we have

$$\|\mathcal{C}(q, n)v\| = e^{\omega_2 n}\|\mathcal{A}(q, n)v\| \leq e^{\omega_1 n}\|\mathcal{A}(q, n)v\| = \|\mathcal{B}(q, n)v\|.$$

Hence, $\sup_{n\geq 0}\|\mathcal{C}(q, n)v\| < +\infty$ and by Proposition 1, we conclude that $v \in \operatorname{Im} R^2(q)$. The proof of the second inclusion is analogous. ☐

Lemma 7. *The map $R^2(q) - R^1(q)$ is a projection for each $q \in M$.*

Proof of the lemma. It follows from Lemma 6 that

$$R^1(q)R^2(q) = R^2(q)R^1(q) = R^1(q)$$

for $q \in M$. Hence,

$$\begin{aligned}(R^2(q) - R^1(q))^2 &= (R^2(q))^2 - R^2(q)R^1(q) - R^1(q)R^2(q) + (R^1(q))^2 \\ &= R^2(q) - R^1(q) - R^1(q) + R^1(q) \\ &= R^2(q) - R^1(q),\end{aligned}$$

which establishes the desired property. ☐

Lemma 8. *For each $q \in M$, we have*

$$\operatorname{Im}(R^2(q) - R^1(q)) = \operatorname{Im} R^2(q) \cap \operatorname{Ker} R^1(q).$$

Proof of the lemma. Take $v \in \operatorname{Im} R^2(q) \cap \operatorname{Ker} R^1(q)$. We have $R^2(q)v = v$ and $R^1(q)v = 0$, and thus,

$$(R^2(q) - R^1(q))v = v.$$

This implies that $v \in \operatorname{Im}(R^2(q) - R^1(q))$. Now take $v \in \operatorname{Im}(R^2(q) - R^1(q))$. Then $R^2(q)v - R^1(q)v = v$. Since $\operatorname{Im} R^1(q) \subset \operatorname{Im} R^2(q)$, we conclude that $v \in \operatorname{Im} R^2(q)$. Moreover,

$$R^1(q)v = R^1(q)R^2(q)v - R^1(q)v = R^1(q)v - R^1(q)v = 0,$$

which implies that $v \in \operatorname{Ker} R^1(q)$. □

We can now complete the proof of the theorem. It follows from (46) that

$$\|\mathcal{A}(q,n)R^1(q)\| \le De^{-(\lambda+\omega_1)n} \quad \text{for } q \in M \text{ and } n \ge 0. \tag{50}$$

Similarly, by (49) we have

$$\|\mathcal{A}(q,-n)(I - R^2(q))\| \le De^{-(\lambda-\omega_2)n} \quad \text{for } q \in M \text{ and } n \ge 0. \tag{51}$$

Moreover, it follows from (47), (48) and Lemma 8 that for every $v \in \operatorname{Im}(R^2(q) - R^1(q))$, we have

$$\|\mathcal{A}(q,n)v\| \le De^{-(\lambda+\omega_2)n}\|v\|$$

and

$$\|\mathcal{A}(q,-n)v\| \le De^{-(\lambda-\omega_1)n}\|v\|$$

for $q \in M$ and $n \ge 0$. On the other hand, it follows from (46) and (47) that $\|R^2(q) - R^1(q)\| \le 2D$. Hence,

$$\|\mathcal{A}(q,n)(R^2(q) - R^1(q))\| \le 2D^2e^{-(\lambda+\omega_2)n} \tag{52}$$

and

$$\|\mathcal{A}(q,-n)(R^2(q) - R^1(q))\| \le 2D^2De^{-(\lambda-\omega_1)n} \tag{53}$$

for $q \in M$ and $n \ge 0$. Finally, it follows from (50), (51), (52) and (53) that the cocycle $\mathcal{A}$ is partially hyperbolic. □

The following result is a version of Theorem 9 for partially hyperbolic cocycles. The proof can be obtained repeating the arguments in the proof of Theorem 9, using Theorems 10 and 11 instead of Theorems 7 and 8.

Theorem 12. *Let $\mathcal{A}$ be a partially hyperbolic cocycle satisfying* (17) *for some $C > 0$, and let $\mathcal{B}$ be a cocycle with generator B such that there exists $c > 0$ satisfying* (29)*. If $c > 0$ is sufficiently small, then the cocycle $\mathcal{B}$ is also partially hyperbolic.*

5. Nonuniformly hyperbolic cocycles

In this section we establish a version of the results in Section 3 for a nonuniformly hyperbolic cocycle. Since some arguments are analogous, at various places we refer to the corresponding arguments in Section 3.

Given an arbitrary function $K\colon M \to (1,\infty)$, a cocycle $\mathcal{A}$ is said to be *K-nonuniformly hyperbolic* if:

1. there exists a family of projections $P(q)$, for $q \in M$, satisfying (6) such that each map in (7) is invertible;

2. there exist constants $\lambda, D > 0$ such that for each $q \in M$ and $n \geq 0$ we have
$$\|\mathcal{A}(q,n)P(q)\| \leq DK(q)e^{-\lambda n} \tag{54}$$
and
$$\|\mathcal{A}(q,-n)Q(q)\| \leq DK(q)e^{-\lambda n}, \tag{55}$$
where $Q(q) = I - P(q)$, with $\mathcal{A}(q,-n)$ as in (10).

The following result is a version of Theorem 7 for nonuniform hyperbolic cocycles.

Theorem 13. *Assume that the cocycle $\mathcal{A}$ is K-nonuniformly hyperbolic and that there exist $C, \mu > 0$ such that*
$$\|\mathcal{A}(q,n)\| \leq Ce^{\mu n} \quad \text{for } q \in M \text{ and } n \geq 0. \tag{56}$$
Then there exist self-adjoint operators $S^1(q) \geq 0$ and $S^2(q) \leq 0$ for $q \in M$ and constants $L, r, B > 0$ such that $S(q) = S^1(q) + S^2(q)$ and $D(q) = S^1(q) - S^2(q)$ are invertible and satisfy
$$I \leq 2D(q) \quad \text{and} \quad \|D(q)\| \leq LK(q)^2, \tag{57}$$
$$A(q)^*S(f(q))A(q) - S(q) \leq -rD(q), \tag{58}$$
$$D(q)A(f^{-1}(q))S(f^{-1}(q))^{-1}A(f^{-1}(q))^*D(q) - D(q)S(q)^{-1}D(q) \leq -rD(q), \tag{59}$$
$$A(q)^*D(f(q))A(q) \leq BD(q), \tag{60}$$
$$S(q)D(q)^{-1}S(q) \leq BD(q) \tag{61}$$
and
$$D(q)S(q)^{-1}D(q)S(q)^{-1}D(q) \leq BD(q). \tag{62}$$

Proof. Without loss of generality, we may assume that $\lambda \leq \mu$ and so one can take $\rho \in (0, \lambda)$ such that
$$\lambda - \rho < \mu + \rho. \tag{63}$$
For each $q \in M$, let
$$S^1(q) = \sum_{n \geq 0} (\mathcal{A}(q,n)P(q))^* \mathcal{A}(q,n)P(q)e^{2(\lambda-\rho)n}.$$
It follows from (54) that
$$\begin{aligned} 0 \leq \langle S^1(q)x, x\rangle &\leq \sum_{n\geq 0} \|\mathcal{A}(q,n)P(q)x\|^2 e^{2(\lambda-\rho)n} \\ &\leq D^2K(q)^2 \sum_{n\geq 0} e^{-2\lambda n}e^{2(\lambda-\rho)n}\|x\|^2 \\ &\leq D'K(q)^2\|x\|^2 \end{aligned}$$
for some constant $D' > 0$. Clearly, $S^1(q)$ is self-adjoint and so
$$\|S^1(q)\| = \sup_{\|x\|=1} \langle S^1(q)x, x\rangle \leq D'K(q)^2. \tag{64}$$

Lemma 9. *For each $q \in M$ we have*
$$A(q)^*S^1(f(q))A(q) - S^1(q) \leq (e^{-2(\lambda-\rho)} - 1)S^1(q). \tag{65}$$

Proof of the lemma. By (6), we have

$$\begin{aligned}
&A(q)^*S^1(f(q))A(q)-S^1(q)\\
&=e^{-2(\lambda-\rho)}\sum_{n\ge 1}(\mathcal{A}(q,n)P(q))^*\mathcal{A}(q,n)P(q)e^{2(\lambda-\rho)n}\\
&\quad-\sum_{n\ge 0}(\mathcal{A}(q,n)P(q))^*\mathcal{A}(q,n)P(q)e^{2(\lambda-\rho)n}\\
&=-e^{-2(\lambda-\rho)}P(q)^*P(q)+(e^{-2(\lambda-\rho)}-1)S^1(q)\\
&\le (e^{-2(\lambda-\rho)}-1)S^1(q),
\end{aligned}$$

which establishes inequality (65). □

Before proceeding, we note that it follows from (55) and (56) that for $q\in M$ and $n\ge 0$ we have

$$\|\mathcal{A}(q,n)Q(q)\|\le CDK(q)e^{\mu n}. \tag{66}$$

For each $q\in M$, let

$$\begin{aligned}
S^2(q)=&-\sum_{n>0}(\mathcal{A}(q,-n)Q(q))^*\mathcal{A}(q,-n)Q(q)e^{2(\lambda-\rho)n}\\
&-\sum_{n\ge 0}(\mathcal{A}(q,n)Q(q))^*\mathcal{A}(q,n)Q(q)e^{-2(\mu+\rho)n}.
\end{aligned}$$

By (55) and (66), we have

$$\begin{aligned}
0\le\langle -S^2(q)x,x\rangle&=\sum_{n>0}\|\mathcal{A}(q,-n)Q(q)x\|^2e^{2(\lambda-\rho)n}\\
&\quad+\sum_{n\ge 0}\|\mathcal{A}(q,n)Q(q)x\|^2e^{-2(\mu+\rho)n}\\
&\le D^2K(q)^2\sum_{n>0}e^{-2\lambda n}e^{2(\lambda-\rho)n}\|x\|^2\\
&\quad+C^2D^2K(q)^2\sum_{n\ge 0}e^{2\mu n}e^{-2(\mu+\rho)n}\|x\|^2\\
&\le D''K(q)^2\|x\|^2
\end{aligned}$$

for some constant $D''>0$. Clearly, $S(q)^2$ is self-adjoint and so

$$\|S^2(q)\|=\sup_{\|x\|=1}\langle -S^2(q)x,x\rangle\le D''K(q)^2. \tag{67}$$

Lemma 10. *Each operator $D(q)$ is invertible and the first inequality in* (57) *holds.*

Proof of the lemma. We have

$$\begin{aligned}
\|D(q)x\|\cdot\|x\|\ge\langle D(q)x,x\rangle&=\langle S^1(q)x,x\rangle+\langle(-S^2(q))x,x\rangle\\
&\ge\|P(q)x\|^2+\|Q(q)x\|^2\ge\frac{1}{2}\|x\|^2
\end{aligned} \tag{68}$$

(see (23)) and hence,

$$\|D(q)x\|\ge\frac{1}{2}\|x\|\quad\text{for } q\in M \text{ and } x\in X. \tag{69}$$

It follows from (69) that $D(q)$ is injective and that $\operatorname{Im} D(q)$ is closed in X. Indeed, assume that $(D(q)x_n)_n$ is a sequence in $\operatorname{Im} D(q)$ that converges to $y \in X$. By (69), we have

$$\|D(q)x_n - D(q)x_m\| \geq \frac{1}{2}\|x_n - x_m\|$$

for $m, n \in \mathbb{N}$, which readily implies that $(x_n)_n$ is a Cauchy sequence in X. Hence, it converges to some point $x \in X$. Since $D(q)$ is continuous, we conclude that $D(q)x_n$ converges to $D(q)x$ and so $y = D(q)x \in \operatorname{Im} D(q)$. Moreover, since $D(q)$ is self-adjoint, we have

$$\operatorname{Im} D(q) = \overline{\operatorname{Im} D(q)} = (\operatorname{Ker} D(q)^*)^\perp = (\operatorname{Ker} D(q))^\perp = \{0\}^\perp = X,$$

which implies that $D(q)$ is onto. Therefore, $D(q)$ is invertible. Moreover, the first inequality in (57) follows directly from (68). □

By (64) and (67), the second inequality in (57) holds with $L = D' + D''$.

Lemma 11. *For each $q \in M$ we have*

$$A(q)^* S^2(f(q))A(q) - S^2(q) \leq (1 - e^{2(\lambda-\rho)})(-S^2(q)). \tag{70}$$

Proof of the lemma. By (6), proceeding as in (21) and (22) we obtain

$$\begin{aligned}
&A(q)^* S^2(f(q))A(q) - S^2(q) \\
&= -e^{2(\lambda-\rho)} \sum_{n\geq 0} (\mathcal{A}(q,-n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{2(\lambda-\rho)n} \\
&\quad + \sum_{n>0} (\mathcal{A}(q,-n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{2(\lambda-\rho)n} \\
&\quad - e^{2(\mu+\rho)} \sum_{n\geq 1} (\mathcal{A}(q,n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{-2(\mu+\rho)n} \\
&\quad + \sum_{n\geq 0} (\mathcal{A}(q,n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{-2(\mu+\rho)n} \\
&= -e^{2(\lambda-\rho)} Q(q)^* Q(q) + e^{2(\mu+\rho)} Q(q)^* Q(q) \\
&\quad + (1 - e^{2(\lambda-\rho)}) \sum_{n>0} (\mathcal{A}(q,-n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{2(\lambda-\rho)n} \\
&\quad + (1 - e^{2(\mu+\rho)}) \sum_{n\geq 0} (\mathcal{A}(q,n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{-2(\mu+\rho)n} \\
&= (1 - e^{2(\lambda-\rho)})(-S^2(q)) \\
&\quad + (e^{2(\lambda-\rho)} - e^{2(\mu+\rho)}) \sum_{n\geq 1} (\mathcal{A}(q,n)Q(q))^* \mathcal{A}(q,n)Q(q)e^{-2(\mu+\rho)n}
\end{aligned}$$

and the desired property follows from (63). □

It follows readily from (65) and (70) that (58) holds taking

$$r = \min\{e^{2(\lambda-\rho)} - 1, 1 - e^{-2(\lambda-\rho)}\} > 0.$$

For each $q \in M$, we define a scalar product on X by

$$\langle x, y\rangle_q = \langle D(q)x, y\rangle \quad \text{for } x, y \in X, \tag{71}$$

and we denote the corresponding norm by $\|\cdot\|_q$.

Lemma 12. *For each $q \in M$, $x \in X$ and $n \geq 0$, we have*

$$\|\mathcal{A}(q,n)P(q)x\|_{f^n(q)} \leq e^{-(\lambda-\rho)n}\|x\|_q \tag{72}$$

and

$$\|\mathcal{A}(q,-n)Q(q)x\|_{f^{-n}(q)} \leq e^{-(\lambda-\rho)n}\|x\|_q. \tag{73}$$

Moreover, there exists $B>0$ satisfying (60)*.*

Proof of the lemma. We have

$$\begin{aligned}\|A(q)P(q)x\|_{f(q)}^2 &= \sum_{n\geq 0}\|\mathcal{A}(f(q),n)P(f(q))A(q)P(q)x\|^2 e^{2(\lambda-\rho)n}\\ &\leq e^{-2(\lambda-\rho)}\sum_{n\geq 1}\|\mathcal{A}(q,n)P(q)x\|^2 e^{2(\lambda-\rho)n}\\ &\leq e^{-2(\lambda-\rho)}\|x\|_q^2\end{aligned}$$

for $q \in M$. Iterating the procedure, we obtain (72). Similarly,

$$\begin{aligned}\|A(q)^{-1}Q(f(q))x\|_q^2 &= \sum_{n>0}\|\mathcal{A}(f(q),-(n+1))Q(f(q))x\|^2 e^{2(\lambda-\rho)n}\\ &\quad+\sum_{n\geq 0}\|\mathcal{A}(f(q),n-1)Q(f(q))x\|^2 e^{-2(\mu+\rho)n}\\ &= e^{-2(\lambda-\rho)}\sum_{n>0}\|\mathcal{A}(f(q),-n)Q(f(q))x\|^2 e^{2(\lambda-\rho)n}\\ &\quad+e^{-2(\mu+\rho)}\sum_{n\geq 0}\|\mathcal{A}(f(q),n)Q(f(q))x\|^2 e^{-2(\mu+\rho)n}\\ &\quad-\|A(q)^{-1}Q(f(q))x\|^2+\|A(q)^{-1}Q(f(q))x\|^2\\ &\leq e^{-2(\lambda-\rho)}\|x\|_{f(q)}^2\end{aligned}$$

and hence inequality (73) holds.

Moreover,

$$\begin{aligned}\|A(q)Q(q)x\|_{f(q)}^2 &= \sum_{n>0}\|\mathcal{A}(q,-(n-1))Q(q)x\|^2 e^{2(\lambda-\rho)n}\\ &\quad+\sum_{n\geq 0}\|\mathcal{A}(q,n+1)Q(q)x\|^2 e^{-2(\mu+\rho)n}\\ &= e^{2(\lambda-\rho)}\sum_{n>0}\|\mathcal{A}(q,-n)Q(q)x\|^2 e^{2(\lambda-\rho)n}\\ &\quad+e^{2(\mu+\rho)}\sum_{n\geq 0}\|\mathcal{A}(q,n)Q(q)x\|^2 e^{-2(\mu+\rho)n}\\ &\quad+e^{2(\lambda-\rho)}\|Q(q)x\|^2-e^{2(\mu+\rho)}\|Q(q)x\|^2\\ &\leq e^{2(\mu+\rho)}\|x\|_q^2\end{aligned}$$

and so,

$$\|A(q)Q(q)x\|_{f(q)} \leq e^{\mu+\rho}\|x\|_q. \tag{74}$$

By (72) and (74), we obtain

$$\|A(q)x\|_{f(q)} \leq \|A(q)P(q)x\|_{f(q)}+\|A(q)Q(q)x\|_{f(q)} \leq 2e^{\mu+\rho}\|x\|_q,$$

which shows that (60) holds with $B=4e^{2(\mu+\rho)}$. □

Now let Y_q and T_q be as in (11) and (13). It follows from (56) that T_q is well-defined.

Lemma 13. *The adjoint* $T_q^*\colon Y_q \to Y_q$ *is given by*

$$(T_q^*\mathbf{x})_n = D(f^n(q))^{-1}A(f^n(q))^*D(f^{n+1}(q))y_{n+1} \tag{75}$$

for $n \in \mathbb{Z}$ *and* $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_q$.

Proof of the lemma. For each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_q$, we have

$$\begin{aligned}
\langle T_q\mathbf{x}, \mathbf{y}\rangle &= \sum_{n\in\mathbb{Z}} \langle (T_q\mathbf{x})_n, y_n\rangle_{f^n(q)} = \sum_{n\in\mathbb{Z}} \langle A(f^{n-1}(q))x_{n-1}, y_n\rangle_{f^n(q)} \\
&= \sum_{n\in\mathbb{Z}} \langle D(f^n(q))A(f^{n-1}(q))x_{n-1}, y_n\rangle \\
&= \sum_{n\in\mathbb{Z}} \langle x_{n-1}, A(f^{n-1}(q))^*D(f^n(q))y_n\rangle \\
&= \sum_{n\in\mathbb{Z}} \langle D(f^{n-1}(q))x_{n-1}, D(f^{n-1}(q))^{-1}A(f^{n-1}(q))^*D(f^n(q))y_n\rangle \\
&= \sum_{n\in\mathbb{Z}} \langle x_{n-1}, D(f^{n-1}(q))^{-1}A(f^{n-1}(q))^*D(f^n(q))y_n\rangle_{f^{n-1}(q)},
\end{aligned}$$

which yields the desired result. □

Lemma 14. *There exists* $B > 0$ *satisfying* (61).

Proof of the lemma. We first note that

$$S(q)P(q)x = D(q)P(q)x = S^1(q)x$$

and

$$S(q)Q(q)x = -D(q)Q(q)x = S^2(q)x.$$

Hence,

$$\begin{aligned}
\|D(q)^{-1}S(q)x\|_q &\le \|D(q)^{-1}S(q)P(q)x\|_q + \|D(q)^{-1}S(q)Q(q)x\|_q \\
&= \|D(q)^{-1}D(q)P(q)x\|_q + \|D(q)^{-1}D(q)Q(q)x\|_q \\
&= \|P(q)x\|_q + \|Q(q)x\|_q \le 2\|x\|_q.
\end{aligned}$$

Since

$$\|D(q)^{-1}S(q)x\|_q^2 = \langle S(q)D(q)^{-1}S(q)x, x\rangle \quad \text{and} \quad \|x\|_q = \langle D(q)x, x\rangle,$$

we conclude that (61) holds with $B = 4$. □

Now we define an operator $W_q\colon Y_q \to Y_q$ by

$$(W_q\mathbf{x})_n = D(f^n(q))^{-1}S(f^n(q))x_n \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_q. \tag{76}$$

It follows from Lemma 14 that W_q is well defined.

Lemma 15. *The operator* W_q *is self-adjoint.*

Proof of the lemma. For each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_q$, we have

$$\begin{aligned}\langle W_q\mathbf{x}, \mathbf{y}\rangle &= \sum_{n\in\mathbb{Z}} \langle D(f^n(q))^{-1}S(f^n(q))x_n, y_n\rangle_{f^n(q)} = \sum_{n\in\mathbb{Z}} \langle S(f^n(q))x_n, y_n\rangle \\ &= \sum_{n\in\mathbb{Z}} \langle x_n, S(f^n(q))y_n\rangle = \sum_{n\in\mathbb{Z}} \langle D(f^n(q))x_n, D(f^n(q))^{-1}S(f^n(q))y_n\rangle \\ &= \sum_{n\in\mathbb{Z}} \langle x_n, D(f^n(q))^{-1}S(f^n(q))y_n\rangle_{f^n(q)} = \langle \mathbf{x}, W_q\mathbf{y}\rangle,\end{aligned}$$

which shows that W_q is self-adjoint. □

It follows from (58) and (75) that

$$\begin{aligned}&\langle T_q^* W_q T_q \mathbf{x}, \mathbf{x}\rangle - \langle W_q\mathbf{x}, \mathbf{x}\rangle \\ &= \sum_{n\in\mathbb{Z}} \langle D(f^n(q))^{-1}A(f^n(q))^* S(f^{n+1}(q))A(f^n(q))x_n, x_n\rangle_{f^n(q)} \\ &\quad - \sum_{n\in\mathbb{Z}} \langle D(f^n(q))^{-1}S(f^n(q))x_n, x_n\rangle_{f^n(q)} \\ &= \sum_{n\in\mathbb{Z}} \langle (A(f^n(q))^* S(f^{n+1}(q))A(f^n(q)) - S(f^n(q)))x_n, x_n\rangle \\ &\le -r\sum_{n\in\mathbb{Z}} \langle D(f^n(q))x_n, x_n\rangle \\ &= -r\sum_{n\in\mathbb{Z}} \langle x_n, x_n\rangle_{f^n(q)} = -r\langle \mathbf{x}, \mathbf{x}\rangle\end{aligned}$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_q$. Hence,

$$T_q^* W_q T_q - W_q \le -rI \quad \text{on } Y_q.$$

It follows from Theorems 1 and 5 together with Lemma 15 that the operator W_q is invertible. Proceeding as in the proof of Lemma 3, one can show that the function $q \to \|W_q^{-1}\|$ is bounded. Moreover, proceeding as in the proof of Lemma 4, one concludes that the operators $S(q)$ are invertible.

Lemma 16. *There exists $B > 0$ satisfying* (62).

Proof of the lemma. We have that

$$(W_q^{-1}\mathbf{x})_n = S(f^n(q))^{-1}D(f^n(q))x_n \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_q. \tag{77}$$

Given $v \in X$, define $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ by $x_0 = v$ and $x_n = 0$ for $n \neq 0$. Then

$$\begin{aligned}\langle D(q)S(q)^{-1}D(q)v, S(q)^{-1}D(q)v\rangle &\le \|W_q^{-1}\mathbf{x}\|^2 \le \|W_q^{-1}\|^2 \cdot \|\mathbf{x}\|^2 \\ &\le \|W_q^{-1}\|^2 \cdot \langle D(q)v, v\rangle.\end{aligned}$$

Hence, (62) holds with $B = \sup_{q\in M}\|W_q^{-1}\|$. □

The remainder of the argument is analogous to that in the proof of Theorem 7 leading to (26), and we conclude that there exists $r' > 0$ such that

$$T_q W_q^{-1} T_q^* - W_q^{-1} \le -r'I \quad \text{on } Y_q$$

for every $q \in M$. This establishes inequality (59). □

Now we establish the converse of Theorem 13.

Theorem 14. *Let $\mathcal{A}$ be a cocycle and assume that there exist self-adjoint operators $S^1(q) \geq 0$ and $S^2(q) \leq 0$, for $q \in M$, and constants $L, r, B > 0$ such that $S(q) = S^1(q) + S^2(q)$ and $D(q) = S^1(q) - S^2(q)$ are invertible and satisfy* (57)*–*(62)*. Then $\mathcal{A}$ is K-nonuniformly hyperbolic.*

Proof. Let Y_q be as in (11), where $\|\cdot\|_q$ is the norm induced by the scalar product in (71). Moreover, let T_q be the operator on Y_q given by (13). It follows from (60) that T_q is well-defined. Finally, let W_q be the operator in (76). It follows from (61) that W_q is also well-defined. Moreover, by (62), W_q is invertible and W_q^{-1} is given by (77). It follows from (58) and (59) that

$$T_q^* W_q T_q - W_q \leq -rI \quad \text{and} \quad T_q W_q^{-1} T_q^* - W_q^{-1} \leq -rI$$

for $q \in M$. Hence, by Theorem 2, the operator T_q is hyperbolic and so in particular $I - T_q$ is invertible. Proceeding as in the proof of Theorem 8, we conclude that the map $q \mapsto \|(I - T_q)^{-1}\|$ is bounded on M. Hence, by Theorem 6, there exist projections $P(q)$, for $q \in M$, satisfying (6) and constants $D, \lambda > 0$ such that

$$\|\mathcal{A}(q, n)x\|_{f^n(q)} \leq De^{-\lambda n}\|x\|_q \tag{78}$$

and

$$\|\mathcal{A}(q, -n)x\|_{f^{-n}(q)} \leq De^{-\lambda n}\|x\|_q, \tag{79}$$

for $q \in M$, $n \geq 0$ and $x \in X$. It follows from (57), (78) and (79) that the cocycle $\mathcal{A}$ is K-nonuniformly hyperbolic. ☐

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