A NOTE ON THE BOX DIMENSION OF DEGENERATE FOCI

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ABSTRACT. We study polynomial planar systems with singularity of focus type without characteristic directions. Simple and natural transformation of weak focus has been used to obtain such degenerate focus. We compute the box dimension of a spiral trajectory, and show connection to cyclicity of the system under a perturbation.

1. INTRODUCTION

Analysis of the Poincaré map, or the first return map is a standard approach in the study of monodromic singular points and limit cycles. The Poincaré map and the normal form of a weak focus has been studied in [6] and [7], from the point of view of fractal geometry. Weak focus is a singular point having pure imaginary eigenvalues of the linear part. The box dimension of a spiral trajectory near weak focus, and also near a limit cycle has been computed. Furthermore, the explicit relation between the box dimension and the leading power in the asymptotic expansion of the Poincaré map of the weak focus has been obtained.

Here we announce our results which extend the investigation to certain classes of degenerate foci. Nilpotent focus and focus with no linear part are called degenerate foci. The nilpotent focus is the focus with nilpotent matrix of the linear part. Here we deal with systems having degenerate focus without linear part and without characteristic directions. Characteristic directions can be seen after blowing up of the system.

Here we use results for weak focus from [6] and [7] and apply them to degenerate focus without characteristic directions. Such degenerate focus has the same asymptotic of the Poincaré map in each direction. In general, the Poincaré map of degenerate focus has different asymptotic expansion depending on the direction. That is the reason why the approach to weak focus and to degenerate focus should be different. Nilpotent focus has two asymptotics, on the characteristic curve and elsewhere. Degenerate focus can have more than one characteristic directions. The asymptotic expansion of the Poincaré map near focus has been computed in [4]. The basic technique, method of blowing-up, shows that these characteristic directions are associated to singular points of the obtained polycycle.

2. DEFINITIONS

For $A \subset \mathbb{R}^N$ bounded we define the ε-*neighbourhood* of A as: $A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$. By the *lower s-dimensional Minkowski content* of A, for $s \geq 0$, we mean

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

and analogously for the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$. If $\mathcal{M}^{*s}(A) = \mathcal{M}_*^s(A)$, we call the common value the *s-dimensional Minkowski content of A*, and

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denote it by $\mathcal{M}^s(A)$. The lower and upper box dimensions of A are

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^s_*(A) = 0\}$$

and analogously $\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}$. If these two values coincide, we call it simply the box dimension of A, and denote it by $\dim_B A$. This will be our situation. If $0 < \mathcal{M}^d_*(A) \leq \mathcal{M}^{*d}(A) < \infty$ for some d, then we say that A is *Minkowski nondegenerate*. In this case obviously $d = \dim_B A$. In the case when the lower or upper d-dimensional Minkowski content of A is equal to 0 or ∞, where $d = \dim_B A$, we say that A is *degenerate*. If there exists $\mathcal{M}^d(A)$ for some d and $\mathcal{M}^d(A) \in (0, \infty)$, then we say that A is *Minkowski measurable*.

We shall use the following notation. For any two sequences of positive real numbers (a_k) and (b_k) converging to zero we write $a_k \simeq b_k$ as $k \to \infty$ if there exist positive real numbers $A < B$ such that $a_k/b_k \in [A, B]$ for all k. Also if $f, g : (0, r) \to (0, \infty)$ are two functions converging to zero as $s \to 0$ and $f(s)/g(s) \in [A, B]$, we write $f(s) \simeq g(s)$ as $s \to 0$. We call such sequences and functions comparable.

Spiral trajectory Γ_α of weak focus is α-*power spiral*, that is spiral $r = f(\varphi)$ satisfying $f(\varphi) \simeq \varphi^{-\alpha}$ for $0 < \alpha \leq 1$, see [6].

3. $F_{n,n}$ TRANSFORMATION OF WEAK FOCUS

We define the map $F_{n,m}$ in order to transform trajectories of weak focus to trajectories of degenerate focus. Let $F_{n,m} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$(1) \qquad F_{m,n}(x,y) = ((\operatorname{sign} x)|x|^{1/m}, (\operatorname{sign} y)|y|^{1/n}).$$

We assume that m and n are positive integers. Since $F_{1,1}$ is identity, we also assume that at least one of these integers is ≥ 2. It is clear that $F_{m,n}$ is a homeomorphism mapping each quadrant onto itself. Note that it is not diffeomorphism. We defined the map $F_{m,n}$ in order to transform trajectories of weak focus to trajectories of degenerate focus. A similar idea is used in e.g. [1], [2], [3] using the quasi-homogeneous polar coordinates in the computation of generalized Lyapunov coefficients.

Theorem 1. *Assume that Γ is a trajectory of the system (2), $k, n \geq 1$*

$$(2) \qquad \begin{aligned} \dot{x} &= -y^{2n-1} \pm x^n y^{n-1}(x^{2n} + y^{2n})^k \\ \dot{y} &= x^{2n-1} \pm x^{n-1} y^n (x^{2n} + y^{2n})^k. \end{aligned}$$

Then

$$(3) \qquad \dim_B \Gamma = 2 - \frac{2}{1 + 2kn},$$

and Γ is Minkowski nondegenerate.

Idea of the proof. Proof is based on application of

$$(4) \qquad F_{n,n}(x,y) = ((\operatorname{sign} x)|x|^{1/n}, (\operatorname{sign} y)|y|^{1/n})$$

to system

$$(5) \qquad \begin{aligned} \dot{x} &= -y \pm x(x^2 + y^2)^k \\ \dot{y} &= x \pm y(x^2 + y^2)^k \end{aligned}$$

with weak focus. For weak focus we apply Theorem 9 from [6], as well as the careful study of degenerate focus defined by system (2), obtained via (5).

REMARK 1. A bifurcation parametar λ could be added in system (2), in order to obtain

$$
(6) \qquad
\begin{aligned}
\dot{x} &= -y^{2n-1} \pm x^n y^{n-1}((x^{2n} + y^{2n})^k + \lambda) \\
\dot{y} &= x^{2n-1} \pm x^{n-1} y^n ((x^{2n} + y^{2n})^k + \lambda).
\end{aligned}
$$

For $\lambda < 0$ a limit cycle is born from the degenerate focus corresponding to parametar $\lambda = 0$, and having the box dimension (3). For $k = 1$ and $n = 1$ we have standard Hopf bifurcation where a limit cycle has been born from weak focus with $\dim_B \Gamma = 4/3$, see [6].

In [7] a flow-sector theorem has been proved for weak focus. The theorem says that weak focus flow in sectors near the singular point is lipeomorphically equivalent to the annulus flow. The flow-sector theorem has been proved for the system with weak focus

$$
\begin{aligned}
\dot{x} &= -y + p(x, y) \\
\dot{y} &= x + q(x, y),
\end{aligned}
$$

in which functions $p(x, y)$ and $q(x, y)$ are given C^1-functions such that $|p(x,y)| \leq C(x^2 + y^2)$ and $|q(x,y)| \leq C(x^2 + y^2)$ for some positive constant C and for (x, y) near the origin. We generalize flow-sector theorem for the focus of the system (2). For a function $F : U \to V$ with $U, V \subset \mathbb{R}^2$, $V = F(U)$, if F and F^{-1} are Lipschitzian we say that F is *lipeomorphism*, and that the sets U and V are lipeomorphic. Here the *annulus flow* is defined by

$$
x^{2n} + y^{2n} = const,
$$

for $n \in \mathbb{N}$.
In the following theorem we claim that the focus flow in sectors near the singular point of (2) is lipeomorphically equivalent to the annulus flow.

Theorem 2. *Let $U_0 \subset \mathbb{R}^2$ be an open sector with the vertex at the origin, such that its opening angle is in $(0, 2\pi)$, and the boundary of U_0 consists of a part of a trajectory and of intervals on two rays emanating from the origin. If the diameter of U_0 is sufficiently small, then system (2) restricted to U_0 is lipeomorphically equivalent to the system*

$$
\begin{aligned}
\dot{x} &= -y^{2n-1} \\
\dot{y} &= x^{2n-1}
\end{aligned}
$$

defined on the sector V_0.

Idea of the proof. The proof is based on the transformation of a weak focus and application of flow sector theorem from [7].
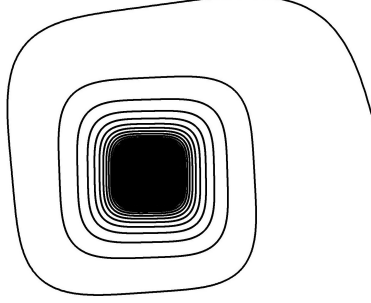
REMARK 2. It is possible to state previous two theorems for a larger class of systems with degenerate focus without characteristic directions.

Characteristic direction for singularity in the origin of a system

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}$$

is linear factor in $\mathbb{R}[x,y]$ of $yP_d(x,y) - xQ_d(x,y)$, where $P_d(x,y)$ and $Q_d(x,y)$ are homogeneous polynomials of the lowest degree. It is obvious that system (2) has no characteristic directions. If there are no characteristic directions, a singular point is either center or focus. The converse is not true.

Now, we consider a different class of degenerate systems.

Proposition 1. *Assume that Γ is a trajectory of the system*

(7)
$$\begin{aligned} \dot{x} &= -y(x^2+y^2)^k + xR_s(x,y) \\ \dot{y} &= x(x^2+y^2)^k + yR_s(x,y), \end{aligned}$$

where $R_s(x,y)$ is a homogeneous polynomial of even degree s and $s > 2k \geq 0$, where s and k are integers. Then

(8)
$$d = \dim_B \Gamma = 2 - \frac{2}{s - 2k + 1},$$

and Γ is Minkowski nondegenerate.

Idea of the proof. Find explicit solution in polar coordinates.

EXAMPLE 3. We reveal Example 4 from [3] dealing with system (7). In polar coordinates we obtain

(9)
$$\begin{aligned} \dot{r} &= r^{s+1} R_s(\cos\varphi, \sin\varphi) \\ \dot{\varphi} &= r^{2k}, \end{aligned}$$

and the origin is a focus if and only if the following integral is different from zero

$$\int_0^{2\pi} R_s(\cos\varphi, \sin\varphi)\, d\varphi \neq 0.$$

We remark that for s odd, the origin is center. In the focus case a spiral trajectory Γ of system (7) is $1/(s - 2k)$-power spiral with

$$d = \dim_B \Gamma = 2 - \frac{2}{s - 2k + 1}.$$

Also, box dimension of an orbit generated by the Poincaré map, at any transversal through the origin, is equal to $d/2$. To system (7) we add analytic perturbation

$(\bar{P}(x,y,\varepsilon), \bar{Q}(x,y,\varepsilon)) = O(\|(x,y)\|)^s$, satisfying $\bar{P}(x,y,0) = \bar{Q}(x,y,0)) = 0$, for $\varepsilon \in \mathbb{R}$, $0 < \|\varepsilon\| \ll 1$. Let the corresponding vector field be denoted by X_ε. Cyclicity of the origin p_0 of the perturbed system will be denoted by $Cycl(X_\varepsilon, p_0)$. By cyclicity we mean the sharp upper bound for the number of limit cycles which can bifurcate from the origin p_0 of the perturbed system. Theorem 1 from [3] says that in this example $Cycl(X_\varepsilon, p_0) > s/2$. For $k = 0$ we obtain weak focus with $Cycl(X_\varepsilon, p_0) = s/2$. According to (8), we can express cyclicity by using box dimension of the trajectory.

4. $F_{m,n}$ TRANSFORMATION OF WEAK FOCUS

The case of $m \neq n$ is more difficult, and by applying transformation (1) to (5), we obtain the following degenerate system, extending (2)

$$
(10) \qquad \begin{aligned} \dot{x} &= -ny^{2n-1} \pm nx^m y^{n-1}(x^{2m} + y^{2n})^k \\ \dot{y} &= mx^{2m-1} \pm mx^{m-1}y^n(x^{2m} + y^{2n})^k. \end{aligned}
$$

Theorem 3. *Assume that Γ is a trajectory of the system (10) where $m, n \geq 1$, $m \geq n$. Then*

$$
(11) \qquad \dim_B \Gamma \geq 2 - \frac{1 + \frac{n}{m}}{1 + 2km}.
$$

Idea of the proof. The proof is based on the careful study of the degenerate spiral $\Gamma = F_{m,n}(\Gamma_1)$, where Γ_1 is a spiral trajectory of the system (5) with weak focus, and $F_{m,n}$ is transformation defined by (1). Also we use suitable Lipschitz transformations.

REMARK 4. We hypothesize that

$$
(12) \qquad \dim_B F_{m,n}(\Gamma) = 2 - \frac{1 + \frac{n}{m}}{1 + 2km}.
$$

5. ACKNOWLEDGMENTS

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