Communications in Contemporary Mathematics
(2015) 1550058 (16 pages)

DOI: 10.1142/S0219199715500583

Tempered exponential dichotomies and Lyapunov exponents for perturbations

Luis Barreira
Departamento de Matemática, Instituto Superior Técnico
Universidade de Lisboa, 1049-001 Lisboa, Portugal
barreira@math.ist.utl.pt

Davor Dragičević
Department of Mathematics, University of Rijeka
51000 Rijeka, Croatia
ddragicevic@math.uniri.hr

Claudia Valls
Departamento de Matemática, Instituto Superior Técnico
Universidade de Lisboa, 1049-001 Lisboa, Portugal
cvalls@math.ist.utl.pt

Received 21 October 2014
Revised 22 April 2015
Accepted 8 May 2015
Published 6 July 2015

We establish a Perron-type result for the perturbations of a linear cocycle in the context of ergodic theory. More precisely, we show that the Lyapunov exponents of a linear cocycle are preserved under sufficiently small nonautonomous perturbations. Our approach is based on the Lyapunov theory of regularity.

Keywords: Exponential dichotomies; Lyapunov exponents; perturbations.

Mathematics Subject Classification 2010: 37D99

1. Introduction

Our main aim is to establish a Perron-type result for the perturbations of a linear cocycle. This means showing that the Lyapunov exponents of the cocycle are preserved under sufficiently small nonautonomous perturbations. For example, for a linear cocycle over a measure-preserving transformation satisfying a natural integrability assumption, it follows from the Multiplicative Ergodic Theorem that the dynamics admits what is sometimes called a tempered exponential dichotomy if and only if all Lyapunov exponents are nonzero almost everywhere. In this particular case, our Perron-type result implies that all Lyapunov exponents are also nonzero under sufficiently small nonautonomous perturbations. We refer to [1] for a detailed exposition of the smooth ergodic theory, which goes back to Oseledets [9] and particularly Pesin [11]. More recently, Lian and Lu [7] considered cocycles with values on the set of bounded linear operators acting on a separable Banach space.

In order to simplify the exposition, in the introduction we mention only the case of ordinary differential equations. Consider the linear equation

$$x' = A(t)x \tag{1.1}$$

and its perturbations

$$x' = A(t)x + f(t, x). \tag{1.2}$$

It is shown in [2] that if the perturbation is so small that

$$\lim_{t\to+\infty} e^{\delta t} \sup_{x\neq 0} \frac{\|f(t,x)\|}{\|x\|} = 0$$

for some $\delta > 0$ and all the Lyapunov exponents of the linear equation in (1.1) are limits, then for any solution $x(t)$ of the nonlinear equation in (1.2) that is not eventually zero, the limit

$$\lambda = \lim_{t\to+\infty} \frac{1}{t} \log \|x(t)\|$$

exists and coincides with a Lyapunov exponent of Eq. (1.1). In the particular case of perturbations of a differential equation $x' = Ax$ with constant coefficients, the corresponding result can be found in Coppel's book [4]. Earlier work is due to Perron [10], Lettenmeyer [6] and Hartman and Wintner [5]. Corresponding results for perturbations of autonomous delay equations were obtained by Pituk [12, 13] (for values in $\mathbb{C}^n$ and finite delay) and Matsui, Matsunaga and Murakami [8] (for values in a Banach space and infinite delay). Related results for perturbations of autonomous difference equations were first obtained by Coffman [3].

Our approach is based on the Lyapunov theory of regularity (see [1]), which allows one to obtain exponential bounds for an evolution operator in terms of the Lyapunov exponents and of the Lyapunov regularity coefficient. The remaining part of the argument is inspired in work of Pituk [12] where he established a corresponding result for perturbations of linear delay equations.

2. Basic Notions

We first introduce some basic notions. Let $\theta : \Omega \to \Omega$ be a measurable map with measurable inverse preserving a probability measure μ on Ω. Then $\mu(\theta(A)) = \mu(A)$ for any measurable set $A \subset \Omega$. Moreover, let GL_d be the set of all invertible $d \times d$ matrices. A measurable map $\Phi : \mathbb{Z} \times \Omega \to \mathrm{GL}_d$ is called a *cocycle over* θ or simply a *cocycle* if:

(1) $\Phi(0, \omega) = \mathrm{Id}$ for $\omega \in \Omega$;
(2) $\Phi(n + m, \omega) = \Phi(n, \theta^m(\omega))\Phi(m, \omega)$ for $m, n \in \mathbb{Z}$ and $\omega \in \Omega$.

The measurable map $A : \Omega \to \mathrm{GL}_d$ defined by $A(\omega) = \Phi(1, \omega)$ is called the *generator* of Φ. On the other hand, given a measurable map $A : \Omega \to \mathrm{GL}_d$, we obtain a cocycle by letting

$$\Phi(n, \omega) = \begin{cases} A(\theta^{n-1}(\omega)) \cdots A(\omega), & n > 0, \\ \mathrm{Id}, & n = 0, \\ A(\theta^n(\omega))^{-1} \cdots A(\theta^{-1}(\omega))^{-1}, & n < 0 \end{cases}$$

for $n \in \mathbb{Z}$ and $\omega \in \mathbb{Z}$.

We say that a cocycle Φ admits a *tempered exponential dichotomy* if there exist projections $P(\omega)$ for $\omega \in \Omega$ and measurable functions $\alpha : \Omega \to (0, +\infty)$ and $K : \Omega \to [1, +\infty)$ such that for μ-almost every $\omega \in \Omega$:

(1) $\alpha(\theta(\omega)) = \alpha(\omega)$ and

$$\limsup_{n \to \pm\infty} \frac{1}{|n|} \log K(\theta^n(\omega)) = 0; \tag{2.1}$$

(2)

$$P(\theta^n(\omega))\Phi(n, \omega) = \Phi(n, \omega)P(\omega) \quad \text{for } n \in \mathbb{Z}; \tag{2.2}$$

(3)

$$\|\Phi(n, \omega)P(\omega)\| \le K(\omega)e^{-\alpha(\omega)n} \quad \text{for } n \ge 0$$

and

$$\|\Phi(n, \omega)(\mathrm{Id} - P(\omega))\| \le K(\omega)e^{\alpha(\omega)n} \quad \text{for } n \le 0.$$

We note that the notion of a tempered exponential dichotomy occurs naturally in the context of ergodic theory. Namely, assume that the generator A of a cocycle Φ satisfies the integrability condition

$$\log^+\|A\|, \quad \log^+\|A^{-1}\| \in L^1(\Omega, \mu), \tag{2.3}$$

where $\log^+ x = \max\{\log x, 0\}$ and $L^1(\Omega, \mu)$ is the set of all μ-integrable functions on Ω. Then the Multiplicative Ergodic Theorem (see, for example, [1]) tells us that μ-almost every point is Lyapunov regular. This means that for μ-almost every $\omega \in \Omega$ there exist numbers $\lambda_1(\omega) < \lambda_2(\omega) < \cdots < \lambda_{s(\omega)}(\omega)$, for some integer $s(\omega) \in [1, d]$, and a decomposition

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_{s(\omega)}(\omega)$$

such that

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \log\|\Phi(n, \omega)v\| = \lambda_i(\omega) \tag{2.4}$$

for $v \in E_i(\omega)\setminus\{0\}$ and $i = 1, \ldots, s(\omega)$, and

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \log|\det \Phi(n, \omega)| = \sum_{i=1}^{s(\omega)} \lambda_i(\omega) \tag{2.5}$$

(that is, the limits in (2.4) and (2.5) exist and are given by the respective right-hand sides). The numbers $\lambda_1(\omega), \ldots, \lambda_{s(\omega)}(\omega)$ are called the *Lyapunov exponents*. We notice that

$$s(\theta(\omega)) = s(\omega) \quad \text{and} \quad \lambda_i(\theta(\omega)) = \lambda_i(\omega)$$

for $i = 1, \ldots, s(\omega)$ and for ω in a set of full μ-measure (that can be assumed to be θ-invariant). The following well-known result shows that the notion of a tempered exponential dichotomy occurs naturally (see, for example, [1]).

Proposition 2.1. *Let Φ be a cocycle whose generator satisfies condition* (2.3). *If for μ-almost every $\omega \in \Omega$ the Lyapunov exponents are nonzero, that is, if $\lambda_i(\omega) \neq 0$ for $i = 1, \ldots, s(\omega)$, then the cocycle Φ admits a tempered exponential dichotomy with the projections $P(\omega)$ obtained from the decomposition $\mathbb{R}^d = E^s(\omega) \oplus E^u(\omega)$, where*

$$E^s(\omega) = \bigoplus_{\lambda_i(\omega)<0} E_i(\omega) \quad \text{and} \quad E^u(\omega) = \bigoplus_{\lambda_i(\omega)>0} E_i(\omega).$$

3. The Case of Discrete Time

3.1. *Main result*

This section contains our main result showing that under certain mild additional assumptions the exponential growth rate of any solution of a nonlinear perturbation of a cocycle Φ is a limit whose value is in fact a Lyapunov exponent of the cocycle. In other words, the only possible exponential growth rates for a nonlinear perturbation of a cocycle are those of the original linear dynamics determined by the cocycle.

More precisely, for each $\omega \in \Omega$, we consider the dynamics

$$x_{n+1} = A(\theta^n(\omega))x_n + f_n(\omega, x_n), \quad n \in \mathbb{Z}, \tag{3.1}$$

for some continuous maps $f_n(\omega, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$. The following is our main result.

Theorem 3.1. *Let Φ be a cocycle whose generator satisfies condition* (2.3) *and let $f_n(\omega, \cdot)$ be continuous functions such that*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \sup_{x\neq 0} \frac{\|f_n(\omega, x)\|}{\|x\|} < 0 \tag{3.2}$$

for μ-almost every $\omega \in \Omega$. Then, for μ-almost every $\omega \in \Omega$, each solution $(x_n)_{n\in\mathbb{Z}}$ of (3.1) *satisfies one of the following alternatives*:

(1) *$x_n = 0$ for any sufficiently large n;*
(2) *there exists $i \in \{1, \ldots, s(\omega)\}$ such that*

$$\lim_{n\to+\infty} \frac{1}{n} \log\|x_n\| = \lambda_i(\omega). \tag{3.3}$$

The proof of Theorem 3.1 is given in Sec. 5.

3.2. *Further results*

In this section we obtain further results related to Theorem 3.1. We start by showing that not only the Lyapunov exponents of a nonlinear perturbation are Lyapunov exponents of the linear system, but also the components along the directions other than the one selected by the perturbation are asymptotically smaller, in some precise sense. Namely, assume that (3.3) holds and consider the projections $P(\omega), Q(\omega)$ and $R(\omega)$ associated to the decomposition

$$\mathbb{R}^d = F(\omega) \oplus G(\omega) \oplus E_i(\omega),$$

where

$$F(\omega) = \bigoplus_{\lambda_j < \lambda_i} E_j(\omega) \quad \text{and} \quad G(\omega) = \bigoplus_{\lambda_j > \lambda_i} E_j(\omega).$$

For each $n \in \mathbb{N}$, we write $x_n = y_n + z_n + w_n$, where

$$y_n = P(\theta^n(\omega))x_n, \quad z_n = Q(\theta^n(\omega))x_n \quad \text{and} \quad w_n = R(\theta^n(\omega))x_n.$$

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *for μ-almost every $\omega \in \Omega$, each solution $(x_m)_{m\in\mathbb{Z}}$ of* (3.1) *satisfying* (3.3) *also satisfies*

$$\lim_{n\to+\infty} \frac{\|y_n\|_{\theta^n(\omega)}}{\|w_n\|_{\theta^n(\omega)}} = 0 \tag{3.4}$$

and

$$\lim_{n\to+\infty} \frac{\|z_n\|_{\theta^n(\omega)}}{\|w_n\|_{\theta^n(\omega)}} = 0. \tag{3.5}$$

The proof of Theorem 3.2 is given in Sec. 6.

By reversing the time direction we obtain similar results to those in Theorems 3.1 and 3.2 when the time goes backwards.

Theorem 3.3. *Let Φ be a cocycle whose generator satisfies condition* (2.3) *and let $f_n(\omega, \cdot)$ be continuous functions such that*

$$\limsup_{n\to-\infty} \frac{1}{|n|} \log \sup_{x\neq 0} \frac{\|f_n(\omega, x)\|}{\|x\|} < 0$$

for μ-almost every $\omega \in \Omega$. Then, for μ-almost every $\omega \in \Omega$, each solution $(x_n)_{n\in\mathbb{Z}}$ of (3.1) *satisfies one of the following alternatives*:

(1) *$x_n = 0$ for any sufficiently small n;*
(2) *there exists $i \in \{1, \ldots, s(\omega)\}$ such that*

$$\lim_{n\to-\infty} \frac{1}{|n|} \log\|x_n\| = \lambda_i(\omega).$$

Moreover, if the second alternative holds, then

$$\lim_{n\to-\infty} \frac{\|y_n\|_{\theta^n(\omega)}}{\|w_n\|_{\theta^n(\omega)}} = 0 \quad \textit{and} \quad \lim_{n\to-\infty} \frac{\|z_n\|_{\theta^n(\omega)}}{\|w_n\|_{\theta^n(\omega)}} = 0.$$

Now we describe a consequence of Theorem 3.1 for linear perturbations. One could also obtain a corresponding result for negative time. For each $\omega \in \Omega$, consider the dynamics

$$x_{n+1} = [A(\theta^n \omega) + B(\theta^n \omega)]x_n, \tag{3.6}$$

for some measurable map $B : \Omega \to M_d$, where M_d is the set of all $d \times d$ matrices. This induces a cocycle Ψ whose generator is $A + B$.

Theorem 3.4. *Let Φ be a cocycle whose generator satisfies condition* (2.3) *and let B be a measurable map such that*

$$\limsup_{n \to +\infty} \frac{1}{n} \log\|B(\theta^n(\omega))\| < 0$$

for μ-almost every $\omega \in \Omega$. Then, for μ-almost every $\omega \in \Omega$, the Lyapunov exponents of the cocycles Φ and Ψ are the same.

Proof. The statement follows from the fact that the dynamics $x_{n+1} = A(\theta^n \omega)x_n$ can also be seen as a perturbation of (3.6), satisfying the same hypotheses. □

In particular, if the dynamics defined by Φ admits a tempered exponential dichotomy, then it follows from Theorem 3.4 that the same happens to the dynamics defined by Ψ.

4. The Case of Continuous Time

In this section we describe how one can obtain corresponding results to those in the former sections for a dynamics with continuous time.

A measurable map $\varphi : \mathbb{R} \times \Omega \to \Omega$ is said to be a *flow* on Ω if:

(1) $\varphi(0, \omega) = \omega$ for $\omega \in \Omega$;
(2) $\varphi(t + s, \omega) = \varphi(t, \varphi(s, \omega))$ for $t, s \in \mathbb{R}$ and $\omega \in \Omega$.

We also consider the maps $\varphi_t = \varphi(t, \cdot)$. A measurable map $\Phi : \mathbb{R} \times \Omega \to \mathrm{GL}_d$ is said to be *a cocycle over* φ or simply a *cocycle* if:

(1) $\Phi(0, \omega) = \mathrm{Id}$ for $\omega \in \Omega$;
(2) $\Phi(t + s, \omega) = \Phi(t, \varphi_s(\omega))\Phi(s, \omega)$ for $t, s \in \mathbb{R}$ and $\omega \in \Omega$.

One can easily verify that Φ is a cocycle over φ if and only if the map

$$(t, \omega, x) \mapsto (\varphi_t(x), \Phi(t, \omega)x)$$

is a flow on $\Omega \times X$.

We say that a cocycle Φ admits a *tempered exponential dichotomy* if there exist projections $P(\omega)$ for $\omega \in \Omega$ and measurable functions $\alpha : \Omega \to (0, +\infty)$ and

$K : \Omega \to [1, +\infty)$ such that for μ-almost every $\omega \in \Omega$:

(1) $\alpha(\theta(\omega)) = \alpha(\omega)$ and

$$\limsup_{t\to\pm\infty} \frac{1}{|t|} \log K(\varphi_t(\omega)) = 0;$$

(2)

$$P(\varphi_t(\omega))\Phi(t,\omega) = \Phi(t,\omega)P(\omega) \quad \text{for } t \in \mathbb{R};$$

(3)

$$\|\Phi(t,\omega)P(\omega)\| \le K(\omega)e^{-\alpha(\omega)t} \quad \text{for } t \ge 0$$

and

$$\|\Phi(t,\omega)(\mathrm{Id} - P(\omega))\| \le K(\omega)e^{\alpha(\omega)n} \quad \text{for } t \le 0.$$

By the Multiplicative Ergodic Theorem for flows (see, for example, [1]), if

$$\log^+ \sup_{-1\le t\le 1} \|\Phi(t,\cdot)\| \in L^1(\Omega,\mu), \tag{4.1}$$

then for μ-almost every $\omega \in \Omega$ there exist numbers $\lambda_1(\omega) < \lambda_2(\omega) < \cdots < \lambda_{s(\omega)}(\omega)$, for some integer $s(\omega) \in [1, d]$, and a decomposition

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_{s(\omega)}(\omega)$$

such that

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log\|\Phi(t,\omega)v\| = \lambda_i(\omega) \tag{4.2}$$

for $v \in E_i(\omega)\setminus\{0\}$ and $i = 1,\dots,s(\omega)$, and

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log|\det \Phi(t,\omega)| = \sum_{i=1}^{s(\omega)} \lambda_i(\omega). \tag{4.3}$$

Again we notice that $s(\theta(\omega)) = s(\omega)$ and $\lambda_i(\theta(\omega)) = \lambda_i(\omega)$ for $i = 1,\dots,s(\omega)$ and for ω in a set of full μ-measure (that can be assumed to be θ-invariant).

Using the information given by (4.2) and (4.3) we can obtain a version of Theorem 3.1 for flows. Namely, for each $\omega \in \Omega$, given a cocycle Φ over a flow φ and a function $f : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, with $(t,x) \mapsto f(t,\omega,x)$ continuous for each $\omega \in \Omega$, consider the unique map $\Psi : \mathbb{R} \times \Omega \to \mathbb{R}^d$ such that

$$\Psi_\omega(t,s) = \Phi_\omega(t,s) + \int_s^t \Phi_\omega(t,\tau) f(\tau,\omega,\Phi_\omega(\tau,s))\,d\tau$$

for $\omega \in \Omega$ and $t,s \in \mathbb{R}$, where

$$\Phi_\omega(t,s) = \Phi(t,\omega)\Phi(s,\omega)^{-1} \quad \text{and} \quad \Psi_\omega(t,s) = \Psi(t,\omega)\Psi(s,\omega)^{-1}.$$

The following result is a continuous time version of Theorem 3.1. The proof follows along the same lines and thus it is omitted.

Theorem 4.1. *Let Φ be a cocycle over a flow satisfying condition* (4.1) *and let f be a function such that*

$$\lim_{t\to+\infty}\frac{1}{t}\log\sup_{x\neq 0}\frac{\|f(t,\omega,x)\|}{\|x\|}<0$$

for μ-almost every $\omega\in\Omega$. Then, for μ almost every $\omega\in\Omega$, the function $x(t)=\Psi_\omega(t,0)x$ satisfies one of the following alternatives:

(1) $x(t)=0$ *for any sufficiently large t*;
(2) *there exists $i\in\{1,\ldots,s(\omega)\}$ such that*

$$\lim_{t\to+\infty}\frac{1}{t}\log\|x(t)\|=\lambda_i(\omega).$$

5. Proof of Theorem 3.1

We define

$$\gamma_n(\omega)=\sup_{x\neq 0}\frac{\|f_n(\omega,x)\|}{\|x\|}.$$

Then

$$\|f_n(\omega,x)\|\le\gamma_n(\omega)\|x\|\quad\text{for } n\in\mathbb{Z}\text{ and } x\in X. \tag{5.1}$$

Moreover, it follows from (3.2) that for μ-almost every $\omega\in\Omega$ there exists $\delta(\omega)>0$ such that

$$\lim_{n\to+\infty}e^{\delta(\omega)n}\gamma_n(\omega)=0. \tag{5.2}$$

We denote by $\tilde{\Omega}\subset\Omega$ the set of full μ-measure formed by all points $\omega\in\Omega$ for which properties (2.4), (2.5) and (5.2) hold. For simplicity of the notation, from now on we shall always write $s(\omega)=s, \lambda_i(\omega)=\lambda_i, \gamma_n=\gamma_n(\omega), \delta=\delta(\omega)$ and $K_n=K(\theta^n(\omega))$.

Lemma 5.1. *For each $\omega\in\tilde{\Omega}$, we have*

$$\limsup_{k\to+\infty}\frac{1}{k}\log\|x_k\|\le\lambda_s.$$

Proof. Take $d>\lambda_s$. Then there exists a measurable function $K:\Omega\to[1,+\infty)$ satisfying (2.1) such that

$$\|\Phi(n,\theta^m(\omega))\|\le K(\theta^m(\omega))e^{dn}\quad\text{for } n\ge 0\text{ and } m\in\mathbb{Z}. \tag{5.3}$$

It follows from (3.1) that

$$x_m=\Phi(m-n,\theta^n(\omega))x_n+\sum_{k=n}^{m-1}\Phi(m-k-1,\theta^{k+1}(\omega))f_k(\omega,x_k)$$

for $m\ge n$. By (5.1) and (5.3), we obtain

$$\|x_m\|\le K_ne^{d(m-n)}\|x_n\|+\sum_{k=n}^{m-1}K_{k+1}e^{d(m-k-1)}\gamma_k\|x_k\| \tag{5.4}$$

for $m \geq n$. On the other hand, by (2.1), given $\varepsilon < \delta$, there exists a constant $D = D(\omega) > 0$ such that

$$K_n = K(\theta^n(\omega)) \leq De^{\varepsilon n} \quad \text{for } n \geq 0. \tag{5.5}$$

It follows from (5.1), (5.4) and (5.5) that

$$\begin{aligned}\|x_m\| &\leq De^{d(m-n)+\varepsilon n}\|x_n\| + D\sum_{k=n}^{m-1} e^{d(m-k-1)+\varepsilon(k+1)}\gamma_k\|x_k\| \\ &= De^{(d(m-n)+\varepsilon n}\|x_n\| + D'e^{d(m-n)}\sum_{k=n}^{m-1} e^{-d(k-n)+\varepsilon k}\gamma_k\|x_k\|\end{aligned}$$

and so,

$$e^{-d(m-n)}\|x_m\| \leq De^{\varepsilon n}\|x_n\| + D'\sum_{k=n}^{m-1} e^{-d(k-n)+\varepsilon k}\gamma_k\|x_k\|$$

for $m \geq n$, where $D' = De^{\varepsilon - d}$. One can now use induction to show that

$$\|x_m\| \leq De^{d(m-n)+\varepsilon n}\|x_n\|e^{\sum_{k=n}^{m-1} D'e^{\varepsilon k}\gamma_k} \tag{5.6}$$

for $m \geq n$. By (5.2), given $\eta > 0$, there exists $n \in \mathbb{N}$ such that $D'e^{\varepsilon k}\gamma_k < \eta$ for $k \geq n$. Hence,

$$\|x_m\| \leq De^{(d+\eta)(m-n)+\varepsilon n}\|x_n\|$$

for $m \geq n$ and so,

$$\limsup_{m\to+\infty} \frac{1}{m}\log\|x_m\| \leq d + \eta.$$

Letting $\eta \to 0$ and $d \searrow \lambda_s$, we obtain the desired result. □

Lemma 5.2. *For each $\omega \in \tilde{\Omega}$ and any solution $(x_n)_{n\in\mathbb{Z}}$ of* (3.1) *not satisfying the first alternative in the theorem, we have*

$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x_k\| \geq \lambda_1.$$

Proof. We first note that $x_1 \neq 0$, since otherwise it would follow from (5.6) that $x_n = 0$ for all $n \in \mathbb{N}$. Take $d < \lambda_1$. Then there exists a measurable function $K : \Omega \to [1, +\infty)$ satisfying (2.1) such that

$$\|\Phi(n, \theta^m(\omega))\| \leq K(\theta^m(\omega))e^{dn} \quad \text{for } n \leq 0 \text{ and } m \in \mathbb{Z}.$$

Proceeding in a similar manner to that in the proof of Lemma 5.1, we find that

$$\|x_m\| \leq D\|x_n\|e^{-d(n-m)+\varepsilon n}e^{\sum_{k=m}^{n-1} D'e^{\delta k}\gamma_k}$$

for $m \leq n$, where $D' = D'(\omega) > 0$ is a constant. Given $\eta > 0$, there exists $m \in \mathbb{N}$ such that $D'e^{\varepsilon k}\gamma_k < \eta$ for $k \geq m$. Hence,

$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x_k\| \geq d - \varepsilon - \eta.$$

Letting $\eta, \varepsilon \to 0$ and $d \nearrow \lambda_1$, we obtain the desired result. □

Now take $c = c(\omega) \neq \lambda_i(\omega)$ for $i \in \{1, \ldots, s\}$ and $\omega \in \tilde{\Omega}$. It follows from Proposition 2.1 that the cocycle $\Psi(n, \omega) = e^{-c(\omega)n}\Phi(n, \omega)$ admits a tempered exponential dichotomy with projections

$$P(\omega) : \mathbb{R}^d \to \bigoplus_{\lambda_i < c} E_i(\omega) \quad \text{and} \quad Q(\omega) : \mathbb{R}^d \to \bigoplus_{\lambda_i > c} E_i(\omega)$$

satisfying $P(\omega) + Q(\omega) = \mathrm{Id}$. Hence, there exist $\alpha = \alpha(\omega) > 0$ and a measurable function $K : \Omega \to [1, +\infty)$ satisfying (2.1) such that

$$\|\Phi(n, \omega)P(\omega)\| \leq K(\omega)e^{(c-\alpha)n}, \quad n \geq 0, \tag{5.7}$$

and

$$\|\Phi(n, \omega)Q(\omega)\| \leq K(\omega)e^{(c+\alpha)n}, \quad n \leq 0, \tag{5.8}$$

for $\omega \in \tilde{\Omega}$. For each $\omega \in \tilde{\Omega}$, we consider the norm $\|\cdot\|_\omega$ on $\mathbb{R}^d$ defined by

$$\|x\|_\omega = \sup_{n \geq 0}(\|\Phi(n, \omega)P(\omega)\|e^{-(c-\alpha)n}) + \sup_{n \leq 0}(\|\Phi(n, \omega)Q(\omega)\|)e^{-(c+\alpha)n}) \tag{5.9}$$

(writing $c = c(\omega)$ and $\alpha = \alpha(\omega)$ for simplicity of the notation).

It follows from (5.7) and (5.8) that

$$\|x\| \leq \|x\|_\omega \leq 2K(\omega)\|x\| \quad \text{for } \omega \in \tilde{\Omega} \text{ and } x \in \mathbb{R}^d. \tag{5.10}$$

Lemma 5.3. *For each $\omega \in \tilde{\Omega}$ and $x \in \mathbb{R}^d$, we have*

$$\|\Phi(n, \omega)P(\omega)x\|_{\theta^n(\omega)} \leq e^{(c-\alpha)n}\|x\|_\omega \quad \text{for } n \geq 0 \tag{5.11}$$

and

$$\|\Phi(n, \omega)Q(\omega)x\|_{\theta^n(\omega)} \leq e^{(c+\alpha)n}\|x\|_\omega \quad \text{for } n \leq 0. \tag{5.12}$$

Proof. We have

$$\begin{aligned}\|\Phi(m, \omega)P(\omega)x\|_{\theta^m(\omega)} &= \sup_{n \geq 0}(\|\Phi(n, \theta^m(\omega))P(\theta^m(\omega))\Phi(m, \omega)P(\omega)x\|e^{-(c-\alpha)n}) \\ &= e^{(c-\alpha)m}\sup_{n \geq 0}(\|\Phi(n + m, \omega)P(\omega)x\|e^{-(c-\alpha)(n+m)}) \\ &\leq e^{(c-\alpha)m}\|x\|_\omega\end{aligned}$$

for $x \in \mathbb{R}^d$ and so (5.11) holds. One can establish (5.12) in a similar manner. □

Now let

$$y_k = P(\theta^k(\omega))x_k \quad \text{and} \quad z_k = Q(\theta^k(\omega))x_k$$

for $k \in \mathbb{Z}$. It follows from (2.2) and (3.1) that

$$y_{k+1} = A(\theta^k(\omega))y_k + P(\theta^{k+1}(\omega))f_k(\omega, x_k)$$

and

$$z_{k+1} = A(\theta^k(\omega))z_k + Q(\theta^{k+1}(\omega))f_k(\omega, x_k)$$

for $k \in \mathbb{Z}$ and $\omega \in \tilde{\Omega}$. Hence, by (5.1), (5.7), (5.10) and (5.11), we obtain

$$\begin{aligned}\|y_{k+1}\|_{\theta^{k+1}(\omega)} &\le e^{c-\alpha}\|y_k\|_{\theta^k(\omega)} + \|P(\theta^{k+1}(\omega))f_k(\omega, x_k)\|_{\theta^{k+1}(\omega)} \\ &\le e^{c-\alpha}\|y_k\|_{\theta^k(\omega)} + 2K_{k+1}\|P(\theta^{k+1}(\omega))f_k(\omega, x_k)\| \\ &\le e^{c-\alpha}\|y_k\|_{\theta^k(\omega)} + 2K_{k+1}^2\|f_k(\omega, x_k)\| \\ &\le e^{c-\alpha}\|y_k\|_{\theta^k(\omega)} + 2K_{k+1}^2\gamma_k\|x_k\| \\ &\le e^{c-\alpha}\|y_k\|_{\theta^k(\omega)} + 2K_{k+1}^2\gamma_k(\|y_k\|_{\theta^k(\omega)} + \|z_k\|_{\theta^k(\omega)}) \end{aligned} \tag{5.13}$$

for $k \in \mathbb{Z}$ and $\omega \in \tilde{\Omega}$. Similarly, it follows from (5.1), (5.8), (5.10) and (5.12) that

$$\|z_{k+1}\|_{\theta^{k+1}(\omega)} \ge e^{c+\alpha}\|z_k\|_{\theta^k(\omega)} - 2K_{k+1}^2\gamma_k(\|y_k\|_{\theta^k(\omega)} + \|z_k\|_{\theta^k(\omega)}) \tag{5.14}$$

for $k \in \mathbb{Z}$ and $\omega \in \tilde{\Omega}$.

Lemma 5.4. *For each $\omega \in \tilde{\Omega}$, we have either*

$$\|z_k\|_{\theta^k(\omega)} \le \|y_k\|_{\theta^k(\omega)} \quad \textit{for all sufficiently large } k \tag{5.15}$$

or

$$\|y_k\|_{\theta^k(\omega)} < \|z_k\|_{\theta^k(\omega)} \quad \textit{for all sufficiently large } k. \tag{5.16}$$

Proof. It follows from (5.13) and (5.14) that

$$\|y_{k+1}\|_{\theta^{k+1}(\omega)} \le (e^{c-\alpha} + 2K_{k+1}^2\gamma_k)\|y_k\|_{\theta^k(\omega)} + 2K_{k+1}^2\gamma_k\|z_k\|_{\theta^k(\omega)} \tag{5.17}$$

and

$$\|z_{k+1}\|_{\theta^{k+1}(\omega)} \ge (e^{c+\alpha} - 2K_{k+1}^2\gamma_k)\|z_k\|_{\theta^k(\omega)} - 2K_{k+1}^2\gamma_k\|y_k\|_{\theta^k(\omega)}. \tag{5.18}$$

Now we assume that (5.15) does not hold. Take $k_0 \in \mathbb{N}$ arbitrarily large such that

$$\|y_{k_0}\|_{\theta^{k_0}(\omega)} < \|z_{k_0}\|_{\theta^{k_0}(\omega)}.$$

We prove by induction on k that if k_0 is sufficiently large, then $\|y_k\|_{\theta^k(\omega)} < \|z_k\|_{\theta^k(\omega)}$ for $k \ge k_0$. So, let us assume that $\|y_k\|_{\theta^k(\omega)} < \|z_k\|_{\theta^k(\omega)}$ for some $k \ge k_0$. It follows from (5.17) and (5.18) that

$$\|y_{k+1}\|_{\theta^{k+1}(\omega)} \le (e^{c-\alpha} + 4K_{k+1}^2\gamma_k)\|z_k\|_{\theta^k(\omega)}$$

and

$$\|z_{k+1}\|_{\theta^{k+1}(\omega)} \ge (e^{c+\alpha} - 4K_{k+1}^2\gamma_k)\|z_k\|_{\theta^k(\omega)}.$$

Combining these inequalities, we obtain

$$\|y_{k+1}\|_{\theta^{k+1}(\omega)} \le \frac{e^{c-\alpha} + 4K_{k+1}^2\gamma_k}{e^{c+\alpha} - 4K_{k+1}^2\gamma_k}\|z_{k+1}\|_{\theta^{k+1}(\omega)}.$$

It follows from (2.1) and (5.2) that $4K_{k+1}^2\gamma_k \to 0$ when $k \to \infty$ and thus,

$$\frac{e^{c-\alpha} + 4K_{k+1}^2\gamma_k}{e^{c+\alpha} - 4K_{k+1}^2\gamma_k} \to \frac{e^{c-\alpha}}{e^{c+\alpha}} < 1.$$

Hence, if k_0 is sufficiently large, then

$$\|y_{k+1}\|_{\theta^{k+1}(\omega)} < \|z_{k+1}\|_{\theta^{k+1}(\omega)}.$$

This completes the proof of the lemma. ◻

Lemma 5.5. *Assume that the first alternative in the theorem does not hold. Then one of the following alternatives holds*:

(1)

$$\limsup_{k\to+\infty} \frac{1}{k} \log\|x_k\| < c \tag{5.19}$$

and

$$\lim_{k\to+\infty} \frac{\|z_k\|_{\theta^k(\omega)}}{\|y_k\|_{\theta^k(\omega)}} = 0; \tag{5.20}$$

(2)

$$\liminf_{k\to+\infty} \frac{1}{k} \log\|x_k\| > c \tag{5.21}$$

and

$$\lim_{k\to+\infty} \frac{\|y_k\|_{\theta^k(\omega)}}{\|z_k\|_{\theta^k(\omega)}} = 0. \tag{5.22}$$

Proof. Assume first that (5.15) holds and let

$$S = \limsup_{k\to+\infty} \frac{\|z_k\|_{\theta^k(\omega)}}{\|y_k\|_{\theta^k(\omega)}}.$$

It follows from (5.15) that $0 \le S \le 1$. On the other hand, by (5.13) and (5.15), we have

$$\|y_{k+1}\|_{\theta^{k+1}(\omega)} \le (e^{c-\alpha} + 4K_{k+1}^2\gamma_k)\|y_k\|_{\theta^k(\omega)} \tag{5.23}$$

for all sufficiently large $k \in \mathbb{N}$. It follows from (5.14) that

$$\frac{\|z_{k+1}\|_{\theta^{k+1}(\omega)}}{\|y_{k+1}\|_{\theta^{k+1}(\omega)}} \ge \frac{e^{c+\alpha} - 2K_{k+1}^2\gamma_k}{e^{c-\alpha} + 4K_{k+1}^2\gamma_k} \cdot \frac{\|z_k\|_{\theta^k(\omega)}}{\|y_k\|_{\theta^k(\omega)}} - \frac{2K_{k+1}^2\gamma_k}{e^{c-\alpha} + 4K_{k+1}^2\gamma_k},$$

for all sufficiently large $k \in \mathbb{N}$. On the other hand, by (2.1) and (5.2) we have

$$\frac{e^{c+\alpha} - 2K_{k+1}^2\gamma_k}{e^{c-\alpha} + 4K_{k+1}^2\gamma_k} \to \frac{e^{c+\alpha}}{e^{c-\alpha}} > 1 \quad \text{and} \quad \frac{2K_{k+1}^2\gamma_k}{e^{c-\alpha} + 4K_{k+1}^2\gamma_k} \to 0$$

when $k \to \infty$ and so, $S = 0$. This establishes (5.20). In order to prove (5.19), take k_0 such that (5.23) holds for all $k \geq k_0$. By (5.23), we obtain

$$\|y_k\|_{\theta^k(\omega)} \leq \|y_{k_0}\|_{\theta^{k_0}(\omega)} e^{(c-\alpha)(k-k_0)} \prod_{j=k_0}^{k} (1 + 4K_{j+1}^2 \gamma_j e^{\alpha - c})$$

for $k \geq k_0$. It follows from (2.1) and (5.2) that

$$\frac{1}{k} \sum_{j=k_0}^{k} \log(1 + 4K_{j+1}^2 \gamma_j e^{\alpha - c}) \leq \frac{1}{k} \sum_{j=k_0}^{k} 4K_{j+1}^2 \gamma_j e^{\alpha - c} \to 0$$

when $k \to \infty$ and so,

$$\limsup_{k \to +\infty} \frac{1}{k} \log\|y_k\|_{\theta^k(\omega)} \leq c - \alpha < c.$$

Finally, by (5.10) and (5.15), we obtain

$$\limsup_{k \to +\infty} \frac{1}{k} \log\|x_k\| \leq \limsup_{k \to +\infty} \frac{1}{k} \log(2\|y_k\|_{\theta^k(\omega)}) = \limsup_{k \to +\infty} \frac{1}{k} \log\|y_k\|_{\theta^k(\omega)}$$

and so inequality (5.19) holds.

Now assume that (5.16) holds. Let

$$S = \limsup_{k \to +\infty} \frac{\|y_k\|_{\theta^k(\omega)}}{\|z_k\|_{\theta^k(\omega)}}.$$

By (5.16), we have $0 \leq S \leq 1$. It follows from (5.14) and (5.16) that

$$\|z_{k+1}\|_{\theta^{k+1}(\omega)} \geq (e^{c+\alpha} - 4K_{k+1}^2 \gamma_k) \|z_k\|_{\theta^k(\omega)} \tag{5.24}$$

for all sufficiently large $k \in \mathbb{N}$. By (5.13) and (5.24), we have

$$\frac{\|y_{k+1}\|_{\theta^{k+1}(\omega)}}{\|z_{k+1}\|_{\theta^{k+1}(\omega)}} \leq \frac{e^{c-\alpha} + 2K_{k+1}^2 \gamma_k}{e^{c+\alpha} - 4K_{k+1}^2 \gamma_k} \cdot \frac{\|y_k\|_{\theta^k(\omega)}}{\|z_k\|_{\theta^k(\omega)}} + \frac{2K_{k+1}^2 \gamma_k}{e^{c+\alpha} - 4K_{k+1}^2 \gamma_k}$$

for all sufficiently large $k \in \mathbb{N}$. It follows from (2.1) and (5.2) that

$$\frac{e^{c-\alpha} + 2K_{k+1}^2 \gamma_k}{e^{c+\alpha} - 4K_{k+1}^2 \gamma_k} \to \frac{e^{c-\alpha}}{e^{c+\alpha}} < 1 \quad \text{and} \quad \frac{2K_{k+1}^2 \gamma_k}{e^{c+\alpha} - 4K_{k+1}^2 \gamma_k} \to 0,$$

when $k \to \infty$ and so, $S = 0$. This establishes (5.22). Now take k_0 such that (5.24) holds for all $k \geq k_0$. Iterating (5.24), we conclude that

$$\|z_k\|_{\theta^k(\omega)} \geq \|z_{k_0}\|_{\theta^{k_0}(\omega)} e^{(c+\alpha)(k-k_0)} \prod_{j=k_0}^{k} (1 - 4K_{j+1}^2 \gamma_j e^{-c-\alpha})$$

for $k \geq k_0$. It follows from (2.1) and (5.2) that

$$\frac{1}{k} \sum_{j=k_0}^{k} \log \frac{1}{1 - 4K_{j+1}^2 \gamma_j e^{-c-\alpha}} \leq \frac{1}{k} \sum_{j=k_0}^{k} \frac{4K_{j+1}^2 \gamma_j e^{-c-\alpha}}{1 - 4K_{j+1}^2 \gamma_j e^{-c-\alpha}} \to 0$$

when $k \to \infty$ and so,

$$\liminf_{k\to+\infty} \frac{1}{k} \log\|z_k\|_{\theta^k(\omega)} \ge c+\alpha > c.$$

Finally, by (2.1) and (5.10), we obtain

$$\liminf_{k\to+\infty} \frac{1}{k} \log\|x_k\| \ge \liminf_{k\to+\infty} \frac{1}{k} \log\left(\frac{1}{2K_k}\|z_k\|_{\theta^k(\omega)}\right) > c$$

and so inequality (5.21) holds. □

In order to complete the proof of the theorem, assume that the first alternative does not hold. Take

$$c_0 < \lambda_1 < c_1 < \cdots < c_{s-1} < \lambda_s < c_s.$$

It follows from Lemma 5.5 that for each $i \in \{0, \ldots, s\}$, we have

$$\limsup_{k\to+\infty} \frac{1}{k} \log\|x_k\| < c_i \quad \text{or} \quad \liminf_{k\to+\infty} \frac{1}{k} \log\|x_k\| > c_i.$$

Together with Lemmas 5.1 and 5.2, this implies that there exists $i \in \{1, \ldots, s\}$ such that

$$\limsup_{k\to+\infty} \frac{1}{k} \log\|x_k\| < c_i \quad \text{and} \quad \liminf_{k\to+\infty} \frac{1}{k} \log\|x_k\| > c_{i-1}.$$

Finally, letting $c_{i-1} \nearrow \lambda_i$ and $c_i \searrow \lambda_i$, we conclude that

$$\lim_{k\to+\infty} \frac{1}{k} \log\|x_k\| = \lambda_i.$$

The proof of the theorem is complete.

6. Proof of Theorem 3.2

Take numbers $a < c$ such that $[a, c] \subset (\lambda_{i-1}, \lambda_i)$. Then

$$\lim_{m\to+\infty} \frac{1}{m} \log\|x_m\| = \lambda_i > \frac{a+c}{2}$$

and it follows from Lemma 5.5 that

$$\lim_{k\to+\infty} \frac{\|y_k\|_{\theta^k(\omega)}}{\|z_k + w_k\|_{\theta^k(\omega)}} = 0, \tag{6.1}$$

with the norms $\|\cdot\|_k$ defined as in (5.9) but with the numbers $c-\alpha$ and $c+\alpha$ replaced, respectively, by a and c.

Now take numbers $a' < c'$ such that $[a', c'] \subset (\lambda_i, \lambda_{i+1})$. Then

$$\lim_{m\to+\infty} \frac{1}{m} \log\|x_m\| = \lambda_i < \frac{a'+c'}{2}$$

and it follows from Lemma 5.5 that

$$\lim_{k\to+\infty} \frac{\|z_k\|'_{\theta^k(\omega)}}{\|y_k + w_k\|'_{\theta^k(\omega)}} = 0, \tag{6.2}$$

with the norms $\|\cdot\|_k'$ defined as in (5.9) but with the numbers $c-\alpha$ and $c+\alpha$ replaced, respectively, by a' and c'. Given $\delta > 0$, take $\eta \in (0,1)$ such that $4K(\omega)\eta/(1-\eta) < \delta$. By (6.2), for all sufficiently large k, we have

$$\|z_k\|_{\theta^k(\omega)}' \leq \eta\|y_k + w_k\|_{\theta^k(\omega)}'. \tag{6.3}$$

Similarly, by (6.1), for all sufficiently large k we have

$$\|y_k\|_{\theta^k(\omega)} \leq \eta\|z_k + w_k\|_{\theta^k(\omega)}. \tag{6.4}$$

On the other hand, by (5.9) (with $c-\alpha$ replaced by a and $c+\alpha$ replaced by c), since $a' > a$ we obtain

$$\begin{aligned}\|y_k\|_{\theta^k(\omega)} &= \sup_{n\geq k}(\|\Phi(n-k,\theta^k(\omega))P(\theta^k(\omega))x_k\|e^{-a(n-k)})\\ &= \sup_{n\geq k}(e^{-(a-a')(n-k)}e^{-a'(n-k)}\|\Phi(n-k,\theta^k(\omega))P(\theta^k(\omega))x_k\|)\\ &\geq \sup_{n\geq k}(e^{-a'(n-k)}\|\Phi(n-k,\theta^k(\omega))P(\theta^k(\omega))x_k\|) = \|y_k\|_{\theta^k(\omega)}'.\end{aligned} \tag{6.5}$$

Analogously, by (5.9) since $c' > c$ we obtain

$$\begin{aligned}\|z_k\|_{\theta^k(\omega)} &= \sup_{n\leq k}(\|\Phi(n-k,\theta^k(\omega))Q(\theta^k(\omega))x_k\|e^{-c(n-k)})\\ &= \sup_{n\leq k}(e^{-(c-c')(n-k)}e^{-c'(n-k)}\|\Phi(n-k,\theta^k(\omega))Q(\theta^k(\omega))x_k\|)\\ &\leq \sup_{n\leq k}(e^{-c'(n-k)}\|\Phi(n-k,\theta^k(\omega))Q(\theta^k(\omega))x_k\|) = \|z_k\|_{\theta^k(\omega)}'.\end{aligned} \tag{6.6}$$

Using (6.5) and (6.6), we deduce from (6.4) that

$$\begin{aligned}\|y_k\|_{\theta^k(\omega)}' \leq \|y_k\|_{\theta^k(\omega)} &\leq \eta\|z_k + w_k\|_{\theta^k(\omega)}\\ &\leq \eta\|z_k\|_{\theta^k(\omega)}' + \eta\|w_k\|_{\theta^k(\omega)}.\end{aligned} \tag{6.7}$$

Therefore, it follows from (6.3) together with (6.7) that

$$\|z_k\|_{\theta^k(\omega)}' \leq \eta^2\|z_k\|_{\theta^k(\omega)}' + \eta^2\|w_k\|_{\theta^k(\omega)} + \eta\|w_k\|_{\theta^k(\omega)}'$$

and

$$\|z_k\|_{\theta^k(\omega)}' \leq \eta(1-\eta^2)^{-1}(\eta\|w_k\|_{\theta^k(\omega)} + \|w_k\|_{\theta^k(\omega)}').$$

Hence, using (5.10),

$$\begin{aligned}\|z_k\|_{\theta^k(\omega)} \leq \|z_k\|_{\theta^k(\omega)}' &\leq 4K(\theta^k(\omega))\eta(1+\eta)(1-\eta^2)^{-1}\|w_k\|_{\theta^k(\omega)}\\ &\leq \delta\|w_k\|_{\theta^k(\omega)}.\end{aligned}$$

Since δ is arbitrary, we obtain

$$\lim_{k\to+\infty}\frac{\|z_k\|_{\theta^k(\omega)}}{\|w_k\|_{\theta^k(\omega)}} = 0,$$

which establishes identity (3.5). Identity (3.4) can be obtained in a similar manner interchanging the roles of y_k and z_k.

Acknowledgments

L.B. and C.V. were supported by FCT/Portugal through UID/MAT/04459/2013. D.D. was supported in part by the Croatian Science Foundation under the project IP-2014-09-2285 and by the University of Rijeka research grant 13.14.1.2.02.

References

[1] L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity*, Encyclopedia of Mathematics and its Applications, Vol. 115 (Cambridge University Press, 2007).

[2] L. Barreira and C. Valls, A Perron-type theorem for nonautonomous differential equations, *J. Differential Equations* **258** (2015) 339–361.

[3] C. Coffman, Asymptotic behavior of solutions of ordinary difference equations, *Trans. Amer. Math. Soc.* **110** (1964) 22–51.

[4] W. Coppel, *Stability and Asymptotic Behavior of Differential Equations* (D. C. Heath and Co., 1965).

[5] P. Hartman and A. Wintner, Asymptotic integrations of linear differential equations, *Amer. J. Math.* **77** (1955) 45–86.

[6] F. Lettenmeyer, Über das asymptotische Verhalten der Lösungen von Differentialgleichungen und Differentialgleichungssystemen, *Stzb. Bayer Akad. Wiss. Munchen, Mathnaiurw. Kl.* **59** (1929) 201–252.

[7] Z. Lian and K. Lu, *Lyapunov Exponents and Invariant Manifolds for Random Dynamical Systems in a Banach Space*, Memoirs of the American Mathematical Society, Vol. 206, No. 967 (American Mathematical Society, 2010).

[8] K. Matsui, H. Matsunaga and S. Murakami, Perron type theorem for functional differential equations with infinite delay in a Banach space, *Nonlinear Anal.* **69** (2008) 3821–3837.

[9] V. Oseledets, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.* **19** (1968) 197–221.

[10] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen, *Math. Z.* **29** (1929) 129–160.

[11] Ya. Pesin, Families of invariant manifolds corresponding to nonzero characteristic exponents, *Math. USSR-Izv.* **10** (1976) 1261–1305.

[12] M. Pituk, A Perron type theorem for functional differential equations, *J. Math. Anal. Appl.* **316** (2006) 24–41.

[13] ———, Asymptotic behavior and oscillation of functional differential equations, *J. Math. Anal. Appl.* **322** (2006) 1140–1158.