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LENGTH OF ε-NEIGHBORHOODS OF ORBITS OF DULAC MAPS

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Abstract. By Dulac maps we mean first return maps of hyperbolic polycycles of analytic planar vector fields. We study the fractal properties of the orbits of a parabolic Dulac map. To this end, we prove that it admits a Fatou coordinate with an asympotic expansion in terms of power-iterated logarithm transseries. This allows to introduce a new notion, the *continuous time length of ε-neighborhoods of orbits*, and to prove that this function of ε admits an asymptotic expansion in the same scale. We show that, under some hypotheses, this expansion determines the class of formal conjugacy of the Dulac map.

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1. Introduction

The present work is a continuation of the study of one-dimensional discrete dynamical systems, based on the fractal properties of their orbits ([13, 15, 16, 20]). Recall that the fractal properties of a bounded subset U of $\mathbb{R}$ or $\mathbb{C}$ reflect the asymptotic behavior at 0 of the function $\varepsilon \mapsto A(U_\varepsilon)$, for $\varepsilon > 0$, where $A(U_\varepsilon)$ denotes the Lebesgue measure of the ε-neighborhood of U. In particular, the *box dimension* of U (see [8] for a precise definition) describes the growth of $A_\varepsilon(U)$ when ε tends to 0.

Consider a germ f in one variable which admits a fixed point a, and a point x_0 close to a. We denote by $A_f(x_0, \varepsilon)$ the Lebesgue mesure of the ε-neighborhood of the orbit of x_0 (or a *directed* version of it in the complex case, see [15]). It is proved in [7] that, for a differentiable germ with an attracting fixed point a, the multiplicity of a is determined by the box dimension of *any* attracted orbit. This result has been generalized to a class of non differentiable germs in [13], where the asymptotic behavior of $A_f(x_0, \varepsilon)$ is given as an explicit function of ε.

In the same spirit, it is proved in [15] that the class of *formal* conjugacy of a (real or complex) parabolic analytic germ is determined by an initial part of the asymptotic expansion of $A_f(x_0, \varepsilon)$ in power-log monomials. Describing the class of *analytic* conjugacy of f would certainly require not only an initial part, but a full asymptotic expansion of $A_f(x_0, \varepsilon)$ in power-log monomials. Unfortunately, it is also proved in [15] that such a complete expansion does not exist. The reason is that the computation of $A_f(x_0, \varepsilon)$ needs the determination of an appropriate

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critical iterate f^{n_ε} of f, where n_ε is obtained with the help of the integer part function. One goal of the present paper is to correct this defect by proposing a convenient modification of the definition of $A_f(x_0,\varepsilon)$. That is, to involve, if they exist, *fractional iterates* f^t of f, for $t\in\mathbb{R}$, instead of usual integer iterates of f. We call this new function the *continuous time length of ε-neighborhoods of orbits of f*, and denote it by $A_f^c(x_0,\varepsilon)$. For the germs considered, it generalizes the classical, *discrete time* length $A_f(x_0,\varepsilon)$ of the ε-neighborhood of the orbits of f in the following sense: for any orbit, the beginning of its asymptotic expansion coincides with the beginning of the asymptotic expansion of $A_f(x_0,\varepsilon)$.

We intend to work with a class of germs not considered yet in the aforementioned articles, the germs of *Dulac maps* (or *Dulac germs* for short). By Dulac map we mean the first return map of a hyperbolic polycycle of an analytic vector field of the real plane. Such a map is analytic on an open interval $(0,b)$, and admits a power-log asymptotic expansion of Dulac type at the origin, see e.g. [9]. We introduced in [14] an algebra $\mathcal{L}$ (here denoted by $\widehat{\mathcal{L}}$, for formal) of formal *power-log transseries* which contains the Dulac asymptotic expansions of Dulac maps. We proved that the elements of $\widehat{\mathcal{L}}$ which are *parabolic* (that is, tangent to the identity) embed into flows of *formal* vector fields, and hence admit fractional iterates. We will restrict to this parabolic situation: given a parabolic element $\widehat{f}\in\widehat{\mathcal{L}}$, using this embedding, we can define a *formal continuous time length of ε-neighborhoods of orbits*, denoted by $\widehat{A}^c_{\widehat{f}}(\varepsilon)$.

We work in a class of *transseries* $\widehat{\mathfrak{L}}$ with well-ordered support in power-iterated logarithm monomials of finite depth in iterated logarithms (see Definition 2.5 for details). Our study of parabolic power-log transseries and Dulac germs leads us to two main results. In Theorem A, we show that, for a parabolic transseries $\widehat{f}\in\widehat{\mathcal{L}}$, the formal continuous time length $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ is a transseries with a well-ordered support of monomials of the type:

$$x^\alpha\left(\frac{1}{-\log x}\right)^{\gamma_1}\left(\frac{1}{\log(-\log x)}\right)^{\gamma_2},\ \alpha\geq 0,\ \gamma_1,\ \gamma_2\in\mathbb{R}.$$

On the other hand, we prove that a parabolic Dulac germ f embeds in a flow (f^t), which allows to consider the function $\varepsilon\mapsto A_f^c(x_0,\varepsilon)$. Denote by $\widehat{f}$ the Dulac expansion of f. In Theorem B, we show that a particular type of a (trans)asymptotic expansion of $A_f^c(x_0,\varepsilon)$ in $\widehat{\mathfrak{L}}$ is unique and equal to the formal length $\widehat{A}^c_{\widehat{f}}(\varepsilon)$, up to a term $C\varepsilon$, $C\in\mathbb{R}$. It is worth noticing that we use in this construction the quasianalyticity result of [9], which states that if a parabolic Dulac germ f is not equal to the identity, then its Dulac asymptotic expansion is different from x. This non trivial asymptotic expansion is a crucial feature in our proofs. We see that the situation considered here is very different from the one studied in [18], in which the author proves the embedding in a flow of a smooth parabolic germ *with a flat contact with the identity*. The methods used in [18] would allow to define the function $A_f^c(x_0,\varepsilon)$, but, unlike ours, would not lead to an asymptotic expansion in $\widehat{\mathfrak{L}}$ of this function.

Having ensured the existence of a complete (trans)asymptotic expansion for the function $\varepsilon\mapsto A_f^c(x_0,\varepsilon)$, we plan in the future to obtain *sectorially* analytic functions in ε. Comparing *sectorial* functions, our goal is to be able to read the analytic class of parabolic Dulac germs. See Example 4 for the computation of the (formal) continuous time length of ε-neighborhoods of orbits in the case of analytic parabolic germs on $\mathbb{R}$.

This paper is organized as follows.

In Section 2, we introduce the notion of the continuous time length $A_f^c(x_0,\varepsilon)$ of ε-neighborhoods of orbits for a germ, and of its formal analogue for power-log transseries. We also recall the notion of *Fatou coordinate* for a germ f, which essentially conjugates f to the translation by 1. It allows to give an explicit formula for the $A_f^c(x_0,\varepsilon)$ for a germ f which embeds as the time-one map of a flow $\{f^t\}$. We also state our two main theorems.

In Section 3, we notice that (trans)asymptotic expansions in $\widehat{\mathfrak{L}}$ of germs following the method of Poincaré are not unique in general. To ensure uniqueness, we introduce the notion of *sectional asymptotic expansions* and define one particular type of sectional asymptotic expansions adapted to Dulac germs and their $A_f^c(x_0,\varepsilon)$.

In Section 4, we recall the classical notion of "embedding as the time-one map of a flow". We state the equivalence between the existence of a Fatou coordinate and an embedding in a flow, for analytic germs on open intervals and for parabolic transseries (the proof of these facts is postponed to Section 10).

In Section 5 we give some examples of sectional asymptotic expansions.

Sections 6 and 7 are dedicated to description of the formal inverse and the formal Fatou coordinate for parabolic Dulac transseries $\widehat{f}$, needed in the formula for formal length $\widehat{A}^c_{\widehat{f}}(\varepsilon)$. Moreover, we prove that the power-iterated logarithm sectional (trans)asymptotic expansions in $\widehat{\mathfrak{L}}$ of the Fatou coordinate and the inverse for a parabolic Dulac map correspond to their formal counterparts. It is worth noticing that the proofs of these sections rely on the particular form of Dulac transseries. Although the proof that the inverse of a transseries is also a transseries is given in full generality in [1], in our framework we need an explicit description of monomials. In particular, in Section 6 we give a precise description of the formal inverse of a Dulac transseries. Part of the proof of the existence and description of the Fatou coordinate for Dulac germs in Section 7 is inspired by the similar classical result for parabolic analytic germs which is explained, for example, in [12].

All these results finally allow to prove Theorem A in Section 8 and Theorem B in Section 9. Finally, the Appendix (Section 10) is dedicated to more technical proofs and definitions.

2. Continuous time length of ε-neighborhoods of orbits and main results

2.1. The continuous time length of ε-neighborhoods of orbits of germs.

Suppose that a germ f is analytic on $(0,d)$, $d>0$, has zero as a fixed point, and that $\mathrm{id}-f>0$ and increasing. Let x_0 belong to the attracting basin of 0, so that the orbit $\mathcal{O}^f(x_0)=\{f^{\circ n}(x_0):\, n\in\mathbb{N}_0\}$ tends to zero. Recall that the function

$$\varepsilon\mapsto A_f(x_0,\varepsilon)$$

denotes the 1-dimensional Lebesgue measure of the ε-neighborhood of the orbit. By [19], $A_f(x_0,\varepsilon)$ is calculated by decomposing the ε-neighborhood of $\mathcal{O}^f(x_0)$ in two parts: the *nucleus* $N(x_0,\varepsilon)$, and the *tail* $T(x_0,\varepsilon)$. The nucleus is the overlapping part of the ε-neighborhood, and the tail is the union of the disjoint intervals of length 2ε. They are determined in function of the *discrete critical time* $n_\varepsilon(x_0)$, which is described by the condition:

$$\begin{cases} f^{n_\varepsilon(x_0)}(x_0) - f^{n_\varepsilon(x_0)+1}(x_0) \leq 2\varepsilon, \\ f^{n_\varepsilon(x_0)-1}(x_0) - f^{n_\varepsilon(x_0)}(x_0) > 2\varepsilon. \end{cases} \tag{2.1}$$

Then

$$A_f(x_0, \varepsilon) = |N(x_0, \varepsilon)| + |T(x_0, \varepsilon)| = \big(f^{n_\varepsilon(x_0)}(x_0) + 2\varepsilon\big) + n_\varepsilon(x_0) \cdot 2\varepsilon. \tag{2.2}$$

Now suppose additionally that f can be *embedded as time-one map in a flow* $\{f^t\}$, f^t analytic on $(0, d)$, of class C^1 in $t \in \mathbb{R}$ (see Section 4 for the precise definition). We show in the next definition how an embedding in a flow allows us to define the *continuous critical time with respect to the flow* $\{f^t\}$, denoted by τ_ε, and, accordingly, what we call the *continuous time length of the ε-neighborhood of orbit $\mathcal{O}^f(x_0)$ with respect to the flow* $\{f^t\}$. As will be shown in Proposition 2.4, it turns out that the discrete critical time $n_\varepsilon(x_0)$ in the standard definition (2.2) is the *ceiling function* of the continuous critical time $\tau_\varepsilon(x_0)$ (the smallest integer bigger than or equal to $\tau_\varepsilon(x_0)$).

It will be proven, as a consequence of the forthcoming Proposition 7.1, that a parabolic Dulac germ embeds as the time one in a unique (up to a translation) flow whose corresponding Fatou coordinate admits a sectional asymptotic expansion in $\mathfrak{L}$. Hence, for Dulac germs, we will get rid of the precision *with respect to a given flow.*

Definition 2.1 (The continuous time length of ε-neighborhoods of orbits)**.** *Assume that f embeds as the time-one map in a flow $\{f^t\}$, f^t analytic on $(0, d)$. The* continuous time length of the ε-neighborhood of orbit $\mathcal{O}^f(x_0)$ with respect to the flow $\{f^t\}$ *is defined as:*

$$A^c_f(x_0, \varepsilon) = \big(f^{\tau_\varepsilon(x_0)}(x_0) + 2\varepsilon\big) + \tau_\varepsilon(x_0) \cdot 2\varepsilon. \tag{2.3}$$

Here, $\tau_\varepsilon(x_0)$ is the continuous ε-critical time *for the ε-neighborhood of orbit $\mathcal{O}^f(x_0)$, defined by the equality:*

$$f^{\tau_\varepsilon(x_0)}(x_0) - f^{\tau_\varepsilon(x_0)+1}(x_0) = 2\varepsilon. \tag{2.4}$$

It turns out, as we will show in Proposition 4.4, that the embedding in a flow is closely related to the existence of a *Fatou coordinate* for f, which is defined as follows:

Definition 2.2 (Fatou coordinate)**.**

1. Let f be an analytic germ on $(0, d)$, $d > 0$. We say that a strictly monotone analytic germ Ψ on $(0, d)$ is a Fatou coordinate *for f if*

$$\Psi(f(x)) - \Psi(x) = 1,\ x \in (0, d),\ d > 0. \tag{2.5}$$

2. Let $\widehat{f} \in \widehat{\mathcal{L}}$ be parabolic. We say that $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ is a formal Fatou coordinate *for $\widehat{f}$ if the following equation is satisfied formally in $\widehat{\mathfrak{L}}$:*

$$\widehat{\Psi}(\widehat{f}) - \widehat{\Psi} = 1. \tag{2.6}$$

We reformulate definition (2.3) of $A^c_f(x_0, \varepsilon)$ in the following corollary.

Proposition 2.3. *Assume that the germ f embeds as the time-one map in a C^1-flow $\{f^t\}$, f^t analytic on $(0,d)$. Let the germ $\xi := \frac{d}{dt}f^t\big|_{t=0}$ be non-oscillatory in the sense from Proposition 4.4. Let $g = \mathrm{id} - f$. Then*

$$A_f^c(x_0,\varepsilon) = \big(g^{-1}(2\varepsilon) + 2\varepsilon\big) + 2\varepsilon \cdot \big[\Psi\big(g^{-1}(2\varepsilon)\big) - \Psi(x_0)\big]. \tag{2.7}$$

We actually prove in Proposition 4.4 that the embedding of f as the time-one map of an analytic C^1-flow with non-oscillatory ξ implies the existence of an analytic Fatou coordinate for f. Therefore Ψ in (2.7) exists by the assumptions of Proposition 2.3. The monotonicity of Ψ is equivalent to the non-oscillatority at 0 of the vector field ξ for the flow $\{f^t\}$ (see the proof of Proposition 4.4).

Proof. Using $g = \mathrm{id} - f$ and g strictly increasing, (2.4) simplifies to

$$g\big(f^{\tau_\varepsilon(x_0)}(x_0)\big) = 2\varepsilon,\ f^{\tau_\varepsilon(x_0)}(x_0) = g^{-1}(2\varepsilon).$$

The equation (4.2) for the Fatou coordinate now gives:

$$\tau_\varepsilon(x_0) = \Psi(f^{\tau_\varepsilon(x_0)}(x_0)) - \Psi(x_0) = \Psi(g^{-1}(2\varepsilon)) - \Psi(x_0).$$

Putting this into definition (2.3), we get the desired formula. □

Proposition 2.4. *Let f and $\mathcal{O}^f(x_0)$ be as above. Let $\{f^t\}$ and Ψ be as in Proposition 2.3. Then:*

$$n_\varepsilon(x_0) = \lceil \tau_\varepsilon(x_0) \rceil,\ \varepsilon > 0.$$

Here, we denote by $\lceil a \rceil := \min\{k \in \mathbb{Z} :\ a \le k\}$, $a \in \mathbb{R}$.

Proof. Let $g = \mathrm{id}$. The relations (2.1) and (2.4) which define n_ε, τ_ε give:

$$g\big(f^{n_\varepsilon(x_0)}(x_0)\big) \le 2\varepsilon,\ g\big(f^{\tau_\varepsilon(x_0)}\big)(x_0) = 2\varepsilon,\ g\big(f^{n_\varepsilon(x_0)-1}(x_0)\big) > 2\varepsilon.$$

Since g is a strictly increasing germ (since f is such), we get that

$$f^{n_\varepsilon(x_0)}(x_0) \le f^{\tau_\varepsilon(x_0)}(x_0) < f^{n_\varepsilon(x_0)-1}(x_0).$$

By monotonicity of Ψ, we get:

$$n_\varepsilon(x_0) \ge \tau_\varepsilon(x_0) > n_\varepsilon(x_0) - 1,$$

which proves the result. □

2.2. **The formal continuous time length of ε-neighborhoods of orbits.** We define here an analogue of the continuous time length of ε-neighborhoods of orbits *in the formal setting.*

We introduce the necessary classes of transseries. We put $\boldsymbol{\ell}_0 := x$, $\boldsymbol{\ell} := \boldsymbol{\ell}_1 := \frac{1}{-\log x}$, and define inductively $\boldsymbol{\ell}_{j+1} = \boldsymbol{\ell} \circ \boldsymbol{\ell}_j$, $j \in \mathbb{N}$, as symbols for iterated logarithms.

Definition 2.5 (The classes $\widehat{\mathcal{L}}_j^\infty$ and $\widehat{\mathfrak{L}}$)**.** *Denote by $\widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}_0$, the set of all transseries of the type:*

$$\widehat{f}(x) = \sum_{i_0=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} a_{i_0\ldots i_j} x^{\alpha_{i_0}} \boldsymbol{\ell}^{\alpha_{i_0 i_1}} \cdots \boldsymbol{\ell}_j^{\alpha_{i_0\cdots i_j}},\ a_{i_0\cdots i_j} \in \mathbb{R},\ x > 0, \tag{2.8}$$

where $(\alpha_{i_0\cdots i_k})_{i_k \in \mathbb{N}}$ is an increasing sequence of real numbers tending to $+\infty$, for every $k = 0, \ldots, j$. If moreover $\alpha_1 > 0$ (the infinitesimal cases), we denote the class by $\widehat{\mathcal{L}}_j$. The subset of $\widehat{\mathcal{L}}_1$ resp. $\widehat{\mathcal{L}}_1^\infty$ of transseries with only integer powers of $\boldsymbol{\ell}$ will be denoted by $\widehat{\mathcal{L}}$ resp. $\widehat{\mathcal{L}}^\infty$.

Put

$$\widehat{\mathfrak{L}} := \bigcup_{j\in\mathbb{N}_0} \widehat{\mathcal{L}}_j^\infty$$

for the class of all power-iterated logarithm transseries of finite depth in iterated logarithms.

Note that the classes $\widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}_0$, are the sub-classes of power-iterated logarithm transseries, whose support is any well-ordered subset of $\mathbb{R}^{j+1}$ (for the lexicographical order). We restrict only to the subclass with strictly increasing exponents. In this paper, we work with Dulac germs and their expansions, for which this restriction is sufficient.

Notice that $x = \boldsymbol{\ell}_0$. The classes $\widehat{\mathcal{L}}_0$ or $\widehat{\mathcal{L}}_0^\infty$ are made of formal power series:

$$\widehat{f}(x) = \sum_{i\in\mathbb{N}} a_i x^{\alpha_i},\ a_i \in \mathbb{R},\ x > 0,$$

such that (α_i) is a strictly increasing real sequence tending to $+\infty$.

For $\widehat{f} \in \widehat{\mathcal{L}}_j^\infty$, we denote by $\mathrm{Lt}(\widehat{f})$ its *leading term* $a_{\gamma_0,\gamma_1,\ldots,\gamma_j}\, x^{\gamma_0}\boldsymbol{\ell}^{\gamma_1}\boldsymbol{\ell}_2^{\gamma_2}\cdots\boldsymbol{\ell}_j^{\gamma_j}$. The tuple $(\alpha,\gamma_1,\ldots,\gamma_j)$ is called *the order* of $\widehat{f}$, and is denoted by $\mathrm{ord}(\widehat{f}) = (\alpha,\gamma_1,\ldots,\gamma_j)$. The transseries $\widehat{f} \in \widehat{\mathcal{L}}_j$, $j \in \mathbb{N}_0$, is called *parabolic* if $\mathrm{ord}(\widehat{f}) = (1, 0, \ldots, 0)$.

Let $\widehat{f} \in \widehat{\mathcal{L}}$ be parabolic. Unlike a germ, a transseries does not have orbits. Indeed, evaluating a transseries $\widehat{f}$ at a point is meaningless. Therefore, the initial Definition 2.1 of the continuous time length of ε-neighborhoods of orbits cannot be directly transported to the formal setting. Fortunately, we can exploit the *equivalent* definition (2.7) from Proposition 2.3. Note that all the functions used in (2.7) have their direct formal analogues.

On the other hand, we have shown in [14] and we recall in Section 4 that any parabolic transseries $\widehat{f} \in \widehat{\mathcal{L}}$ can be embedded in a *unique* C^1-flow $\{\widehat{f}^t\}_{t\in\mathbb{R}}$ of elements of $\widehat{\mathcal{L}}$. That is, its formal Fatou coordinate $\widehat{\Psi}$ in $\widehat{\mathcal{L}}_2^\infty$, as defined in Definition 2.2, is *unique* in $\widehat{\mathfrak{L}}$ (up to an additive constant) (see Proposition 4.4). Therefore, unlike a germ, the definition of the formal continuous time length in the formal setting is unambiguous (it does not depend on the chosen flow). Also, the formal definition is independent of the orbit.

Definition 2.6 (The formal continuous time length of ε-neighborhoods of orbits)**.** *Let $\widehat{f} \in \widehat{\mathcal{L}}$ be parabolic. Let $\widehat{g} = \mathrm{id} - \widehat{f}$. We define the* formal continuous time length of ε-neighborhoods of orbits of $\widehat{f}$ *by:*

$$\widehat{A}^c_{\widehat{f}}(\varepsilon) = \widehat{g}^{-1}(2\varepsilon) + 2\varepsilon\cdot\widehat{\Psi}\big(\widehat{g}^{-1}(2\varepsilon)\big). \tag{2.9}$$

Here, $\widehat{g}^{-1}$ denotes the formal inverse of $\widehat{g}$, $\widehat{\Psi}$ is the formal Fatou coordinate for $\widehat{f}$, which exists by Proposition 4.3.

Note that $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ defined by (2.9) is unique up to a term $K\cdot\varepsilon$, $K \in \mathbb{R}$, due to the fact that the formal Fatou coordinate $\widehat{\Psi}$ is unique up to an additive constant.

Theorem A. *Let $\widehat{f} \in \widehat{\mathcal{L}}$ be parabolic. Let $\widehat{g} = \mathrm{id} - \widehat{f}$. Then $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ is well-defined and belongs to the class $\widehat{\mathcal{L}}_2$.*

Note that *well-defined* means that the formal Fatou coordinate $\widehat{\Psi}$ and the formal inverse $\widehat{g}^{-1}$ exist in $\widehat{\mathfrak{L}}$. The proof of Theorem A and more precise form of transseries $\widehat{A}^c_{\hat{f}}(\varepsilon)$ will be given in Section 8.

In particular, it will be proved that, if the leading term of $\widehat{g}$ does not involve a logarithm, then there exists at most one term of $\widehat{A}^c_{\hat{f}}(\varepsilon)$ which contains a "double logarithm".

2.3. **Continuous and discrete time length of ε-neighborhoods of orbits for Dulac germs.** By $\mathcal{G}$ we denote the set of all one-dimensional germs defined on some positive open neighborhood of the origin. Furthermore, by $\mathcal{G}_{AN} \subset \mathcal{G}$ we denote the set of all germs which are moreover analytic on some positive open neighborhood of the origin.

Let $f \in \mathcal{G}_{AN}$ be a parabolic Dulac germ and let $\mathcal{O}^f(x_0)$ be an orbit of f accumulating at 0. Let $\widehat{f} \in \widehat{\mathcal{L}}$ be its Dulac expansion.

Theorem B is twofold. On one hand, it expresses an equality, up to a certain order, between the classical (discrete time) length of the ε-neighborhood of the orbit $\mathcal{O}^f(x_0)$ and its continuous time length as defined in Definition 2.1. On the other hand, it states that the continuous time length of the ε-neighborhood of the orbit $\mathcal{O}^f(x_0)$ admits the sectional asymptotic expansion (unique, with respect to an *integral section*, see Section 3) in power-iterated logarithm scale (in $\widehat{\mathfrak{L}}$) equal to the formal continuous length of ε-neighborhoods of orbits for $\widehat{f}$, up to a term εK, where K is an arbitrary constant.

Let us explain shortly the importance of Theorem B. It was shown in [16] that for analytic parabolic germs at 0, which form a subclass of the class of parabolic Dulac germs, the classical length of the ε-neighborhood of orbit $\mathcal{O}^f(x_0)$, the germ $\varepsilon \mapsto A_f(x_0, \varepsilon)$, does not have a complete asymptotic expansion in ε in power-logarithm scale. Moreover, the germ $\varepsilon \mapsto A_f(x_0, \varepsilon)$ does not belong to $\mathcal{G}_{AN}$ (there is an accumulation of singularities at $\varepsilon = 0$). This is due to the discontinuity of the integer part function, which is involved in the definition of the discrete critical time n_ε. Theorem B shows that the continuous length of the ε-neighborhood of orbit $\mathcal{O}^f(x_0)$, which is an *analytic* generalization of the standard length and belongs to $\mathcal{G}_{AN}$, gives a continuation of the asymptotic expansion of the classical length from the moment where the former expansion ceases to exist. On the other hand, both asymptotic expansions coincide in the first finitely many terms which characterize the germ formally. See [15] for formal recognition of germs from finitely many terms in the asymptotic expansion of the length of ε-neighborhoods of their orbits. Therefore we expect that our new notion of the continuous length, admitting a complete asymptotic expansion, will let us recognize the analytic class of the germ.

For a Dulac germ $f \in \mathcal{G}_{AN}$, we consider a function $\varepsilon \mapsto A^c_f(x_0, \varepsilon) \in \mathcal{G}_{AN}$ defined in (2.7). As already stressed before Definition 2.1, it is implicitely assumed that $A^c_f(x_0, \varepsilon)$ is defined with respect to the flow whose corresponding Fatou coordinate admits a sectional asymptotic expansion in $\widehat{\mathfrak{L}}$. We show later in Proposition 7.1 that such a Fatou coordinate is unique up to an additive constant. Consequently, $\varepsilon \mapsto A^c_f(x_0, \varepsilon)$ is unique up to an additive term $\varepsilon \cdot K$, $K \in \mathbb{R}$, depending on the choice of the constant in the Fatou coordinate.

We notice in Remark 3.1 and in Example 1 in Section 3 that the asymptotic expansion of germs in $\widehat{\mathfrak{L}}$ is not unique, and thus, not well-defined. This is due to the presence of limit ordinals in $\widehat{\mathfrak{L}}$ and to exponentially small terms ($x = e^{-\frac{1}{\boldsymbol{\ell}}}$, $\boldsymbol{\ell} = e^{-\frac{1}{\boldsymbol{\ell}_2}}$ etc). To ensure uniqueness of the asymptotic expansion in $\widehat{\mathfrak{L}}$, we introduce in Section 3 the notion of *sectional asymptotic expansions*, see Definition 3.3. We describe a transfinite Poincaré algorithm for this expansion. In particular, we define sectional asymptotic expansions with respect to *integral* sections (Definition 3.14), adapted to the function $\varepsilon \mapsto A_f^c(x_0, \varepsilon) \in \mathcal{G}_{AN}$ for Dulac germs f.

Theorem B. *Let $f \in \mathcal{G}_{AN}$ be a parabolic Dulac germ and $\widehat{f} \in \widehat{\mathcal{L}}$ its Dulac expansion. Let $\mathcal{O}^f(x_0)$ be any orbit of f. Let $A_f(x_0, \varepsilon) \in \mathcal{G}$ and $A_f^c(x_0, \varepsilon) \in \mathcal{G}_{AN}$ as above. Then:*

1. $\widehat{A}^c_{\widehat{f}}(\varepsilon) \in \widehat{\mathcal{L}}_2$.

2. The sectional asymptotic expansions of $A_f^c(x_0, \varepsilon)$ with respect to different integral sections **s** *are unique up to a term εK, $K \in \mathbb{R}$, where different choices of constant K correspond to different choices of the integral section. Moreover, up to εK, they are equal to $\widehat{A}^c_{\widehat{f}}(\varepsilon)$.*

3. $A_f^c(x_0, \varepsilon) - A_f(x_0, \varepsilon) = O(\varepsilon)$, $\varepsilon \to 0$.

That is, the formal continuous length $\widehat{A}^c_{\widehat{f}}(\varepsilon) \in \widehat{\mathcal{L}}_2$ extends the beginning of the sectional asymptotic expansion (with respect to any integral section) of function $\varepsilon \mapsto A_f(x_0, \varepsilon)$ after the order $O(\varepsilon)$.

Theorem B claims the following commutative diagram (Figure 2.3). In the diagram, $\Psi \in \mathcal{G}_{AN}$ denotes the Fatou coordinate of the germ f used in the definition of $A_f^c(x_0, \varepsilon)$.

$$\begin{tikzcd}
 & f \arrow[r, "\text{Dulac expansion}"] \arrow[d] & \widehat{f} \arrow[d] & \\
\text{in } \mathcal{G}_{\mathrm{AN}} & \psi \arrow[r, "\text{asymptotic expansion}", "\text{w.r.t. an integral section}"'] \arrow[d] & \widehat{\psi} \arrow[d] & \text{in } \widehat{\mathfrak{L}} \\
 & A_f^c\,(x_0, \varepsilon) \arrow[r, "\text{asymptotic expansion}", "\text{w.r.t. an integral section}"'] & \widehat{A}^c_{\widehat{f}}\,(\varepsilon) &
\end{tikzcd}$$

FIGURE 2.1. The commutative diagram for a Dulac germ $f \in \mathcal{G}_{AN}$.

It will be proven in Proposition 7.1 that a Dulac germ $f \in \mathcal{G}_{AN}$ admits a *unique* (up to a constant) Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ that admits a sectional asymptotic expansion in $\widehat{\mathfrak{L}}$. Moreover, that the (unique) sectional asymptotic expansion of Ψ in $\widehat{\mathfrak{L}}$ with respect to an integral section is equal to the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$, up to a constant term. It is implicitely understood in Theorem B that $A_f^c(x_0, \varepsilon)$ is taken with respect to the flow that corresponds to this coordinate Ψ.

The proof of Theorem B is given in Section 9. Moreover, we show in Corollary 9.2 that the formal class of a simple Dulac germ f (i.e. the formal class of its Dulac expansion $\widehat{f}$) deduced in [14, Theorem A] can be read from two terms in the initial part of the sectional asymptotic expansion in ε of the function of the length of the ε-neighborhood of any orbit with respect to any integral section.

3. Sectional asymptotic expansions

This paper is dedicated to the study of Dulac germs (see Definition 3.9) and their continuous-time lengths of epsilon-neighborhoods of orbits. The Dulac asymptotic expansions involve monomials of the form $x^\alpha \log^p x$, $\alpha \in \mathbb{R}$, $p \in \mathbb{N}$. However, it turns out that the *asymptotic expansions* of the germs which are necessary to state and prove our results are *transfinite* and involve also *iterated logarithms.* Hence, first we have to explain what it means for a germ to have a power-log transseries as an asymptotic expansion.

Let the classes $\widehat{\mathcal{L}}_j^\infty$, $j \in \mathbb{N}$, and $\widehat{\mathfrak{L}}$ be as in Definition 2.5.

Remark 3.1 (Non-uniqueness of asymptotic expansions in $\widehat{\mathcal{L}}_j^\infty$ of germs from $\mathcal{G}$)**.** Recall the standard Poincaré method for deducing the asymptotic expansion of order type ω of a germ $f \in \mathcal{G}$ *term-by-term* in some scale, if the expansion exists. We describe here a generalization of the Poincaré method to transfinite asymptotic expansions, which we call the *transfinite Poincaré algorithm.* In particular, we consider germs with asymptotic expansions in the class $\widehat{\mathfrak{L}}$ (Definition 2.5), consisting of all power-log transseries of finite depth in iterated logarithms, with strictly increasing exponents. If the transfinite Poincaré algorithm applied to f gives the asymptotic expansion in the class $\widehat{\mathfrak{L}}$ of order type ω, the asymptotic expansion of f in $\widehat{\mathfrak{L}}$ will be unique. In general, the asymptotic expansion in $\widehat{\mathfrak{L}}$ is not unique, due to presence of limit ordinals in the algorithm and exponentially small corrections.

The transfinite Poincaré algorithm.

Let $f \in \mathcal{G}$. We construct an asymptotic expansion of f in $\widehat{\mathfrak{L}}$ of order type $\theta \in \mathbf{On}$, following the algorithm of Poincaré. To this end, we construct a transfinite series of *partial expansions* $(\widehat{f}_\nu)_{\nu<\theta}$ and a transfinite series of auxiliary cut-off germs $(f_\nu)_{\nu<\theta}$, following the algorithm:

First, put $f_{\nu_0} = f$, $\widehat{f}_{\nu_0} = 0$, where ν_0 is the smallest ordinal. The step of the algorithm at step ν is:

1. Find the *leading term* of f_ν, that is, $a\boldsymbol{\ell}_0^{\beta_0}\boldsymbol{\ell}_1^{\beta_1}\cdots\boldsymbol{\ell}_j^{\beta_j}$, $a \in \mathbb{R}$, $j \in \mathbb{N}_0$, such that:
$$\lim_{x\to 0}\frac{f_\nu(x)}{a\boldsymbol{\ell}_0^{\beta_0}\boldsymbol{\ell}_1^{\beta_1}\cdots\boldsymbol{\ell}_j^{\beta_j}} = 1.$$
Denote $\mathrm{Lt}(f_\nu) := a\boldsymbol{\ell}_0^{\beta_0}\boldsymbol{\ell}_1^{\beta_1}\cdots\boldsymbol{\ell}_j^{\beta_j}$.
2. We distinguish two cases:
   (i) *The successor ordinal case.* If $\nu+1$ is a successor ordinal, put
$$\widehat{f}_{\nu+1} := \widehat{f}_\nu + \mathrm{Lt}(f_\nu);\ f_{\nu+1} = f_\nu - \mathrm{Lt}(f_\nu).$$
   (ii) *The limit ordinal case.* If $\alpha = \lim_{\nu<\alpha}\nu$ is a limit ordinal, put:
$$\widehat{f}_\alpha := \lim_{\nu<\alpha}\widehat{f}_\nu,$$

implicitely assuming that $(\widehat{f}_\nu)_{\nu<\alpha}$ converge in some of the formal topologies introduced in [14]. Let $g_\alpha \in \mathcal{G}$ be a germ admitting the transfinite Poincaré expansion $\widehat{f}_\alpha$. Then put:

$$f_\alpha = f - g_\alpha.$$

Finally, we say that $\widehat{f} := \widehat{f}_\theta \in \widehat{\mathfrak{L}}$ is a *transfinite Poincaré asymptotic expansion of* f. Note that the successor ordinal step 2.(i) of the algorithm is well-defined by the Poincaré method. However, the *limit-ordinal step* 2.(ii) of the algorithm is not well-defined, due to non-unique choice of germ g_α. Since $\boldsymbol{\ell}_k$ are exponentially small with respect to $\boldsymbol{\ell}_{k+1}$, $k \in \mathbb{N}_0$ $(e^{-\frac{1}{\boldsymbol{\ell}_{k+1}}} = \boldsymbol{\ell}_k)$, different choices of germs g_α in step 2.(ii) in general lead to different asymptotic expansions. This is illustrated in Example 1 below.

To make the limit-ordinal step well-defined and unique, we introduce *section functions* and the notion of *sectional asymptotic expansions* in Definition 3.2 below.

Example 1.

(1) $f(x) = x(\boldsymbol{\ell} + \boldsymbol{\ell}^2) + x^2$. We can also write it as $f(x) = x(\boldsymbol{\ell} + \boldsymbol{\ell}^2 + e^{-\frac{1}{\boldsymbol{\ell}}})$. By the algorithm described in Remark 3.1,

$$\widehat{f}(x) = x(\boldsymbol{\ell} + \boldsymbol{\ell}^2) + x^2, \text{ and } \widehat{f}(x) = x(\boldsymbol{\ell} + \boldsymbol{\ell}^2),$$

are its (equally good) asymptotic expansions in $\widehat{\mathcal{L}} \subset \widehat{\mathfrak{L}}$.

(2) $f(x) = x\boldsymbol{\ell} + x^2\boldsymbol{\ell}_2$. Obviously, f admits an asymptotic expansion $\widehat{f} \in \widehat{\mathcal{L}}_2$, $\widehat{f} = x\boldsymbol{\ell} + x^2\boldsymbol{\ell}_2$. However, equally legitimate asymptotic expansions by the transfinite Poincaré algorithm from Remark 3.1 are, for example:

$$f(x) = x(\boldsymbol{\ell} + x\boldsymbol{\ell}_2) = x(\boldsymbol{\ell} + e^{-1/\boldsymbol{\ell}}\boldsymbol{\ell}_2) \Rightarrow \widehat{f} = x\boldsymbol{\ell} \in \widehat{\mathcal{L}}_1,$$
$$f(x) = x\boldsymbol{\ell} + x^2\boldsymbol{\ell}_2 + x^2\boldsymbol{\ell}_3 - x^2\boldsymbol{\ell}_3 = x\big(\boldsymbol{\ell} + e^{-1/\boldsymbol{\ell}}(\boldsymbol{\ell}_2 + \boldsymbol{\ell}_3)\big) - x^2\boldsymbol{\ell}_3$$
$$\Rightarrow \ \widehat{f} = x\boldsymbol{\ell} - x^2\boldsymbol{\ell}_3 \in \widehat{\mathcal{L}}_3, \text{ etc.}$$

In this manner, we can easily generate non-unique asymptotic expansions of $f \in \mathcal{G}$ in $\widehat{\mathfrak{L}}$. Moreover, they can belong to any $\widehat{\mathcal{L}}_j$, $j \in \mathbb{N}$.

The step toward well-defined expansions is to restrict the class of germs to some subclass of $\mathcal{G}$ which admit asymptotic expansions in $\widehat{\mathfrak{L}}$, but with additional requests on the choice of germs on limit ordinal levels of the transfinite Poincaré algorithm. We will call such expansions *sectional asymptotic expansions*, with in advance fixed *section function* (function which attributes *unique* germs to expansions on limit ordinal steps of the algorithm).

The definition of section function is inductive. First, let $\mathcal{S}_0 \subset \mathcal{G}$ be the set of all germs from $\mathcal{G}$ that admit the power asymptotic expansion in $\widehat{\mathcal{L}}_0^\infty$. This expansion is obviously unique. That is, $f \in \mathcal{S}_0$ if there exists a strictly increasing sequence $(\alpha_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ tending to $+\infty$ and a sequence of reals (a_i), such that

$$f(x) - \sum_{i=1}^{n} a_i x^{\alpha_i} = o(x^{\alpha_n}),\ n \in \mathbb{N}.$$

Equivalently, $f \in \mathcal{S}_0$ if it admits the Poincaré asymptotic expansion

$$\widehat{f}(x) = \sum_{i\in\mathbb{N}} a_i x^{\alpha_i} \in \widehat{\mathcal{L}}_0^\infty,\ x \to 0.$$

Denote by $\widehat{\mathcal{S}}_0 \subset \widehat{\mathcal{L}}_0^\infty$ the set of all power asymptotic expansions of germs from $\mathcal{S}_0$.

Definition 3.2 (Section function)**.** *A* section function *is a function* $\mathbf{s}$ *defined on some subset of* $\widehat{\mathfrak{L}}$ *which attributes germs to transseries, where its domain of the definition, as well as the section function itself, is inductively defined on increasing subsets of* $\widehat{\mathfrak{L}}$ *as follows:*

(1) *For* $\widehat{f} \in \widehat{\mathcal{S}}_0$, $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_0}(\widehat{f})$ *is a uniquely chosen germ* $f \in \mathcal{G}$ *that admits* $\widehat{f} \in \widehat{\mathcal{S}}_0$ *as its Poincaré asymptotic expansion. Let*

$$\mathcal{S}_0^{\mathbf{s}} := \mathbf{s}(\widehat{\mathcal{S}}_0).$$

(2) *The induction step* ($j \in \mathbb{N}$)*. Assume that the sets* $\widehat{\mathcal{S}}_k^{\mathbf{s}}$, $\mathcal{S}_k^{\mathbf{s}}$, *and the restriction* $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_k^{\mathbf{s}}}$, $k < j$, *are already defined. Then we define the the sets* $\widehat{\mathcal{S}}_j^{\mathbf{s}}$, $\mathcal{S}_j^{\mathbf{s}}$, *and the restriction* $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_j^{\mathbf{s}}}$ *as follows:*

(2.1) $\widehat{\mathcal{S}}_j^{\mathbf{s}} \subset \widehat{\mathcal{L}}_j^\infty$ *is a set of all transseries* $\widehat{f} \in \widehat{\mathcal{L}}_j^\infty$ *of the form:*

$$\widehat{f}(x) = \sum_{i \in \mathbb{N}} \widehat{f}_i\big(\boldsymbol{\ell}(x)\big) x^{\alpha_i},$$

such that (α_i) *is a strictly increasing sequence of real numbers tending to* $+\infty$ *and* $\widehat{f}_i \in \widehat{\mathcal{S}}_{j-1}^{\mathbf{s}}$ *(resp.* $\widehat{\mathcal{S}}_0$ *if* $j = 1$*),* $i \in \mathbb{N}$*, for which there exists (at least one) germ* $f \in \mathcal{G}$ *such that:*

$$f - \sum_{i=1}^{n} f_i\big(\boldsymbol{\ell}(x)\big) x^{\alpha_i} = o(x^{\alpha_n}),\ n \in \mathbb{N}, \tag{3.1}$$

where $f_i := \mathbf{s}(\widehat{f}_i)$.

(2.2) *For* $\widehat{f} \in \widehat{\mathcal{S}}_j^{\mathbf{s}}$ *as in* (2.1)*, define* $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_j^{\mathbf{s}}}(\widehat{f}) \in \mathcal{G}$ *as a germ* f *satisfying* (3.1)*.*

(2.3) *Let*

$$\mathcal{S}_j^{\mathbf{s}} := \mathbf{s}(\widehat{\mathcal{S}}_j^{\mathbf{s}}).$$

Note that, since $\widehat{\mathcal{L}}_j^\infty \subset \widehat{\mathcal{L}}_{j+1}^\infty$, $j \in \mathbb{N}_0$, we have that

$$\widehat{\mathcal{S}}_j^{\mathbf{s}} \subset \widehat{\mathcal{S}}_{j+1}^{\mathbf{s}} \text{ and } \widehat{\mathcal{S}}_0 \subset \widehat{\mathcal{S}}_j^{\mathbf{s}},\ j \in \mathbb{N}.$$

Analogously,

$$\mathcal{S}_j^{\mathbf{s}} \subset \mathcal{S}_{j+1}^{\mathbf{s}} \text{ and } \mathcal{S}_0 \subset \mathcal{S}_j^{\mathbf{s}},\ j \in \mathbb{N}.$$

Just keep in mind that we are always interested in the smallest $j \in \mathbb{N}_0$ such that $\widehat{f} \in \widehat{\mathcal{L}}_j^\infty$.

Obviously, by Definition 3.2, $\mathbf{s} : \widehat{\mathcal{S}}_0 \cup \big(\cup_{j\in\mathbb{N}} \widehat{\mathcal{S}}_j^{\mathbf{s}}\big) \subset \widehat{\mathfrak{L}} \to \cup_{j\in\mathbb{N}_0} \mathcal{S}_j^{\mathbf{s}}$. For a section $\mathbf{s}$, denote its domain by $\widehat{\mathcal{T}}^{\mathbf{s}} \subset \widehat{\mathfrak{L}}$,

$$\widehat{\mathcal{T}}^{\mathbf{s}} := \widehat{\mathcal{S}}_0 \cup \big(\cup_{j\in\mathbb{N}} \widehat{\mathcal{S}}_j^{\mathbf{s}}\big).$$

Denote by $\widehat{\mathcal{T}} \subset \widehat{\mathfrak{L}}$ the union of $\widehat{\mathcal{T}}^{\mathbf{s}}$ over all section functions $\mathbf{s}$:

$$\widehat{\mathcal{T}} := \cup_{\mathbf{s}} \widehat{\mathcal{T}}^{\mathbf{s}}.$$

Analogously, denote by $\mathcal{T}^{\mathbf{s}} \subset \mathcal{G}$ the image of a section $\mathbf{s}$:

$$\mathcal{T}^{\mathbf{s}} = \cup_{j\in\mathbb{N}_0} \mathcal{S}_j^{\mathbf{s}}.$$

By $\mathcal{T} \subset \mathcal{G}$ we denote the set of all germs that admit sectional asymptotic expansions (with respect to all possible section functions) in $\widehat{\mathfrak{L}}$:

$$\mathcal{T} = \cup_{\mathbf{s}} \mathcal{T}^{\mathbf{s}}.$$

Note that it is necessary to take the union over all possible section functions. Indeed, there exist germs that admit sectional asymptotic expansions in $\widehat{\mathfrak{L}}$ with respect to some section function, but which do not admit sectional asymptotic expansion in $\widehat{\mathfrak{L}}$ with respect to some other section function. Take, for example, the germ

$$f(x) = x \cdot \Big(\frac{1}{1-\boldsymbol{\ell}} + e^{-\frac{1}{\boldsymbol{\ell}^2}}\Big).$$

Take any section function $\mathbf{s}$ such that $\mathbf{s}\big(\sum_{k=0}^{\infty} y^k\big) = \frac{1}{1-y}$. Obviously, $f(x) - x \cdot \frac{1}{1-\boldsymbol{\ell}}$ does not admit asymptotic expansion in $\widehat{\mathfrak{L}}$. On the contrary, take any section function $\mathbf{s}$ such that $\mathbf{s}\big(\sum_{k=0}^{\infty} y^k\big) = \frac{1}{1-y} + e^{-\frac{1}{y^2}}$. The sectional asymptotic expansion of f with respect to the section $\mathbf{s}$ is then equal to $\widehat{f}(x) = x \sum_{k=0}^{\infty} \boldsymbol{\ell}^k$.

Definition 3.3 (Sectional asymptotic expansions)**.** *Let $\mathbf{s} : \widehat{\mathcal{T}}^{\mathbf{s}} \to \mathcal{T}^{\mathbf{s}}$ be a section function defined as in Definition 3.2. Let $\widehat{f} \in \widehat{\mathcal{S}}_j^{\mathbf{s}}$, $j \in \mathbb{N}$, or $\widehat{f} \in \widehat{\mathcal{S}}_0$. We say that the germ $f = \mathbf{s}(\widehat{f}) \in \mathcal{S}_j^{\mathbf{s}}$, $j \in \mathbb{N}_0$, is the $\mathbf{s}$-sectional sum of $\widehat{f}$. Equivalently, we say that $\widehat{f}$ is the $\mathbf{s}$-sectional asymptotic expansion of f.*

Remark 3.4 (Sectional asymptotic expansions and transfinite Poincaré algorithm)**.** Let $\mathbf{s}$ be a fixed section function and let $f \in \mathcal{T}$ admit $\widehat{f} \in \widehat{\mathfrak{L}}$ as its $\mathbf{s}$-sectional asymptotic expansion. Then $\widehat{f}$ is obtained algorithmically, starting from f and following the *transfinite Poincaré algorithm* described in Remark 3.1 in the following manner:

1. In the *successor ordinal case* 2.(i), simply follow the Poincaré method.
2. In the *limit ordinal case* 2.(ii), the choice of germ g_α is uniquely dictated by the chosen section function $\mathbf{s}$.

Proposition 3.5 (Uniqueness of the $\mathbf{s}$-sectional asymptotic expansion)**.** *Every fixed section function $\mathbf{s}$ gives a bijective correspondence between $\widehat{\mathcal{T}}^{\mathbf{s}}$ and $\mathcal{T}^{\mathbf{s}}$. Consequently, the $\mathbf{s}$-sectional asymptotic expansion of a germ, if it exists, is unique.*

The proof is obvious by the construction of section function.

Definition 3.6. *Let $f \in \mathcal{T}$. The* leading term of f *is the leading term $Lt(\widehat{f})$ of its (any) sectional asymptotic expansion $\widehat{f} \in \widehat{\mathcal{T}}$ (the leading term is independent of the choice of section $\mathbf{s}$ by the Poincaré algorithm). In particular, $f \in \mathcal{T}$ is parabolic with if $Lt(\widehat{f}) = (1, 0, \ldots, 0)$. We say that a Dulac germ f is parabolic if the order of its Dulac expansion $\widehat{f} \in \widehat{\mathcal{L}}$ is $ord(\widehat{f}) = (1, 0)$.*

Definition 3.7. *Let $\widehat{f} \in \widehat{\mathcal{L}}_j^{\infty}$, $j \in \mathbb{N}_0$. We will say that $\widehat{f}$ given by* (2.8) *is a* convergent transseries *if* (2.8) *is a summable family of monomials* (*summable pointwise on some open interval* $(0,d)$*,* $d > 0$)*. Equivalently, if there exists a $d > 0$ such that*

$$\sum_{i_0=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} \left|a_{i_0 \ldots i_j}\right| x^{\alpha_{i_0}} \boldsymbol{\ell}^{\alpha_{i_0 i_1}} \cdots \boldsymbol{\ell}_j^{\alpha_{i_0 \cdots i_j}} < \infty, \quad x \in (0, d)\,. \tag{3.2}$$

In that case, by $f \in \mathcal{G}$ we denote the sum of $\widehat{f}$ on $(0, d)$ in the sense of summable families:

$$
\begin{aligned}
f(x) = &\sum_{i_0=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} a^{+}_{i_0 \ldots i_j} x^{\alpha_{i_0}} \boldsymbol{\ell}^{\alpha_{i_0 i_1}} \cdots \boldsymbol{\ell}_j^{\alpha_{i_0 \cdots i_j}} \\
&- \sum_{i_0=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_j=0}^{\infty} a^{-}_{i_0 \ldots i_j} x^{\alpha_{i_0}} \boldsymbol{\ell}^{\alpha_{i_0 i_1}} \cdots \boldsymbol{\ell}_j^{\alpha_{i_0 \cdots i_j}}, \quad x \in (0, d)
\end{aligned}
\tag{3.3}
$$

where $a^{+}_{i_0 \ldots i_j} := \max\{a_{i_0 \ldots i_j}, 0\} > 0$, $a^{-}_{i_0 \ldots i_j} := \max\{-a_{i_0 \ldots i_j}, 0\} > 0$.

For definition and properties of summable families, see e.g. [3]. Note that, due to positivity, the order of summation in (3.2) and in summands of (3.3) is not important.

Definition 3.8 (Coherent sections)**.** *We say that a section function* $\mathbf{s}$ *is* coherent *if it respects* convergence. *That is, if, for every convergent* $\widehat{f} \in \widehat{\mathcal{L}}_j^{\infty}$, $j \in \mathbb{N}$, *it holds:*

1. $\widehat{f} \in \widehat{\mathcal{S}}_j^{\mathbf{s}}$, $j \in \mathbb{N}$,
2. $\mathbf{s}(\widehat{f}) = f$, *where* f *is the sum of* $\widehat{f}$ *in the sense of Definition 3.7.*

Note that all convergent transseries from $\widehat{\mathcal{L}}_0$ trivially belong to $\widehat{\mathcal{S}}_0$. Let $f \in \mathcal{G}$ be a sum of a convergent transseries $\widehat{f} \in \widehat{\mathcal{L}}_j^{\infty}$, $j \in \mathbb{N}$, as in (3.3). Then obviously the sectional asymptotic expansion of f with respect to any coherent section $\mathbf{s}$ is equal to $\widehat{f}$ (by the Poincaré algorithm with respect to the section $\mathbf{s}$).

The notion of *sectional asymptotic expansions* is motivated by the well-known definition of the *Dulac asymptotic expansion*:

Definition 3.9 (Dulac series, see [4], [9] or [17])**.**

1. *We say that* $\widehat{f} \in \widehat{\mathcal{L}}_1$ *is a* Dulac series *if it is of the form:*

$$
\widehat{f} = \sum_{i=1}^{\infty} P_i(\boldsymbol{\ell}) x^{\alpha_i}, \tag{3.4}
$$

where $(\alpha_i)_i$, $\alpha_i > 0$, *is a strictly increasing, finitely generated sequence tending to* $+\infty$ *(or finite), and* P_i *is a sequence of polynomials.*

2. *We say that* $f \in \mathcal{G}_{AN}$ *is a* Dulac germ *if there exist a sequence of* polynomials *and a strictly increasing, finitely generated sequence* (α_i) *tending to* $+\infty$, *such that*

$$
f - \sum_{i=1}^{n} P_i(\boldsymbol{\ell}) x^{\alpha_i} = o(x^{\alpha_n}), \ n \in \mathbb{N}.
$$

If moreover $P_1 \equiv 1$, $\alpha_1 = 1$, *and at least one of the polynomials* P_i, $i > 1$, *is not zero, then* f *is called a* parabolic Dulac germ.

By the definition of sectional asymptotic expansions and the definition of coherent section functions, Dulac series are sectional asymptotic expansions of Dulac germs with respect to any coherent choice of section. They are called the *Dulac expansions* of f. Note that, unlike its asymptotic expansions in general, the Dulac expansion of a Dulac germ is *unique*, due to the additional request that P_i be polynomial. As was already mentioned in Section 1, the germs of first return maps of hyperbolic polycycles of planar analytic vector fields are Dulac germs, see e.g. [4, 9]. In particular, the quasianalyticity statement of [9] guarantees that such a first return map, if it is tangent to the identity, is a parabolic Dulac germ according to Definition 3.9.

For the purpose of Theorem B, we define here a particular family of coherent sections $\mathbf{s}$ which we will call the *integral sections*. We will prove later in Sections 6, 7 and 9 that the sectional asymptotic expansions of the inverse g^{-1}, of the Fatou coordinate Ψ, and of $A_f^c(x_0,\varepsilon)$ for f Dulac with respect to an integral section are unique and belong to $\widehat{\mathcal{L}}_2^\infty$.

Definition 3.10 provides a way of attributing a unique sum to a divergent series in $\widehat{\mathcal{L}}_0^\infty$, in the way adapted to the Fatou coordinate constructed in Section 7. Furthermore, Definition 3.13 provides a way of attributing a unique sum to a particular transseries in $\widehat{\mathcal{L}}_1^\infty$, as needed for function $A^c(x_0,\varepsilon)$ in the proof of Theorem B in Section 9.

Definition 3.10 (Integrally summable series in $\widehat{\mathcal{L}}_0^\infty$)**.**
(1) By $\widehat{\mathcal{L}}_0^I \subset \widehat{\mathcal{L}}_0^\infty$ we denote the set of all series

$$\widehat{f}(y)=\sum_{n=N}^{\infty} a_n y^n \in \widehat{\mathcal{L}}_0^\infty,\ N\in\mathbb{Z},\ a_n\in\mathbb{R},$$

satisfying the following:
1. $\widehat{f}$ is a divergent power series, in the sense that the series does not converge on any $(0,d)$, $d>0$.
2. There exists an $\alpha\in\mathbb{R}$, $\alpha\neq 0$, such that, formally in $\widehat{\mathcal{L}}^\infty$,

$$\frac{d}{dx}\Big(x^\alpha \widehat{f}(\boldsymbol{\ell})\Big)=x^{\alpha-1}R(\boldsymbol{\ell}), \tag{3.5}$$

where $R(y)$ is convergent Laurent series.
We call $\widehat{\mathcal{L}}_0^I$ the set of integrally summable series of $\widehat{\mathcal{L}}_0^\infty$*.*

(2) For $\widehat{f}\in\widehat{\mathcal{L}}_0^I$, put $g(y):=e^{-\frac{\alpha-1}{y}}R(y)\in\mathcal{G}_{AN}$. The germ $f\in\mathcal{G}_{AN}$ defined by:

$$f(y):=\begin{cases}\dfrac{\int_0^{e^{-1/y}} g(t)\,d(e^{-1/t})}{e^{-\frac{\alpha}{y}}}, & \alpha>0,\\[2ex] \dfrac{-\int_{e^{-1/y}}^{d} g(t)\,d(e^{-1/t})}{e^{-\frac{\alpha}{y}}}, & \alpha<0,\ d>0.\end{cases} \tag{3.6}$$

will be called the integral sum *of $\widehat{f}$.*

Proposition 3.11. *$\alpha\in\mathbb{R}$ in Definition 3.10 is unique. We call such α the* exponent of integration *of $\widehat{f}$.*

The proof is in the Appendix. Note that if (3.5) was true for $\alpha=0$, it would imply that $\widehat{f}(y)$ was convergent, so we can suppose $\alpha\neq0$ in the definition. Note also that by definition the convergent Laurent series are not a subset of $\widehat{\mathcal{L}}_0^I$.

Remark 3.12.
(1) Note that the integral sum as defined by (3.6) of $\widehat{f}\in\widehat{\mathcal{L}}_0^I$ with the exponent of integration $\alpha>0$ is *unique*. On the other hand, for $\widehat{f}\in\widehat{\mathcal{L}}_0^I$ with the exponent of integration $\alpha<0$ it is *unique only up to $Cx^{-\alpha}$, $C\in\mathbb{R}$*, due to the arbitrary choice of $d>0$ in (3.6).

(2) Proposition 10.3 in the Appendix shows that $\widehat{f} \in \widehat{\mathcal{L}}_0^I$ is the power asymptotic expansion of its integral sum f.

Definition 3.13 (Integrally summable series in $\widehat{\mathcal{L}}_1^\infty$)**.**
1. By $\widehat{\mathcal{L}}_1^I \subset \widehat{\mathcal{L}}_1^\infty$ we denote the set of all transseries $\widehat{F} \in \widehat{\mathcal{L}}_1^\infty$ which can be regrouped as:

$$\widehat{F}(y) = \widehat{G}_1(y)\widehat{f}\big(\boldsymbol{\ell}(e^{-\frac{\gamma}{y}}\widehat{h}(y))\big) + \widehat{G}_0(y), \tag{3.7}$$

where $\widehat{f} \in \widehat{\mathcal{L}}_0^I$, and $\widehat{G}_0,\ \widehat{G}_1,\ \widehat{h}$ and their first derivatives $\widehat{G}_0',\ \widehat{G}_1',\ \widehat{h}'$ are convergent transseries in $\widehat{\mathcal{L}}_1^\infty$ in the sense of Definition 3.7, with the sums commuting with the derivative. Moreover, $\widehat{h}$ does not contain $\boldsymbol{\ell}(y)$ in the first term, $\gamma \in \mathbb{R}$, $\gamma > 0$.

We call the class $\widehat{\mathcal{L}}_1^I \subset \widehat{\mathcal{L}}_1^\infty$ the class of integrally summable series in $\widehat{\mathcal{L}}_1^\infty$.

2. Let f be the integral sum *of $\widehat{f}$ from Definition 3.10 (defined up to a certain exponential factor), $h,\ G_0,\ G_1$ the sums of $\widehat{h},\ \widehat{G}_0,\ \widehat{G}_1$ respectively. We call*

$$F(y) = G_1(y)f\big(\boldsymbol{\ell}(e^{-\frac{\gamma}{y}}h(y))\big) + G_0(y) \tag{3.8}$$

the integral sum *of $\widehat{F}$.*

Note that $\widehat{\mathcal{L}}_0^I \subset \widehat{\mathcal{L}}_1^I$ (put $\widehat{G}_1 \equiv 1,\ \gamma = 1,\ \widehat{h} \equiv 1,\ \widehat{G}_0 \equiv 0$). Note also that the set of all convergent transseries from $\widehat{\mathcal{L}}_1^\infty$ is a subset of $\widehat{\mathcal{L}}_1^I$ (put $\widehat{G}_1 \equiv 0$).

Note that in general the decomposition (3.7) of $\widehat{F} \in \widehat{\mathcal{L}}_1^I$ is not unique, so its integral sum F is not well-defined. However, we prove in Proposition 10.5 in the Appendix that the integral sum F of $\widehat{F}$ is *unique* if $\alpha > 0$ or *unique up to an additive factor* $cG_1(y) \cdot \big(e^{-\frac{\gamma}{y}}h(y)\big)^{-\alpha},\ c \in \mathbb{R}$ if $\alpha < 0$, where $\alpha,\ \gamma,\ \widehat{h},\ \widehat{G}_1$ (α is the exponent of integration of $\widehat{f}$) are elements of an arbitrary decomposition (3.7) of $\widehat{F}$.

Definition 3.14 (The integral sections)**.** *Every coherent section $\mathbf{s} : \widehat{\mathcal{T}}^{\mathbf{s}} \to \mathcal{T}^{\mathbf{s}}$ whose restriction to $\widehat{\mathcal{L}}_1^I \subset \widehat{\mathcal{L}}_1^\infty$ is given by $\mathbf{s}\big|_{\widehat{\mathcal{L}}_1^I}(\widehat{F}) = F$, where F is one integral sum of $\widehat{F}$ given in* (3.8)*, is called an* integral section*.*

The definition implies that the restrictions of integral sections $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_1^{\mathbf{s}}} : \widehat{\mathcal{S}}_1^{\mathbf{s}} \to \mathcal{T}$ are *coherent*. Indeed, the set of all convergent transseries in $\widehat{\mathcal{L}}_1^\infty$ is trivially a subset of $\widehat{\mathcal{L}}_1^I$ (just put $\widehat{G}_1 \equiv 0$). By (3.8), the integral section gives exactly their sum.

In particular, the restriction $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_0}$ is coherent. Note that the integral sections in Definition 3.14 are well-defined, since by Proposition 10.4 in the Appendix $\widehat{\mathcal{L}}_1^I \subset \widehat{\mathcal{S}}_1^{\mathbf{s}} \subset \widehat{\mathcal{T}}^{\mathbf{s}}$.

Remark 3.15. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be two different integral sectons from Definition 3.14. Let $\widehat{F} \in \widehat{\mathcal{L}}_1^I$. Then $F_1 := \mathbf{s_1}(\widehat{F})$ and $F_2 := \mathbf{s_2}(\widehat{F})$ differ by:

$$F_1(y) - F_2(y) = \begin{cases} cG_1\big(e^{-\frac{\gamma}{y}}h(y)\big)^{\alpha},\ c \in \mathbb{R}, & \alpha < 0, \\ 0, & \alpha > 0. \end{cases}$$

Here, $\alpha,\ \gamma,\ \widehat{h},\ \widehat{G}_1$ (α is the exponent of integration of $\widehat{f}$) are elements of an arbitrary decomposition (3.7) of $\widehat{F}$.

Let $\mathbf{s}$ be an integral section, as in Definition 3.14. Obviously, if a germ $f \in \mathcal{G}$ admits a sectional asymptotic expansion in $\widehat{\mathcal{L}}_2^\infty$ with respect to the integral section $\mathbf{s}$, the expansion is *unique*.

Let $f \in \mathcal{G}_{AN}$ be a Dulac germ, as in Definition 3.9. Note that the (unique) sectional asymptotic expansion of f with respect to any fixed integral section $\mathbf{s}$ corresponds to its Dulac expansion, since the integral sections are coherent. We will show in Section 6 that the (unique) sectional asymptotic expansion of the inverse $g^{-1} \in \mathcal{G}_{AN}$, $g = \mathrm{id} - f$, with respect to any fixed integral section $\mathbf{s}$ is equal to the formal inverse $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_2$. Similarly, we will show in Section 7 that the (unique) sectional asymptotic expansion of the Fatou coordinate for a Dulac germ, $\Psi \in \mathcal{G}_{AN}$, with respect to any fixed integral section $\mathbf{s}$ is, *up to a constant*, equal to the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$. Finally, in Section 9 we prove Theorem B: we show that the (unique) sectional asymptotic expansion of the continuous length of ε-neighborhoods of orbits for f Dulac, $A_f^c(x_0, \varepsilon) \in \mathcal{G}_{AN}$, with respect to any fixed integral section $\mathbf{s}$ is, *up to a term $C\varepsilon$, $C \in \mathbb{R}$,* equal to the formal length $\widehat{A}^c_{\widehat{f}}(\varepsilon) \in \widehat{\mathcal{L}}_2$.

Moreover, we will show that the sectional asymptotic expansions of Ψ resp. $A_f^c(x_0, \varepsilon)$ with respect to different integral sections lead to different choices of constants $C \in \mathbb{R}$ resp. $C\varepsilon$, $C \in \mathbb{R}$. This is in fact neglectible, since the Fatou coordinate is defined only up to an additive constant term. As a consequence, $A_c^f(x_0, \varepsilon)$ is defined only up to $C\varepsilon$, $C \in \mathbb{R}$, due to the term $\varepsilon \cdot \Psi \circ g^{-1}(2\varepsilon)$ in its definition. Therefore, any choice of the integral section for the sectional asymptotic expansions is *equally good*.

4. Embedding in a one-parameter flow

In this subsection (Propositions 4.3 and 4.4) we discuss a close relation between the existence of a (formal) Fatou coordinate for a germ f (resp. $\widehat{f}$) and of an embedding of f (resp. $\widehat{f}$) in a (formal) one-parameter flow. The results of this section are used in Section 2. For the explicit relation, consult also the constructive proofs of propositions in the Appendix.

Definition 4.1 (One-parameter flow, standard definition)**.** *A family* $\{f^t\}_{t\in\mathbb{R}}$ *of germs defined on some interval* $(0, d)$ *is called a* one-parameter flow *if:*

(1) $f^0 = \mathrm{id}$,
(2) $f^{t+s} = f^t \circ f^s$, $t, s \in \mathbb{R}$.

We say that the one-parameter flow $\{f^t\}$ *is a* C^1-flow *if the mapping* $t \mapsto f^t(x)$ *is of class* $C^1(\mathbb{R})$*, for every* $x \in (0, d)$.

Let f be a germ defined on some interval $(0, d)$*. We say that f* embeds as the time-one map in a flow $\{f^t\}$ *if* $f^1 = f$.

Definition 4.2 (Formal one-parameter flow, see [14])**.** *We say that a one-parameter family* $\{\widehat{f}^t\}_{t\in\mathbb{R}}$, $\widehat{f}^t \in \widehat{\mathcal{L}}$, *is a* C^1-formal flow *if:*

(1) $\widehat{f}^0 = \mathrm{id}$, $\widehat{f}^{t+s} = \widehat{f}^t \circ \widehat{f}^s$, $s, t \in \mathbb{R}$,
(2) $S := \cup_{t\in\mathbb{R}} \mathcal{S}(\widehat{f}^t)$ *is a well-ordered subset of* $\mathbb{R}_{>0} \times \mathbb{Z}$,
(3) $t \mapsto [\widehat{f}^t]_{(\alpha,m)}$ *is of the class* $C^1(\mathbb{R})$*, for every* $(\alpha, m) \in S$.

Here, $[\widehat{f}^t]_{(\alpha,m)}$ *denotes the coefficient (which is a function of* t*) of the monomial* $x^\alpha \boldsymbol{\ell}^m$ *in* $\widehat{f}^t$*. Again, we say that* $\widehat{f} \in \widehat{\mathcal{L}}$ *embeds in the flow* $\{\widehat{f}^t\}_{t\in\mathbb{R}}$ *as the time-one map if* $\widehat{f} = \widehat{f}^1$*.*

In [14], we have proved that any *parabolic* transseries $\widehat{f} \in \widehat{\mathcal{L}}$ can be embedded as the time-one map in the unique C^1-flow in $\widehat{\mathcal{L}}$. The next proposition states that, accordingly, the formal Fatou coordinate exists and is unique.

Proposition 4.3 (Existence and uniqueness of the formal Fatou coordinate)**.** *Let* $\widehat{f} \in \widehat{\mathcal{L}}$ *parabolic. The formal Fatou coordinate* $\widehat{\Psi}$ *exists and is unique in* $\widehat{\mathfrak{L}}$*, up to an additive constant. Moreover,* $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$*. Let* $\{\widehat{f}^t\}$ *be the (unique)* C^1*-flow in* $\widehat{\mathcal{L}}$ *in which* $\widehat{f}$ *embeds as the time-one map. Then:*

$$\widehat{\Psi}(\widehat{f}^t(x)) - \widehat{\Psi}(x) = t,\ \ t \in \mathbb{R}. \tag{4.1}$$

For preciser description of the formal Fatou coordinate of a parabolic $\widehat{f} \in \widehat{\mathcal{L}}$, see Remark 10.1 in the Appendix. If $\widehat{f}$ is moreover parabolic Dulac, the description is provided in Proposition 7.4.

The next proposition establishes, for an analytic germ, the equivalence between the existence of an analytic Fatou coordinate and the embedding of the germ in a C^1-flow of analytic germs:

Proposition 4.4. *Let* f *be an analytic germ on* $(0,d)$*. The following are equivalent:*

1. there exists an analytic Fatou coordinate germ Ψ *for* f *on* $(0,d)$*,*

2. there exists a C^1*-flow* $\{f^t\}$ *of analytic germs on* $(0,d)$ *in which* f *can be embedded as the time one-map, and such that the germ* ξ *defined by*

$$\xi := \frac{d}{dt} f^t \Big|_{t=0}$$

is a non-oscillatory[1] *germ on some open interval* $(0,d)$*.*

In this case, for the Fatou coordinate Ψ *and for the corresponding flow* $\{f^t\}$*, it holds that:*

$$\Psi(f^t(x)) - \Psi(x) = t,\ \ x \in (0,d),\ \ t \in \mathbb{R}. \tag{4.2}$$

The proofs of Propositions 4.3 and 4.4, are given in the Appendix. Also, see Remark 10.6 in the Appendix for the explanation of the importance of the *non-oscillatority assumption* in Proposition 4.4.

In the particular case of a *parabolic Dulac germ* f, the existence, uniqueness and description of the Fatou coordinate for f, as well as of the formal Fatou coordinate for its Dulac expansion $\widehat{f}$, are given in Section 7 by mean of an explicit construction (see Proposition 7.1 in Section 7).

5. Examples

Example 2 (Examples of sectional asymptotic expansions in $\widehat{\mathfrak{L}}$)**.** Let $\mathbf{s}$ be a coherent section function (see Definition 3.8).

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[1] non-oscillatority means that there is no accumulation of zero points at 0

(1) The sectional asymptotic expansion of a Dulac germ $f \in \mathcal{G}_{AN}$ with respect to $\mathbf{s}$ is unique and equal to its Dulac expansion:

$$\widehat{f} = \sum_{i=1}^{\infty} x^{\alpha_i} P_i(\boldsymbol{\ell}) \in \widehat{\mathcal{L}},$$

where $(\alpha_i)_i$, $\alpha_i > 0$ is a strictly increasing, finitely generated sequence tending to $+\infty$ (or finite), and P_i is a sequence of polynomials. It is not really transfinite.

(2) It is not very difficult to check that the time-one map of the vector field $X = \xi(x)\frac{d}{dx}$, $\xi(x) = x^2\boldsymbol{\ell}(x)^{-1}$, is a Dulac germ $f \in \mathcal{G}_{AN}$. Its Dulac expansion $\widehat{f} \in \widehat{\mathcal{L}}$ is given by the formal exponential $\widehat{f}(x) = e^{x^2\boldsymbol{\ell}^{-1}\frac{d}{dx}}$.

The Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ of f can be computed, up to a constant, by

$$\Psi(x) = \int_x^d \frac{1}{\xi(x)}\,dx = \int_x^d x^{-2}\boldsymbol{\ell}(x)\,dx,\ \ d > 0.$$

On the other hand, the formal Fatou coordinate of its Dulac expansion $\widehat{f}$ can be computed as the formal integral $\widehat{\Psi}(x) = \int \frac{dx}{\xi(x)} = \int x^{-2}\boldsymbol{\ell}\,dx$, see the proof of Proposition 4.3. By formal integration by parts,

$$\widehat{\Psi}(x) = -x^{-1}\sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n.$$

The above series $\widehat{T}(\boldsymbol{\ell}) := \sum_{n=1}^{\infty} n!\boldsymbol{\ell}^n$ is a divergent series in $\widehat{\mathcal{L}}_0$. There are many ways to find a germ $T \in \mathcal{G}_{AN}$ which admits $\widehat{T}$ as its power asymptotic expansion. We choose one particular sum. That is, we choose any section function $\mathbf{s}$ as defined in Definition 3.2, such that

$$\mathbf{s}(\widehat{T}) = \frac{\int_d^x x^{-2}\boldsymbol{\ell}(x)\,dx}{-x^{-1}} \in \mathcal{G}_{AN}.$$

We put $T = \mathbf{s}(\widehat{T})$ as the unique sum with respect to this section. Then $\widehat{\Psi}$ is the unique (up to a constant) sectional asymptotic expansion of $\Psi \in \mathcal{G}_{AN}$ with respect to $\mathbf{s}$, in the sense described in Definition 3.3.

(3) Let $\mathbf{s}$ be a coherent section as defined in Definition 3.8. The following examples of germs in $\mathcal{G}_{AN}$ admit trivially the sectional asymptotic expansions with respect to $\mathbf{s}$ in $\widehat{\mathfrak{L}}$:

$$f(x) := F(x, \boldsymbol{\ell}, \boldsymbol{\ell}_2),\ \ g(x) := g(x, \frac{x}{\boldsymbol{\ell}}),$$

where F is a germ of three variables analytic at $(0,0,0)$ and G a germ of two variables analytic at $(0,0)$.

Let us show the statement for f (for g analogously). For y, z small enough, the germ $x \mapsto F(x, y, z)$ is analytic at 0. Therefore, for every $n \in \mathbb{N}$, we have:

$$F(x, y, z) = \sum_{k=0}^{n} g_k(y, z)x^k + R_n(x, y, z),$$

where $x \mapsto R_n(x, y, z)$ is analytic, for *fixed* small y, z. Take the *Lagrange form of the remainder* $R_n(x, y, z)$, for y, z fixed:

$$R_n(x, y, z) = \int_0^x \partial_t^{n+1} F(t, y, z)\frac{(x-t)^n}{n!}dt$$

Since F is analytic in y, z, we get that R_n is analytic in the three variables, and, moreover, we get the following *uniform* bound in y, z:

$$|R_n(x,y,z)| \leq C\frac{x^{n+1}}{(n+1)!},$$

on some small ball around $(0,0,0)$. Since R_n is analytic in y, z, $n \in \mathbb{N}$, we conclude that $g_k(y,z)$ are also analytic in y, z, for all $k = 0,\ldots,n$, and we can repeat the same procedure for their expansions at the next level. Finally, f admits the unique sectional asymptotic expansion with respect to $\mathbf{s}$ equal to $\widehat{F}(x,\boldsymbol{\ell},\boldsymbol{\ell}_2) \in \widehat{\mathcal{L}}_2$, where $\widehat{F}$ is the Taylor expansion of F at $(0,0,0)$.

As a simple example of this type of germ, take e.g.

$$f(x) = \frac{1}{1-\frac{x}{1-\frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}_2}}}.$$

Its sectional asymptotic expansion in $\widehat{\mathcal{L}}_2$ with respect to $\mathbf{s}$ is given by the transseries:

$$\widehat{f} = \sum_{k=0}^{\infty} x^k\Big(1-\frac{\boldsymbol{\ell}}{1-\boldsymbol{\ell}_2}\Big)^{-k} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\binom{-k}{l}\binom{-l}{r}(-1)^{l+r}x^k\boldsymbol{\ell}^l\boldsymbol{\ell}_2^r.$$

Consider a formal Dulac series $\widehat{f} \in \widehat{\mathcal{L}}$. It is proven in Proposition 6.1 in Section 6 that, if the leading term of $\widehat{g} = \mathrm{id} - \widehat{f}$ involves a logarithm, then its formal inverse contains a double logarithm. This will imply that it is also the case for the formal continuous time length $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ defined in Definition 2.6. The following example shows another possible cause of the presence of a double logarithm in $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ – the existence of a *nonzero residual invariant* of $\widehat{f}$.

Example 3 (The '*residual*' double logarithm in the formal continuous time length of a Dulac series)**.** According to the results of [14], every parabolic element $\widehat{f} \in \widehat{\mathcal{L}}$ admits a normal form which is the exponential of a simple vector field. It is convenient to work with an example which is already given in this normal form. Consider the vector field $X = \xi(x)\frac{\mathrm{d}}{\mathrm{d}x}$, where $\xi(x) = -\frac{x^2}{1-x+bx\boldsymbol{\ell}}$ with $b \in \mathbb{R}\setminus\{0\}$. Its formal time-one map $\widehat{f} \in \widehat{\mathcal{L}}$ is given by the formula:

$$\begin{aligned}\widehat{f}(x) &= \mathrm{Exp}(X)\cdot \mathrm{id} = x + \xi(x) + \frac{1}{2}\xi'(x)\xi(x) + \text{h.o.t.}\\ &= x - x^2 + bx^3\boldsymbol{\ell} + \text{h.o.t.}\end{aligned}$$

The coefficient $b \in \mathbb{R}$ is the *residual invariant* of $\widehat{f}$. In order to compute the formal continuous time length $A^c_{\widehat{f}}(\varepsilon)$ given by Definition 2.6, we need to compute the formal Fatou coordinate $\widehat{\Psi}(x)$ of $\widehat{f}(x)$, and the formal inverse of $\widehat{g}(x) = x - \widehat{f}(x)$. By the proof of Proposition 4.3, we have:

$$\begin{aligned}\widehat{\Psi}'(x) &= \frac{1}{\xi(x)} = -\frac{1}{x^2} + \frac{1}{x} + \frac{b}{x\log x}, \text{ hence}\\ \widehat{\Psi}(x) &= \frac{1}{x} + \log x + b\,\log(-\log x).\end{aligned}$$

On the other hand:

$$\begin{aligned}\widehat{g}\,(x) &= x - \widehat{f}\,(x) = x^2 - bx^3\boldsymbol{\ell} + \text{h.o.t.}\\ &= x^2 \circ \widehat{\varphi}\,(x)\,,\end{aligned}$$

for some parabolic element $\widehat{\varphi}\,(x) \in \widehat{\mathcal{L}}$ of the form $\widehat{\varphi}(x) = x - \frac{b}{2}x^2\boldsymbol{\ell} + h.o.t.$ Hence:

$$\widehat{g}^{-1}\,(x) = \widehat{\varphi}^{-1}\left(x^{1/2}\right) = x^{1/2} + bx\boldsymbol{\ell} + \text{h.o.t.}$$

belongs to $\widehat{\mathcal{L}}$. In particular, by Proposition 6.1 (2), it does not contain any double logarithm. Let us now compute the following component of $\widehat{A}^c_{\widehat{f}}(\varepsilon)$:

$$\begin{aligned}\widehat{\Psi}\left(\widehat{g}^{-1}\,(2\varepsilon)\right) &= \frac{1}{\sqrt{2}\varepsilon^{1/2}(1+o(1))} + \log\left(\sqrt{2}\varepsilon^{1/2}(1+o(1))\right) + \\ &\qquad + b\log\left(-\log\left(\sqrt{2}\varepsilon^{1/2}(1+o(1))\right)\right) = \\ &= \varepsilon^{-1/2} + \frac{1}{2}\log\varepsilon + b\log\left(-\log\varepsilon\right) + \text{h.o.t.}\end{aligned}$$

Here, $o(1) \in \widehat{\mathcal{L}}$ is of order higher that 0 in ε, and *h.o.t.* belongs to $\widehat{\mathcal{L}}$ (no double logarithms). Hence we observe the presence of a *unique* iterated logarithm in $\widehat{A}^c_{\widehat{f}}(\varepsilon)$, caused by the presence of a *residual term* $bx^3\boldsymbol{\ell}$ (that is, $b \neq 0$) in the formal normal form of $\widehat{f}$.

Example 4 (The formal continuous time length is in general a transfinite series)**.** By [14], a parabolic transseries $\widehat{f} \in \widehat{\mathcal{L}}$ can be reduced to its normal form $\widehat{f}_0$ by an action of a parabolic change of variables $\widehat{\varphi} \in \widehat{\mathcal{L}}$, whose support in general has the order type strictly bigger than ω. If $\widehat{\Psi}_0$ is the (unique, up to a constant) formal Fatou coordinate of $\widehat{f}_0$ in $\widehat{\mathfrak{L}}$, then $\widehat{\Psi} = \widehat{\Psi}_0 \circ \widehat{\varphi}$ is the formal Fatou coordinate of $\widehat{f}$. We deduce that the formal Fatou coordinate of $\widehat{f}$ and, consequently, its continuous time length $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ are transseries whose support has the order type strictly bigger than ω.

Consider for example the Dulac series $\widehat{f}(x) = x - x^2\boldsymbol{\ell}^{-1} + x^2$. We have shown in Example 1 in [14] that the change of variables reducing $\widehat{f}$ to its formal normal form $f_0(x) = x - x^2\boldsymbol{\ell}^{-1} + \rho x^3\boldsymbol{\ell}$ is a transseries $\widehat{\varphi} \in \widehat{\mathcal{L}}$ whose support is indexed by a *transfinite* ordinal (*strictly* bigger than ω). Hence it applies also to $\widehat{\Psi}$ as described here and then to the formal continuous area $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ by its definition.

Example 5 (*The regular case:* Theorem B for germs analytic at the origin)**.** Let $f(x) = x - x^{k+1} + a_2x^{k+2} + \ldots$, $k \in \mathbb{N}$, $a_i \in \mathbb{R}$, be a germ of a parabolic diffeomorphism on $\mathbb{R}_+$ (prenormalized for simplicity). Let $\mathcal{O}^f(x_0)$ be its orbit with initial point $x_0 > 0$ close to the origin. Put $g := \text{id} - f$.

The germ f can be considered as the restriction on the positive real axis of a complex parabolic germ $f(z)$ with real coefficients. It is known that there exist two sectorial Fatou coordinates $\Psi_\pm(z)$, with the common asymptotic expansion:

$$\widehat{\Psi}(z) = \left(\frac{1}{kz^k} + \rho\log z\right) \circ \widehat{\varphi}, \quad \rho \in \mathbb{R},\ \widehat{\varphi} \in z + z^2\mathbb{R}[[z]]. \tag{5.1}$$

The series $\widehat{\Psi}$ has real coefficients. Note that on the real line there exists an analytic Fatou coordinate $\Psi:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$, with the asymptotic expansion $\widehat{\Psi}(x)$, as $x\to 0$, which is the restriction to $\mathbb{R}_{>0}$ of the complex sectorial Ψ_+.

Recall the definitions of the standard and of the continuous time length of the ε-neighborhood of $\mathcal{O}^f(x_0)$ (with respect to the flow corresponding to the Fatou coordinate Ψ):

$$A_f(x_0,\varepsilon)=n_\varepsilon\cdot 2\varepsilon+f^{n_\varepsilon}(x_0)+2\varepsilon=\Big(\Psi\big(f^{n_\varepsilon}(x_0)\big)-\Psi(x_0)\Big)\cdot 2\varepsilon+f^{n_\varepsilon}(x_0)+2\varepsilon,$$
$$A_f^c(x_0,\varepsilon)=\tau_\varepsilon\cdot 2\varepsilon+g^{-1}(2\varepsilon)+2\varepsilon=\Big(\Psi\big(g^{-1}(2\varepsilon)\big)-\Psi(x_0)\Big)\cdot 2\varepsilon+g^{-1}(2\varepsilon)+2\varepsilon.$$

Recall that $f^{\tau_\varepsilon}(x_0)=g^{-1}(2\varepsilon)$. The asymptotic expansion of $g^{-1}(2\varepsilon)$, as $\varepsilon\to 0$, is given by (since $g(x)=x^{k+1}\circ\varphi_1(x),\ \varphi_1\in x+x^2\mathbb{R}\{x\}$):

$$\widehat{g}^{-1}(2\varepsilon)=\widehat{\varphi}_1^{-1}\big((2\varepsilon)^{\frac{1}{k+1}}\big),\ \widehat{\varphi}_1^{-1}\in x+x^2\mathbb{C}[[x]]. \tag{5.2}$$

As in the proof of Theorem B, point (3), we have $f^{n_\varepsilon}(x_0)=h_0(\varepsilon)+h(\varepsilon)$, where $h_0(\varepsilon)$ is the sum of (finitely many) terms of $g^{-1}(2\varepsilon)$, up to the order $O(\varepsilon)$, and $h(\varepsilon)=O(\varepsilon)$. We will show now that $h(\varepsilon)=O(\varepsilon)$ in $f^{n_\varepsilon}(x_0)$ does not have an asymptotic behavior in $\widehat{\mathfrak{L}}$ (power-iterated logarithm scale), as $\varepsilon\to 0$. Suppose the contrary, and consider the sequence $\varepsilon_m=\frac{f^m(x_0)-f^{m+1}(x_0)}{2}\to 0$, as $m\to\infty$. We will denote by $n_{\varepsilon_m^+}$ and $h(\varepsilon_m^+)$ (resp. $n_{\varepsilon_m^-}$ and $h(\varepsilon_m^-)$) the limits of the critical index n_ε and $h(\varepsilon)$ when $\varepsilon\to\varepsilon_m$ on the right (resp. on the left). We get:

$$\begin{aligned}f^{n_{\varepsilon_m^+}}(x_0)-f^{n_{\varepsilon_m^-}}(x_0)&=\Psi^{-1}(n_{\varepsilon_m^+}+\Psi(x_0))-\Psi^{-1}(n_{\varepsilon_m^-}+\Psi(x_0))\\&\sim(\Psi^{-1})'(n_{\varepsilon_m})(n_{\varepsilon_m^+}-n_{\varepsilon_m^-})\\&\sim n_{\varepsilon_m}^{-\frac{1}{k}-1}(n_{\varepsilon_m^+}-n_{\varepsilon_m^-})\sim\varepsilon_m\cdot(n_{\varepsilon_m^+}-n_{\varepsilon_m^-}),\ m\to\infty.\end{aligned}\tag{5.3}$$

The last row is deduced since $n_\varepsilon\sim\Psi(g^{-1}(2\varepsilon))\sim\varepsilon^{-\frac{k}{k+1}}$, see also [15].

On the other hand, if $h(\varepsilon)=O(\varepsilon)$ has asymptotic behavior in $\widehat{\mathfrak{L}}$, then $h(\varepsilon)=C\varepsilon+o(\varepsilon),\ \varepsilon\to 0,\ C\in\mathbb{R}$ (possibly also 0). Then,

$$f^{n_{\varepsilon_m^+}}(x_0)-f^{n_{\varepsilon_m^-}}(x_0)=h(\varepsilon_m^+)-h(\varepsilon_m^-)=o(\varepsilon_m),\ m\to\infty. \tag{5.4}$$

Indeed, the first part $h_0(\varepsilon)+C\varepsilon$ is finite sum of monomials, thus continuous at ε_m, and cancels.

From (5.3) and (5.4) it follows that $n_{\varepsilon_m^+}-n_{\varepsilon_m^-}\to 0$ as $m\to\infty$. But $\varepsilon\mapsto n_\varepsilon$ is a step function at ε_m, and therefore $n_{\varepsilon_m^+}-n_{\varepsilon_m^-}=1$, for every $m\in\mathbb{N}$. This leads to a contradiction. The asymptotic expansion in $\widehat{\mathfrak{L}}$ of $A_f(x_0,\varepsilon)$ from the term $O(\varepsilon)$ onwards is therefore non-existent.

On the other hand, using (5.1) and (5.2) and Definition 2.6 of $\widehat{A}^c_{\widehat{f}}(\varepsilon)$, we compute the formal length $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ for the formal Taylor expansion $\widehat{f}\in\mathbb{R}[[x]]$ of f:

$$\begin{aligned}\widehat{A}^c_{\widehat{f}}(\varepsilon)&=b_1\varepsilon^{\frac{1}{k+1}}+b_2\varepsilon^{\frac{2}{k+1}}+\cdots+b_k\varepsilon^{\frac{k}{k+1}}+b_{k+1}\varepsilon\log\varepsilon+\big(b_{k+2}+\Psi(x_0)\big)\varepsilon+\\&\quad+b_{k+3}\varepsilon^{1+\frac{1}{k+1}}+b_{k+4}\varepsilon^{1+\frac{2}{k+1}}+\cdots\in\widehat{\mathcal{L}}_1.\end{aligned}\tag{5.5}$$

Here, $b_i\in\mathbb{R}$ are real numbers depending only on the coefficients of $\widehat{f}$ and not on the initial condition. Note that in this analytic case the formal series (5.5) is not really transfinite, but just a formal series indexed by ω (since $\widehat{\Psi}$ and $\widehat{g}^{-1}$ are just series).

By Theorem B *2.*, (5.5) is in fact the asymptotic expansion in $\widehat{\mathfrak{L}}$ of $A_f^c(x_0,\varepsilon)$, up to a term εK, $K\in\mathbb{R}$.

Finally, the expansion (5.5) is by Theorem B *3.* the continuation of the non-existent asymptotic expansion in $\widehat{\mathfrak{L}}$ of the standard length $A_f(x_0,\varepsilon)$, after $O(\varepsilon)$:

$$A_f(x_0,\varepsilon)=b_1\varepsilon^{\frac{1}{k+1}}+b_2\varepsilon^{\frac{2}{k+1}}+\cdots+b_k\varepsilon^{\frac{k}{k+1}}+b_{k+1}\varepsilon\log\varepsilon+\big(b_{k+2}+\Psi(x_0)\big)\varepsilon+R(\varepsilon,x_0).$$

Here, the remainder $R(\varepsilon,x_0)=O(\varepsilon)$, $\varepsilon\to 0$, cannot be further expanded in $\widehat{\mathfrak{L}}$. It depends on the initial point x_0 of the orbit. The similar expansion of $A_f(x_0,\varepsilon)$ was obtained before in [15], but for diffeomorphisms in $\mathbb{C}$. In that case, the area is computed instead of the length and the exponents are bigger by 1.

Note that in the analytic case it is sufficient to work with ordinary series indexed by ω and with standard asymptotic expansions. That is, the introduction of transseries and of sectional (trans)asymptotic expansions from Section 3 is not needed to treat the analytic case.

6. The formal inverse of a transseries

We recall that $\widehat{\mathcal{L}}$ is the class of transseries in $\widehat{\mathcal{L}}_1$ which involve only integer powers of the variable $\boldsymbol{\ell}$. Let $\widehat{g}\in\widehat{\mathcal{L}}$,

$$\widehat{g}=ax^{\alpha}\boldsymbol{\ell}^m+\text{h.o.t.},\ a\in\mathbb{R},\ \alpha>0,\ m\in\mathbb{Z}. \tag{6.1}$$

Let us define the set $\widetilde{\mathcal{R}_{\widehat{g}}}$ as the sub-semigroup of $\mathbb{R}_{\geq 0}\times\mathbb{Z}\times\mathbb{Z}$, resp. of $\mathbb{R}_{\geq 0}\times\mathbb{Z}$, generated by:

(1) $(\beta-\alpha,\ell-m,0)$ for $(\beta,\ell)\in\mathcal{S}(\widehat{g})\setminus\{(\alpha,m)\}$, $(0,1,-1)$ and $(0,0,1)$, if $m\neq 0$,
(2) $(\beta-\alpha,\ell)$ for $(\beta,\ell)\in\mathcal{S}(\widehat{g})\setminus\{(\alpha,0)\}$ and $(0,1)$, if $m=0$.

Let $\mathcal{R}_{\widehat{g}}=\begin{cases}\widetilde{\mathcal{R}_{\widehat{g}}}+(1,0,0), & m\neq 0,\\ \widetilde{\mathcal{R}_{\widehat{g}}}+(1,0), & m=0.\end{cases}$

Proposition 6.1 (Inverse of a transseries from $\widehat{\mathcal{L}}$)**.** *Let $\widehat{g}\in\widehat{\mathcal{L}}$ as in* (6.1)*. Then its formal inverse $\widehat{g}^{-1}$ belongs to $\widehat{\mathcal{L}}_2$. If $\widehat{g}$ moreover contains no logarithm in the leading term ($m=0$), then its formal inverse $\widehat{g}^{-1}$ belongs to $\widehat{\mathcal{L}}_1$. More precisely,*

(1) $m\neq 0$

$$\widehat{g}^{-1}(x)=(a\alpha^m)^{-\frac{1}{\alpha}}(x\boldsymbol{\ell}^{-m})^{\frac{1}{\alpha}}\cdot\Big(1+\widehat{R}(x)\Big),$$

where $\widehat{R}\in\widehat{\mathcal{L}}_2$ with ord$(\widehat{R})\succ(0,0,0)$ and its support $\mathcal{S}(\widehat{R})$ is made of monomials of the type

$$(x\boldsymbol{\ell}^{-m})^{\frac{\gamma}{\alpha}}\boldsymbol{\ell}^r\boldsymbol{\ell}_2^s,\quad(\gamma,r,s)\in\widetilde{\mathcal{R}_{\widehat{g}}}.$$

(2) $m=0$

$$\widehat{g}^{-1}(x)=a^{-\frac{1}{\alpha}}x^{\frac{1}{\alpha}}\cdot\Big(1+\widehat{R}(x)\Big),$$

where $\widehat{R}\in\widehat{\mathcal{L}}$ with ord$(\widehat{R})\succ(0,0)$ and $\mathcal{S}(\widehat{R})$ is made of monomials of the type

$$x^{\frac{\gamma}{\alpha}}\boldsymbol{\ell}^r,\ (\gamma,r)\in\widetilde{\mathcal{R}_{\widehat{g}}}.$$

Here, $\widetilde{\mathcal{R}_{\widehat{g}}}$ is as above defined.

Note that a similar theorem about the formal inverse in a more general setting of transseries was proved in [1].

Outline of the proof. Write

$$\widehat{g} = g_{\alpha,m} \circ \widehat{\varphi}, \tag{6.2}$$

where $g_{\alpha,m}(x) = ax^{\alpha}\boldsymbol{\ell}^m$ and $\widehat{\varphi}(x) = x + \text{h.o.t.}$ If $\widehat{g}(x) = ax + \text{h.o.t.}$ is parabolic or hyperbolic, we simply have $g_{1,0} = a \cdot id$. Therefore,

$$\widehat{g}^{-1} = \widehat{\varphi}^{-1} \circ \widehat{g}_{\alpha,m}^{-1}. \tag{6.3}$$

Thus, in order to compute the formal inverse of the initial transseries $\widehat{g}$, one needs to compute the formal inverse of the monomial $g_{\alpha,m}$ and the formal inverse of the parabolic transseries $\widehat{\varphi}$. In the following three auxiliary lemmas, we control the support in each step. The proof of Proposition 6.1 is finally given at the end of the section.

Lemma 6.2. *Let $g_{\alpha,m}(x) = ax^{\alpha}\boldsymbol{\ell}^m$, $a \in \mathbb{R},\ \alpha > 0,\ m \in \mathbb{Z}$. Then:*

$$g_{\alpha,m}^{-1}(x) = (a\alpha^m)^{-\frac{1}{\alpha}} \cdot x^{\frac{1}{\alpha}}\, \boldsymbol{\ell}^{-\frac{m}{\alpha}} \Big(1 + F\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big),$$
$$\widehat{g}_{\alpha,m}^{-1}(x) = (a\alpha^m)^{-\frac{1}{\alpha}} \cdot x^{\frac{1}{\alpha}}\, \boldsymbol{\ell}^{-\frac{m}{\alpha}} \Big(1 + \widehat{F}\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big) \in \widehat{\mathcal{L}}_2.$$

Here, $F(t,s)$ is a germ of two variables analytic at $(0,0)$, $F(0,0) = 0$, with Taylor expansion $\widehat{F}$.

In particular, if $m = 0$, then

$$g_{\alpha,0}^{-1}(x) = \widehat{g}_{\alpha,0}^{-1}(x) = a^{-\frac{1}{\alpha}} x^{\frac{1}{\alpha}}.$$

Proof. First, we estimate the leading term of $g_{\alpha,m}^{-1}$. Put

$$y = ax^{\alpha}\boldsymbol{\ell}(x)^m. \tag{6.4}$$

Applying the logarithm function to both sides of this equality leads to:

$$\log y = \log a + \alpha \log x + m \log \boldsymbol{\ell}(x).$$

It follows that $\log y \sim \alpha \log x$ when $y \to 0$, and, consequently, that $\boldsymbol{\ell}(x) \sim \alpha\boldsymbol{\ell}(y)$. From (6.4),

$$x(y) \sim (a\alpha^m)^{-\frac{1}{\alpha}} y^{\frac{1}{\alpha}} \boldsymbol{\ell}(y)^{-\frac{m}{\alpha}},\ y \to 0.$$

Therefore,

$$g_{\alpha,m}^{-1}(x) = (a\alpha^m)^{-\frac{1}{\alpha}} x^{\frac{1}{\alpha}} \boldsymbol{\ell}^{-\frac{m}{\alpha}} \big(1 + h(x)\big),\ h(x) = o(1),\ x \to 0. \tag{6.5}$$

Putting (6.5) in the equation $g_{\alpha,m}(g_{\alpha,m}^{-1}(x)) = x$, after some simplification, we get

$$\big(1 + h(x)\big)^{\alpha} \cdot \Big(\frac{1}{1 + \log(a\alpha^m) \cdot \boldsymbol{\ell} - m\frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2} - \alpha\boldsymbol{\ell}\log(1 + h(x))}\Big)^m = 1.$$

By the *analytic implicit function theorem*,

$$h(x) = F_1\Big(\boldsymbol{\ell}, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\Big) = F\Big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\Big),\ F_1,\ F \text{ analytic germs at } (0,0).$$

Since $h(x) = o(1)$ and $\boldsymbol{\ell}_2,\ \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2} \to 0$, as $x \to 0$, we conclude that $F(0,0) = 0$. $\square$

Lemma 6.3. *Let $\widehat{\varphi}$ be as defined in (6.2). Then $\widehat{\varphi}$ is parabolic and $\widehat{\varphi} \in \widehat{\mathcal{L}}_2$. In particular, if $m = 0$, then $\widehat{\varphi} \in \widehat{\mathcal{L}}$. Moreover,*

$$\mathcal{S}(\widehat{\varphi} - \mathrm{id}) \subseteq \mathcal{R}_{\widehat{g}}.$$

Proof. The case $m \neq 0$. By (6.2),

$$\widehat{\varphi} = \widehat{g}_{\alpha,m}^{-1} \circ \widehat{g}.$$

Let us write $\widehat{g}(x) = ax^\alpha \boldsymbol{\ell}^m \big(1 + \widehat{T}(x)\big)$. Then $\widehat{T} \in \widehat{\mathcal{L}}$, $\mathrm{ord}(\widehat{T}) \succ (0,0,0)$ and

$$\mathcal{S}(\widehat{T}) = \Big\{ \big(\beta - \alpha, \ell - m\big) \big|\ (\beta, \ell) \in \mathcal{S}(g) \Big\}.$$

We now compute

$$\begin{aligned} \widehat{\varphi}(x) &= \widehat{g}_{\alpha,m}^{-1}\big(\widehat{g}(x)\big) = \widehat{g}_{\alpha,m}^{-1}\Big(ax^\alpha \boldsymbol{\ell}^m \big(1 + \widehat{T}(x)\big)\Big) = \\ &= x \cdot \Big(1 + F_2(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}, \widehat{T})\Big), \end{aligned} \tag{6.6}$$

where F_2 is an analytic germ of three variables vanishing at 0.

In the computation we use the expansions:

$$\begin{aligned} \boldsymbol{\ell}(\widehat{g}(x)) &= \frac{1}{\alpha} \boldsymbol{\ell} \cdot \Big(1 + F_3\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}, \widehat{T}\big)\Big), \\ \boldsymbol{\ell}_2(\widehat{g}(x)) &= \boldsymbol{\ell}_2 \Big(1 + F_4\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}, \widehat{T}\big)\Big), \end{aligned}$$

where F_3, F_4 are analytic germs of three variables vanishing at 0.

Obviously, $\mathcal{S}(\widehat{\varphi} - \mathrm{id}) \subseteq \mathcal{R}_{\widehat{g}}$, with $\mathcal{R}_{\widehat{g}}$ defined at the beginning of the section. By *Neumann's Lemma*, the set $\mathcal{S}(\widehat{\varphi})$ is well-ordered. Therefore, $\widehat{\varphi} \in \widehat{\mathcal{L}}_2$.

The case $m = 0$. As the proof is similar and simpler, we omit it. $\square$

Lemma 6.4.

1. Let $\widehat{\varphi} \in \widehat{\mathcal{L}}_2$ be parabolic. The formal inverse $\widehat{\varphi}^{-1}$ is parabolic and belongs to $\widehat{\mathcal{L}}_2$. Moreover, let $R \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}$ be the semigroup generated by $(\beta - 1, p, q)$ for $(\beta, p, q) \in \mathcal{S}(\widehat{\varphi} - \mathrm{id})$, $(0,1,0)$ and $(0,1,1)$. Then $\mathcal{S}(\widehat{\varphi}^{-1} - \mathrm{id}) - (1,0,0) \subseteq R$.

2. Let $\widehat{\varphi} \in \widehat{\mathcal{L}}$ be parabolic. The formal inverse $\widehat{\varphi}^{-1}$ is parabolic and belongs to $\widehat{\mathcal{L}}$. Moreover, let $R \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}$ be the semigroup generated by $(\beta - 1, p)$ for $(\beta, p) \in \mathcal{S}(\widehat{\varphi} - \mathrm{id})$ and $(0,1)$. Then $\mathcal{S}(\widehat{\varphi}^{-1} - \mathrm{id}) - (1,0) \subseteq R$.

Proof. We prove the statement 1. for an element of $\widehat{\mathcal{L}}_2$. The statement 2. for an element of $\widehat{\mathcal{L}}$ is handled in the same way.

Let $\widehat{\varphi} \in \widehat{\mathcal{L}}_2$ be parabolic. Put $\widehat{h} = \widehat{\varphi} - \mathrm{id}$. Then $\mathrm{ord}(\widehat{h}) \succ (1,0,0)$. Let us define the *Schröder's operator* $\Phi_{\widehat{\varphi}} : \widehat{\mathcal{L}}_2 \to \widehat{\mathcal{L}}_2$,

$$\Phi_{\widehat{\varphi}} \cdot \widehat{f} = \widehat{f} \circ \widehat{\varphi},\ \widehat{f} \in \mathcal{L}_2. \tag{6.7}$$

Put $\Phi_{\widehat{\varphi}} = Id + H_{\widehat{\varphi}}$, $H_{\widehat{\varphi}} : \widehat{\mathcal{L}}_2 \to \widehat{\mathcal{L}}_2$. By formal Taylor expansion (see [1], $\widehat{\varphi}$ parabolic):

$$H_{\widehat{\varphi}} \cdot \widehat{f} = \Phi_{\widehat{\varphi}} \cdot \widehat{f} - \widehat{f} = \widehat{f} \circ \widehat{\varphi} - \widehat{f} = \widehat{f}'\widehat{h} + \frac{1}{2!}\widehat{f}''\widehat{h}^2 + \cdots,\ \widehat{f} \in \widehat{\mathcal{L}}_2. \tag{6.8}$$

Since $\mathrm{ord}(\widehat{h}) \succ (1,0,0)$, the operator $H_{\widehat{\varphi}}$ is a *small operator*, see [1] or [14, Section 5.1] for definition and properties of small operators. Therefore, the inverse operator

$\Phi_{\widehat{\varphi}}^{-1} : \widehat{\mathcal{L}}_2 \to \widehat{\mathcal{L}}_2$ is well-defined by the series (formally convergent in the product topology with respect to the discrete topology, see [14]) :

$$\Phi_{\widehat{\varphi}}^{-1} = (\mathrm{Id} + H_{\widehat{\varphi}})^{-1} := \sum_{k=0}^{\infty} (-1)^k H_{\widehat{\varphi}}^k \tag{6.9}$$

Consequently, $\widehat{\varphi}^{-1} = \Phi_{\widehat{\varphi}}^{-1} \cdot \mathrm{id} \in \widehat{\mathcal{L}}_2$.

We analyze now the support of the formal inverse $\widehat{\varphi}^{-1} = \Phi_{\widehat{\varphi}}^{-1} \cdot \mathrm{id}$ more precisely. Differentiating a monomial from a $\widehat{f} \in \widehat{\mathcal{L}}_2$, we get:

$$(x^\gamma \boldsymbol{\ell}^m \boldsymbol{\ell}_2^n)' = \gamma x^{\gamma-1} \boldsymbol{\ell}^m \boldsymbol{\ell}_2^n + m x^{\gamma-1} \boldsymbol{\ell}^{m+1} \boldsymbol{\ell}_2^n + n x^{\gamma-1} \boldsymbol{\ell}^{m+1} \boldsymbol{\ell}_2^{n+1},\ \gamma \geq 1,\ m,\ n \in \mathbb{Z}.$$

It is then easy to deduce from (6.8) that:

$$\mathcal{S}(H_{\widehat{\varphi}} \cdot \widehat{f}) \subseteq R + \mathcal{S}(\widehat{f}),\ f \in \widehat{\mathcal{L}}_2, \tag{6.10}$$

where $R \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}$ is a sub-semigroup generated by $(\beta - 1, p, q)$ for $(\beta, p, q) \in \mathcal{S}(\widehat{h})$ and $(0, 1, 0)$ and $(0, 1, 1)$. Iterating (6.10) we get that $\mathcal{S}(H_{\widehat{\varphi}}^n \cdot \widehat{f}) \subseteq R + \mathcal{S}(\widehat{f})$, $n \in \mathbb{N}$. Consequently, by (6.9),

$$\mathcal{S}\Big(\Phi_{\widehat{\varphi}}^{-1} \cdot \widehat{f} - \widehat{f}\Big) \subseteq R + \mathcal{S}(\widehat{f}),\ \widehat{f} \in \widehat{\mathcal{L}}_2.$$

Since $\widehat{\varphi}^{-1} = \Phi_{\widehat{\varphi}}^{-1} \cdot \mathrm{id}$, we get that $\mathcal{S}(\widehat{\varphi}^{-1} - \mathrm{id}) - (1, 0, 0) \subseteq R$. □

Corollary 6.5. *Let $\widehat{\varphi} \in \widehat{\mathcal{L}}_2$ (resp. $\widehat{\mathcal{L}}$, if $m = 0$) be as in (6.2). Let $\mathcal{R}_{\widehat{g}}$ be as defined at the beginning of the section. Then $\widehat{\varphi}^{-1} \in \widehat{\mathcal{L}}_2$ (resp. $\widehat{\mathcal{L}}$, if $m = 0$) and*

$$\mathcal{S}(\widehat{\varphi}^{-1} - \mathrm{id}) \subseteq \mathcal{R}_{\widehat{g}}.$$

Proof. By (6.2), $\widehat{\varphi}$ is parabolic. By Lemma 6.3, $\mathcal{S}(\widehat{\varphi} - \mathrm{id}) \subseteq \mathcal{R}_{\widehat{g}}$. By Lemma 6.4, $\mathcal{S}(\widehat{\varphi}^{-1} - \mathrm{id}) - (1, 0, 0) \subseteq \widetilde{\mathcal{R}_{\widehat{g}}}$. □

Proof of Proposition 6.1.

The case $m \neq 0$. By (6.3), we have that

$$\widehat{g}^{-1} = \widehat{\varphi}^{-1} \circ \widehat{g}_{\alpha,m}^{-1}.$$

By Lemma 6.2 and Corollary 6.5, $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_2$. Let us analyse the support. We have, using Lemma 6.2:

$$\begin{aligned} \widehat{g}^{-1}(x) &= \widehat{g}_{\alpha,m}^{-1}(x) + \big(\widehat{\varphi}^{-1} - \mathrm{id}\big)\big(\widehat{g}_{\alpha,m}^{-1}(x)\big) = \\ &= (a\alpha^m)^{-\frac{1}{\alpha}} \big(x \boldsymbol{\ell}^{-m}\big)^{\frac{1}{\alpha}} \big(1 + \widehat{F}(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2})\big) + \\ &\quad + \big(\widehat{\varphi}^{-1} - \mathrm{id}\big)\Big((a\alpha^m)^{-\frac{1}{\alpha}} \big(x \boldsymbol{\ell}^{-m}\big)^{\frac{1}{\alpha}} \big(1 + \widehat{F}(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2})\big)\Big). \end{aligned} \tag{6.11}$$

Let $x^\gamma \boldsymbol{\ell}^r \boldsymbol{\ell}_2^s \in \mathcal{S}(\widehat{\varphi}^{-1} - \mathrm{id})$. By Corollary 6.5, $(\gamma, r, s) \in \mathcal{R}_{\widehat{g}}$. We compute:

$$(x^\gamma \boldsymbol{\ell}^r \boldsymbol{\ell}_2^s) \circ g_{\alpha,m}^{-1}(x) = (a\alpha^m)^{-\frac{\gamma}{\alpha}} \alpha^r \cdot \big(x \boldsymbol{\ell}^{-m}\big)^{\frac{\gamma}{\alpha}} \cdot \boldsymbol{\ell}^r \boldsymbol{\ell}_2^s \cdot \Big(1 + \widehat{F}_5\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big), \tag{6.12}$$

where $\widehat{F}_5$ is the Taylor expansion of an analytic germ F_5 vanishing at the origin. In the computation we use the following:

$$\boldsymbol{\ell}(\widehat{g}_{\alpha,m}^{-1}) = \alpha \cdot \boldsymbol{\ell} \cdot \Big(1 + \widehat{F}_6\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big),$$

$$\boldsymbol{\ell}_2(\widehat{g}_{\alpha,m}^{-1}) = \boldsymbol{\ell}_2 \cdot \Big(1 + \widehat{F}_7\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big), \tag{6.13}$$

where $\widehat{F}_6,\ \widehat{F}_7$ are Taylor expansions of analytic germs $F_6,\ F_7$ vanishing at the origin. Combining (6.11) and (6.12), we get the statement in the case $m \neq 0$.

The case $m = 0$. Lemmas 6.2 - 6.4 are simpler in this case. The proof is a simple exercise. □

6.1. **The inverse of a Dulac series and of a Dulac germ.** Let $g \in \mathcal{G}_{AN}$ be a Dulac germ and $\widehat{g} \in \widehat{\mathcal{L}}$ its Dulac series. Put

$$\widehat{g}(x) = x^{\alpha} P_m(\boldsymbol{\ell}^{-1}) + o(x^{\alpha}),$$

where P_m is a polynomial of degree $m \in \mathbb{N}_0$. Let $\widetilde{\mathcal{A}} \subset \mathbb{R}_{\geq 0}$ be a sub-semigroup generated by $\{\beta - \alpha :\ \beta \in \mathcal{S}_x(\widehat{g})\}$. Here, $\mathcal{S}_x$ denotes the support of $\widehat{g}$ with respect only to powers of x. Put

$$\mathcal{A} = \{\gamma + 1 :\ \gamma \in \widetilde{\mathcal{A}}\} \subset \mathbb{R}_{>0}. \tag{6.14}$$

Note that $\widetilde{\mathcal{A}}$ and thus $\mathcal{A}$ are countable, of order type ω (in fact, finitely generated).

We compute in Proposition 6.6 the formal inverse of a *Dulac series* $\widehat{g} \in \widehat{\mathcal{L}}$. We refine the statement of Proposition 6.1 in this special case, using the particular, polynomial form of the coefficient functions in Dulac series.

We show further in Proposition 6.7 that the formal inverse $\widehat{g}^{-1}$ is the sectional asymptotic expansion of the inverse of the Dulac germ g with respect to an integral section in the sense of Definition 3.2.

The results will be used in the proof of Theorem B.

Proposition 6.6 (A refinement of Proposition 6.1 for Dulac series)**.** *Let $\widehat{g} \in \widehat{\mathcal{L}}$ be a Dulac series.*

1. $m \neq 0$. Then the formal inverse $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_2$. More precisely,

$$\widehat{g}^{-1}(x) = \sum_{i=1}^{\infty} \widehat{f}_{\beta_i}(\boldsymbol{\ell}) x^{\frac{\beta_i}{\alpha}},\ \beta_i \in \mathcal{A},\ \textit{with}\ \text{convergent (by Definition 3.7)}\ \textit{coefficients}$$

$$\widehat{f}_{\beta_i}(y) = y^{-M_{\beta_i} + \frac{m\beta_i}{\alpha}} \widehat{G}_{\beta_i}\Big(\boldsymbol{\ell}(y), \frac{y}{\boldsymbol{\ell}(y)}\Big) \in \widehat{\mathcal{L}}_1^{\infty},\ M_{\beta_i} \in \mathbb{N}_0. \tag{6.15}$$

Here, $\widehat{G}_{\beta_i}$ are Taylor expansions of analytic germs G_{β_i} of two variables. In particular, the leading term of $\widehat{g}^{-1}(x)$ does not contain $\boldsymbol{\ell}_2$ (the double logarithm).

2. $m = 0$. Then $\widehat{g}^{-1} \in \widehat{\mathcal{L}}$ is a Dulac series*. In particular, the coefficients $\widehat{f}_{\beta_i}(y) \in \widehat{\mathcal{L}}_0^{\infty}$ are* polynomial *in y^{-1}.*

Proof. Define $g_\alpha(x) := x^{\alpha} P_m(\boldsymbol{\ell}^{-1})$ as the *leading block* of g (unlike $g_{\alpha,m}(x)$ in Proposition 6.1 which represented only the leading term). Put

$$\widehat{g} = g_\alpha \circ \widehat{\varphi}. \tag{6.16}$$

1. The case $m \neq 0$. Exactly as in the proof of Lemma 6.2, we get the following:

$$g_\alpha^{-1}(x) = (a\alpha^{-m})^{-\frac{1}{\alpha}} \cdot x^{\frac{1}{\alpha}} \boldsymbol{\ell}^{\frac{m}{\alpha}} \Big(1 + F\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big),$$

$$\widehat{g}_\alpha^{-1}(x) = (a\alpha^{-m})^{-\frac{1}{\alpha}} \cdot x^{\frac{1}{\alpha}} \boldsymbol{\ell}^{\frac{m}{\alpha}} \Big(1 + \widehat{F}\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big) \in \widehat{\mathcal{L}}_2. \tag{6.17}$$

Here, $F(t,s)$ is a germ of two variables analytic at $(0,0)$, $F(0,0)=0$, with Taylor expansion $\widehat{F}$. Since $\widehat{g}_\alpha^{-1}$ is of the 'same' form as was $\widehat{g}_{\alpha,m}^{-1}$ from Lemma 6.2, by the proof of Lemma 6.3 we immediately get that $\widehat{\varphi} := \widehat{g}_\alpha^{-1} \circ \widehat{g}$ belongs to $\widehat{\mathcal{L}}_2$ and is parabolic. Moreover, that:

$$\widehat{\varphi}(x) = x \cdot \Big(1 + \widehat{F}_1(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}, \widehat{T})\Big). \tag{6.18}$$

Here, $\widehat{F}_1$ is the Taylor expansion of an analytic germ of three variables vanishing at 0. The improvement of Lemma 6.3 in the Dulac case is the special form of $\widehat{T}$ in (6.18). Since $\widehat{g}(x) = ax^\alpha \boldsymbol{\ell}^{-m} \cdot (1 + \widehat{T}(x))$, $\widehat{T}(x) = o(1)$, is a Dulac series, we get:

$$\widehat{T}(x) = \boldsymbol{\ell}^m P_0(\boldsymbol{\ell}^{-1}) + \sum_{i=1}^{\infty} x^{\alpha_i - \alpha} \boldsymbol{\ell}^m P_i(\boldsymbol{\ell}^{-1}). \tag{6.19}$$

Here, P_0 is a polynomial of degree strictly smaller than m, $(P_i)_{i\geq 1}$ is a sequence of polynomials, and $\alpha_i > \alpha$, $i \in \mathbb{N}$.

We now show that $\widehat{\varphi} := \widehat{g}_\alpha^{-1} \circ \widehat{g}$ is *strictly* parabolic: that the order of $\widehat{\varphi} - \mathrm{id}$ in x is *strictly* bigger than 1. Indeed, suppose the contrary, that:

$$\widehat{\varphi}(x) = x + cx\boldsymbol{\ell}^k + \text{h.o.t.},$$

where $c \neq 0$ and $k \in \mathbb{N}$. Computing the formal composition $\widehat{g}_\alpha(x + cx\boldsymbol{\ell}^k + \text{h.o.t.})$, we obtain

$$\begin{aligned}
\widehat{g}_\alpha(x + cx\boldsymbol{\ell}^k + \text{h.o.t.}) &= (x + cx\boldsymbol{\ell}^k + \text{h.o.t.})^\alpha P_m\Big(-\log\big(x + cx\boldsymbol{\ell}^k + \text{h.o.t.}\big)\Big) \\
&= x^\alpha (1 + c\boldsymbol{\ell}^k + o(\boldsymbol{\ell}^k))^\alpha \cdot P_m\Big(\boldsymbol{\ell}^{-1} + c\boldsymbol{\ell}^k + o(\boldsymbol{\ell}^k)\Big) = \\
&= x^\alpha (1 + c\boldsymbol{\ell}^k + o(\boldsymbol{\ell}^k))^\alpha \Big(P_m(\boldsymbol{\ell}^{-1}) + P_m'(\boldsymbol{\ell}^{-1})(c\boldsymbol{\ell}^k + \text{h.o.t.}) + \ldots\Big) = \\
&= x^\alpha P_m(\boldsymbol{\ell}^{-1}) + acx^\alpha \boldsymbol{\ell}^{-m+k} + \text{h.o.t.}
\end{aligned}$$

Since by (6.16) this composition should equal $\widehat{g}(x)$, which is the sum of $x^\alpha P_m(\boldsymbol{\ell}^{-1})$ and the terms of strictly higher order than x^α in x, it necessarily follows that $c = 0$.

We now invert formally $\widehat{\varphi} \in \widehat{\mathcal{L}}_2$, which is *strictly* parabolic. By Lemma 6.4 and Corollary 6.5, $\widehat{\varphi}^{-1} \in \widehat{\mathcal{L}}_2$, and the support in x is inherited from more general Corollary 6.5. To estimate the precise form of coefficients in front of any power of x in $\widehat{\varphi}^{-1}$ in the Dulac case, we use the Neumann inverse series formula (6.9), as in the proof of Lemma 6.4:

$$\begin{aligned}
\widehat{\varphi}^{-1} =& (\mathrm{Id} + H_{\widehat{\varphi}})^{-1} \cdot \mathrm{id} := \sum_{k=0}^{\infty} (-1)^k H_{\widehat{\varphi}}^k \cdot \mathrm{id} = \\
=& \mathrm{id} - \widehat{h} + \Big(\widehat{h} \circ \widehat{\varphi} - \widehat{h}\Big) + \Big(\big(\widehat{h} \circ \widehat{\varphi} - \widehat{h}\big) \circ \widehat{\varphi} - \big(\widehat{h} \circ \widehat{\varphi} - \widehat{h}\big)\Big) + \ldots
\end{aligned} \tag{6.20}$$

Here, $H_{\widehat{\varphi}} \cdot \widehat{f} = \widehat{f} \circ \widehat{\varphi} - \widehat{f}$, $\widehat{f} \in \widehat{\mathcal{L}}_2$, and $\widehat{h} := \widehat{\varphi} - \mathrm{id}$. Due to the *strict* parabolicity of $\widehat{\varphi}$, the above series converges in the *formal topology.*

It can be seen from (6.18) and (6.19) that the coefficients in front of any power of x in $\widehat{\varphi}(x)$, that is, in $\widehat{h}(x)$, are finite sums of terms of the form:

$$P(\boldsymbol{\ell}^{-1})\widehat{G}(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}), \tag{6.21}$$

where P is a polynomial and $\widehat{G}$ a Taylor expansion of an analytic germ G of two variables. Consequently, by (6.20), computing formal compositions with $\widehat{\varphi}$ strictly parabolic, we conclude that the *coefficient* of a fixed power x^β in $\widehat{\varphi}^{-1} - \mathrm{id}$ is of the same form:

$$\sum_{i=1}^{n_\beta} P_i(\boldsymbol{\ell}^{-1}) \cdot \widehat{F}_i(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}),\ \beta \geq 1,\ n_\beta \in \mathbb{N},$$

Here, P_i are polynomials and $\widehat{F}_i$ are the Taylor expansions of analytic germs F_i of two variables. Putting $M_\beta := \max_{j=1\ldots n_\beta} \deg(P_j)$, the coefficient can be written in a reduced form:

$$\boldsymbol{\ell}^{-M_\beta} \cdot \widehat{G}_\beta(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}),\ M_\beta \in \mathbb{N}_0. \tag{6.22}$$

Here, $\widehat{G}_\beta$ is the Taylor expansion of an analytic germ G_β of two variables.

Finally,

$$\widehat{g}^{-1} = \widehat{\varphi}^{-1} \circ \widehat{g}_\alpha^{-1} = \widehat{g}_\alpha^{-1} + (\widehat{\varphi}^{-1} - \mathrm{id}) \circ \widehat{g}_\alpha^{-1}.$$

Note that, since $\widehat{\varphi}$ and thus $\widehat{\varphi}^{-1}$ are strictly parabolic, the leading block of $\widehat{g}^{-1}$ is exactly $\widehat{g}_\alpha^{-1}$. The support in x of $\widehat{g}^{-1} - \widehat{g}_\alpha^{-1}$ is inherited from more general Proposition 6.1. We analyse the blocks in the particular Dulac case. Using (6.17), (6.22), and (6.13), we conclude that $\widehat{g}^{-1} - \widehat{g}_\alpha^{-1}$ contains blocks the type:

$$\boldsymbol{\ell}^{-M_\beta} \widehat{H}_\beta\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big) \cdot \big(x\boldsymbol{\ell}^m\big)^{\frac{\beta}{\alpha}} = \boldsymbol{\ell}^{-M_\beta + \frac{m\beta}{\alpha}} \widehat{H}_\beta\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big) \cdot x^{\frac{\beta}{\alpha}},\ \beta \in \mathcal{A},\quad M_\beta \in \mathbb{N}_0.$$

Here, $\widehat{H}_\beta$ are Taylor expansions of analytic germs H_β of two variables. The exponents β belong to the support of x in $\widehat{\varphi}^{-1} - \mathrm{id}$, which is by Corollary 6.5 described by $\mathcal{A}$ in (6.14).

Since the first block of $\widehat{g}^{-1}$ is $\widehat{g}_\alpha^{-1}$ given in (6.17), $\widehat{g}^{-1}$ does not contain the double logarithm in the leading term.

2. The case $m = 0$. We follow the same steps, but the computation is easier. In this case, $g_\alpha(x) = ax^\alpha,\ a \in \mathbb{R}$, and $\widehat{\varphi}$ is strictly parabolic and Dulac. It follows that $\widehat{\varphi}^{-1}$ and consequently $\widehat{g}^{-1} = \widehat{\varphi}^{-1} \circ \widehat{g}_\alpha^{-1}$ are also Dulac. $\square$

Proposition 6.7. *Let $g \in \mathcal{G}_{AN}$ be a Dulac germ and $\widehat{g} \in \widehat{\mathcal{L}}$ its Dulac expansion. Then the formal inverse $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_2$ from* (6.15) *is the* (*unique*) *sectional asymptotic expansion of the inverse germ $g^{-1} \in \mathcal{G}_{AN}$ with respect to any integral section. More precisely, for every $n \in \mathbb{N}$,*

$$g^{-1}(x) - \sum_{i=1}^{n} f_{\beta_i}(\boldsymbol{\ell}) x^{\frac{\beta_i}{\alpha}} = o(x^{\frac{\beta_n}{\alpha}}),\ x \to 0,$$

$$f_{\beta_i}(y) = y^{-M_{\beta_i} + \frac{m\beta_i}{\alpha}} G_{\beta_i}\Big(\boldsymbol{\ell}(y), \frac{y}{\boldsymbol{\ell}(y)}\Big) \in \mathcal{G}_{AN},\ M_{\beta_i} \in \mathbb{N}_0, \tag{6.23}$$

where $\beta_i \in \mathcal{A}$, $i \in \mathbb{N}$, are as in (6.15) *and G_{β_i} are analytic counterparts of $\widehat{G}_{\beta_i}$ from* (6.15).

Proof. The blocks of the formal inverse $\widehat{g}^{-1}$ are by (6.15) *convergent transseries*. Since every integral section is coherent (respects convergence), it is sufficient to prove that $\widehat{g}^{-1}$ can be expanded in increasing powers in x in the form (6.23), where f_{β_i} are the *sums* of convergent $\widehat{f}_{\beta_i}$. Coarsely, this is proven by repeating the same steps of construction as described in the proof of Proposition 6.6, but this time on germs in $\mathcal{G}_{AN}$.

Let $g_\alpha(x) = x^\alpha P_m(\boldsymbol{\ell}^{-1})$ as before be the first block in the Dulac expansion. Then

$$g = g_\alpha \circ \varphi,\ g^{-1} = \varphi^{-1} \circ g_\alpha^{-1}.$$

Computing as in the formal case, we get:

$$g_\alpha^{-1}(x) = (a\alpha^{-m})^{-\frac{1}{\alpha}} \cdot x^{\frac{1}{\alpha}}\, \boldsymbol{\ell}^{\frac{m}{\alpha}} \Big(1 + F\big(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}\big)\Big), \tag{6.24}$$

$$\varphi(x) = g_\alpha^{-1}\big(g(x)\big) = x \cdot \big(1 + F_2(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}, T(x))\big). \tag{6.25}$$

Here, $F,\ F_2$ are analytic germs in two variables, with Taylor expansions $\widehat{F}$ and $\widehat{F}_2$ from $\widehat{g}_\alpha^{-1}$ resp. $\widehat{\varphi}$. The germ $T \in \mathcal{G}_{AN}$ is defined by $g(x) = ax^\alpha \boldsymbol{\ell}^{-m}(1 + T(x))$. Since g is a Dulac germ with Dulac expansion $\widehat{g}$, it follows that:

$$T(x) - \boldsymbol{\ell}^m P_0(\boldsymbol{\ell}^{-1}) - \sum_{i=1}^{n} x^{\alpha_i - \alpha} \boldsymbol{\ell}^m P_i(\boldsymbol{\ell}^{-1}) = o(x^{\alpha_n - \alpha}),\ \forall n \in \mathbb{N}, \tag{6.26}$$

with P_i as in (6.19).

Putting (6.26) in (6.25), and expanding F_2, we get immediately that:

$$\widehat{\varphi}(x) = \sum_{i=1}^{\infty} x^{\beta_i} \boldsymbol{\ell}^{m_{\beta_i}} \widehat{G}_{\beta_i}(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}),$$

$$\varphi(x) = \sum_{i=1}^{n} x^{\beta_i} \boldsymbol{\ell}^{m_{\beta_i}} G_{\beta_i}(\boldsymbol{\ell}_2, \frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2}) + o(x^{\beta_n}),\ n \in \mathbb{N}, \tag{6.27}$$

where G_{β_i} are analytic germs of two variables with Taylor expansion $\widehat{G}_{\beta_i}$, $i \in \mathbb{N}$.

In particular, as was the case for $\widehat{\varphi}$, the leading term of $\varphi(x) - x$ is of power strictly bigger than 1 in x (φ is strictly parabolic).

We now analyse the *blocks* in the expansion by increasing powers of x of φ^{-1}, using the Neumann inverse series. We prove that they are the sums of the corresponding convergent blocks (see (6.21)) of $\widehat{\varphi}^{-1}$. By coherence of integral sections, this implies that $\widehat{\varphi}^{-1}$ is the sectional asymptotic expansion of φ^{-1} with respect to any integral section.

Recall the *Schröder operator* $\widehat{\Phi}_\varphi$ from Lemma 6.4, used for obtaining the formal inverse $\widehat{\varphi}^{-1}$ of $\widehat{\varphi}$. Define similarly here the linear operator Φ_φ acting on $\mathcal{G}_{AN}$, $\Phi_\varphi \in L(\mathcal{G}_{AN})$, by:

$$\Phi_\varphi \cdot f = f \circ \varphi,\ f \in \mathcal{G}_{AN}.$$

Denote here $h = \varphi - \mathrm{id} \in \mathcal{G}_{AN}$. Furthermore, let us introduce the linear operator $H_\varphi := \Phi_\varphi - \mathrm{Id} \in L(\mathcal{G}_{AN})$,

$$H_\varphi \cdot f = f \circ \varphi - h,\ f \in \mathcal{G}_{AN}.$$

Let us consider the *Neumann series*:

$$\sum_{k=0}^{\infty}(-1)^k H_\varphi^k \cdot id. \tag{6.28}$$

Denote its partial sums by

$$S_n := \sum_{k=0}^{n}(-1)^k H_\varphi^k \cdot id \in \mathcal{G}_{AN},\ n \in \mathbb{N}.$$

We prove that the Neumann partial sums S_n *approximate* φ^{-1}, as $n \to \infty$. More precisely, we prove that, for every $\gamma > 0$, there exists a $n_\gamma \in \mathbb{N}$, such that

$$S_{n_\gamma}(x) = \varphi^{-1}(x) + O(x^\gamma),\ x \to 0. \tag{6.29}$$

In other words, we prove that:

$$S_{n_\gamma}\big(\varphi(x)\big) = x + O(x^\gamma),\ x \to 0.$$

Indeed,

$$\begin{aligned} S_{n_\gamma}\big(\varphi(x)\big) &= \big(\Phi_\varphi \cdot S_{n_\gamma}\big)(x) = \big(\mathrm{Id} + H_\varphi\big) \cdot \big(\sum_{k=0}^{n_\gamma}(-1)^k H_\varphi^k \cdot \mathrm{id}\big) = \\ &= \sum_{k=0}^{n_\gamma}(-1)^k H_\varphi^k \cdot \mathrm{id} + \sum_{k=0}^{n_\gamma}(-1)^k H_\varphi^{k+1} \cdot \mathrm{id} = x + (-1)^{n_\gamma} H_\varphi^{n_\gamma+1} \cdot \mathrm{id}. \end{aligned} \tag{6.30}$$

Since φ is *strictly* parabolic, there exists some $\delta > 0$ such that $H_\varphi \cdot \mathrm{id} = o(x^{1+\delta})$. Inductively, there exists a $n_\gamma \in \mathbb{N}$ such that $H_\varphi^{n_\gamma+1} \cdot \mathrm{id} = O(x^\gamma)$. Now (6.30) transforms to:

$$S_{n_\gamma}\big(\varphi(x)\big) = x + O(x^\gamma),$$

that is

$$S_{n_\gamma}(x) = \varphi^{-1}(x) + O\big((\varphi^{-1}(x))^\gamma\big) = \varphi^{-1}(x) + O(x^\gamma). \tag{6.31}$$

By (6.31) we have, for $\gamma \to \infty$, the following expansion of φ^{-1} in strictly increasing powers of x:

(6.32)

$$\varphi^{-1} = \mathrm{id} - h + \Big(h \circ \varphi - h\Big) + \Big(\big(h \circ \varphi - h\big) \circ \varphi - \big(h \circ \varphi - h\big)\Big) + \ldots + O(x^\gamma),$$

as compared with its formal analogue (6.20). In (6.32), the number of summands in the second line is finite (and equal to n_γ for a fixed $\gamma > 0$). We now expand the compositions in summands of φ^{-1} by the increasing powers of x, using expansion for φ given in (6.27) and the fact that φ is *strictly* parabolic. Since φ is strictly parabolic, the order of x in the consecutive brackets of φ^{-1} is strictly increasing. Thus only finitely many terms contribute to a block with a fixed power of x. The blocks in x of φ^{-1} are the sums of the corresponding convergent blocks of $\widehat{\varphi}^{-1}$.

Finally, we analyse the blocks with increasing powers of x in the composition $g^{-1} = \varphi^{-1} \circ g_\alpha^{-1}$, and show similarly using the expansion *by blocks* of φ^{-1} and (6.24) that they are the sums of the corresponding convergent blocks of $\widehat{g}^{-1} = \widehat{\varphi}^{-1} \circ \widehat{g}_\alpha^{-1}$. ∎

7. Fatou coordinate of a Dulac germ and of a Dulac series

In this section, we prove *simultaneously* the three points of the following statement:

Proposition 7.1. *Let $f \in \mathcal{G}_{AN}$ be a parabolic Dulac germ and let $\widehat{f} \in \widehat{\mathcal{L}}$ be its Dulac expansion.*

1. There exists a unique (up to an additive constant) formal Fatou coordinate $\widehat{\Psi}$ for $\widehat{f}$ in $\widehat{\mathfrak{L}}$. It belongs to $\widehat{\mathcal{L}}_2^\infty$.

2. There exists a unique (up to an additive constant) Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ for the germ f which admits a sectional asymptotic expansion in the class $\widehat{\mathfrak{L}}$ (in the sense of Definition 3.2).

3. Let $\mathbf{s}$ be a fixed integral section and $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ the formal Fatou coordinate (with a fixed choice of the additive constant). Then there exists a choice of the additive constant in $\Psi \in \mathcal{G}_{AN}$ from 2. such that the formal Fatou coordinate $\widehat{\Psi}$ is the (unique) sectional asymptotic expansion of $\Psi \in \mathcal{G}_{AN}$ with respect to $\mathbf{s}$. The Fatou coordinate Ψ is of the form:

$$\Psi = \Psi_\infty + R,$$

where $\Psi_\infty \to \infty$ and $R = o(1)$, as $x \to 0$.

4. $\widehat{\Psi} = \widehat{\Psi}_\infty + \widehat{R}$, where $\widehat{\Psi}_\infty \in \widehat{\mathcal{L}}_2^\infty$ is the sectional asymptotic expansion of Ψ_∞ with respect to $\mathbf{s}$ and $\widehat{R} \in \widehat{\mathcal{L}}$ is the sectional asymptotic expansion of R with respect to $\mathbf{s}$.

Note that different choices of integral sections $\mathbf{s}$ in *3.* lead to change in Ψ only by an additive constant $C \in \mathbb{R}$.

Note that $\Psi_\infty \to \infty$, $x \to 0$, is the *infinite* part, and $R = o(1)$, $x \to 0$, is the *infinitesimal* part. We call Ψ_∞ the *principal part* of Ψ.

Remark 7.2. Let $\alpha_1 > 1$ be such that $\operatorname{ord}(\mathrm{id} - f) = (\alpha_1, m)$, $m \in \mathbb{N}_0^-$. The function R from Proposition 7.1 satisfies the *modified* Abel difference equation:

$$R(f(x)) - R(x) = \delta(x). \tag{7.1}$$

Here, δ is an analytic germ at open $0+$ and $\delta(x) = O(x^\gamma)$, with $\gamma > \alpha_1 - 1$.

Remark 7.3 (*Non-uniqueness* of the Fatou coordinate in $\mathcal{G}_{AN}$, without requesting the existence of the expansion in $\widehat{\mathfrak{L}}$ in Proposition 7.1)**.**

Note that any strictly monotone $\Psi \in \mathcal{G}_{AN}$ whose inverse Ψ^{-1}, as a germ at infinity, satisfies $\Psi^{-1}(w+1) = f(\Psi^{-1}(w))$, is a Fatou coordinate for f. This gives us freedom of choice of Ψ^{-1} on the fundamental domain $[0,1)$ and the rule for its extension at the neighborhood of ∞, thus, non-unicity of a Fatou coordinate for f.

In particular, let $\Psi_1 \in \mathcal{G}_{AN}$ be the Fatou coordinate constructed in Proposition 7.1 admitting a sectional expansion $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$. Let $\Psi_2 \in \mathcal{G}_{AN}$ defined by $\Psi_2 := \Psi_1 + T_1 \circ \Psi_1$, where T_1 is any periodic function on $\mathbb{R}$ of period 1 whose derivative T_1' is bounded in $(-1,1)$ (e.g. $T_1(x) = \frac{1}{4\pi}\sin(2\pi x)$, $x \in \mathbb{R}$). It can be easily checked that Ψ_2 is also a Fatou coordinate for f (by Definition 2.2). It does not admit the expansion in $\widehat{\mathfrak{L}}$, due to periodicity of T_1.

The proof of Proposition 7.1 is constructive, so in the course of the proof we also prove the precise form of $\widehat{\Psi}$ stated in Proposition 7.4. In addition, we prove that there is only one monomial in $\widehat{\Psi}$ containing the double logarithm in Remark 10.1 in the Appendix.

Proposition 7.4 (Formal Fatou coordinate for a parabolic Dulac germ)**.** *Let $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^{\infty}$ be the (unique) formal Fatou coordinate for a parabolic Dulac germ $f \in \mathcal{G}_{AN}$. Then there exists $\rho \in \mathbb{R}$ such that $\widehat{\Psi} - \rho \boldsymbol{\ell}_2^{-1} \in \widehat{\mathcal{L}}^{\infty}$, and*

$$\widehat{\Psi} - \rho \boldsymbol{\ell}_2^{-1} = \sum_{i=1}^{\infty} x^{\alpha_i} \widehat{f}_i(\boldsymbol{\ell}). \tag{7.2}$$

Here,

1. *$\alpha_1 < 0$,*
2. *α_i is a strictly increasing real sequence tending to $+\infty$ (finitely generated),*
3. *$\widehat{f}_i$ is a formal Laurent series such that, formally in $\widehat{\mathcal{L}}^{\infty}$,*

$$\widehat{f}_i(\boldsymbol{\ell}) = \frac{\int x^{\alpha_i - 1} \widehat{R}_i(\boldsymbol{\ell})\, dx}{x^{\alpha_i}}, \tag{7.3}$$

where $\widehat{R}_i$ is a convergent series *(more precisely, a power asymptotic expansion of a rational function).*

Note that, in general, $\widehat{f}_i(y)$ is a *divergent power series* for every positive value $y > 0$. Nevertheless, it is what we call *integrally summable* in Definition 3.14. We define its *integral sum* $f_i \in \mathcal{G}_{AN}$ as:

$$f_i(y) = \begin{cases} \dfrac{\int_0^{e^{-1/y}} e^{-\frac{\alpha_i - 1}{y}} R_i(y)\, d(e^{-1/y})}{e^{-\frac{\alpha_i}{y}}}, & \alpha_i > 0, \\ -\dfrac{\int_{e^{-1/y}}^{d} e^{-\frac{\alpha_i - 1}{y}} R_i(y)\, d(e^{-1/y})}{e^{-\frac{\alpha_i}{y}}}, & \alpha_i < 0,\ d > 0. \end{cases}$$

Here, $R_i(y)$ is the sum of convergent $\widehat{R}_i(y)$. In the special case $\alpha_i = 0$, $\widehat{f}_i(y)$ is obviously convergent Laurent and we take simply its sum $f_i(y)$. Indeed, by (7.3) $\widehat{f}_i'(y) = \widehat{R}_i(y) y^{-2}$ and $\widehat{R}_i(y)$ is convergent Laurent. It is proven in Proposition 10.3 in the Appendix that $\widehat{f}_i$ is the power asymptotic expansion of its integral sum f_i. These integral sums will be the building blocks in the parallel construction of the Fatou coordinate germ $\Psi \in \mathcal{G}_{AN}$ from Proposition 7.1.

Before proving Propositions 7.1 and 7.4, we give a motivating example illustrating the construction of the Fatou coordinate.

Example 6. Take, for example, a Dulac germ $f \in \mathcal{G}_{AN}$ with the expansion:

$$\widehat{f}(x) = x - x^2 \boldsymbol{\ell}^{-1} + o(x^3).$$

The algorithm which will be described in this section is a block-by-block construction the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ satisfying formally the Fatou equation in $\widehat{\mathfrak{L}}$:

$$\widehat{\Psi}\big(x - x^2 \boldsymbol{\ell}^{-1} + o(x^3)\big) - \widehat{\Psi}(x) = 1.$$

Let $\widehat{\Psi}_1$ be the first block of $\widehat{\Psi}$, that is, $\widehat{\Psi} = \widehat{\Psi}_1 + h.o.b.$ Here, *h.o.b.* stands for *blocks of strictly higher order* (*in* x). Applying the formal Taylor expansion, the lowest-order block on the left-hand side is the $-\widehat{\Psi}_1'(x) \cdot x^2 \boldsymbol{\ell}^{-1}$, so it should equal 1 on the right-hand side. Therefore, $\widehat{\Psi}_1$ is given as the formal integral:

$$\widehat{\Psi}_1(x) = \int \frac{dx}{x^2 \log x}.$$

To continue, put $\widehat{\Psi}=\widehat{\Psi}_1+\widehat{R}$ in the Fatou equation and repeat the procedure for the following blocks.

We try the parallel construction of the Fatou coordinate $\Psi\in\mathcal{G}_{AN}$. By formal integration by parts, we see that

$$\widehat{\Psi}_1(x)=x^{-1}\sum_{n=1}^{\infty}n!\boldsymbol{\ell}^n. \tag{7.4}$$

For every $\boldsymbol{\ell}\in(0,d)$, $\widehat{f}(\boldsymbol{\ell}):=\sum_{n=1}^{\infty}n!\boldsymbol{\ell}^n$ is obviously a divergent series (which is Borel summable). However, following our construction, this series is *uniquely real summable* in the natural way. Indeed, let

$$\Psi_1(x):=-\int_x^d\frac{dt}{t^2\log t}$$

be the analytic analogon of $\widehat{\Psi}_1$ from (7.4). Then $\Psi_1\in\mathcal{G}_{AN}$. The germs $x\mapsto x\Psi(x)$ and consequently $f(\boldsymbol{\ell}):=\boldsymbol{\ell}\mapsto e^{-1/\boldsymbol{\ell}}\Psi(e^{-1/\boldsymbol{\ell}})$ also belong to $\mathcal{G}_{AN}$. By Proposition 10.3, the power asymptotic expansion of $f\in\mathcal{G}_{AN}$ is equal to $\widehat{f}$. This procedure is formalized in Definition 3.14.

The proof of Propositions 7.1 and 7.4. The point (1) in Proposition 7.1 has already been proven in Proposition 4.3. We now offer a constructive proof of the existence of $\widehat{\Psi}\in\widehat{\mathcal{L}}_2^{\infty}$ from (1), that is, the proof of Proposition 7.4. As illustrated in Example 6, simultaneously with formal Fatou coordinate $\widehat{\Psi}$, we construct the Fatou coordinate $\Psi\in\mathcal{G}_{AN}$ from point (2) and prove the relation (3) between $\widehat{\Psi}$ and Ψ. We follow in large part the construction of the Fatou coordinate for parabolic diffeomorphisms as explained, for example, in [11].

We proceed in four steps:

Step 1. In Section 7.1, we construct the formal Fatou coordinate $\widehat{\Psi}$, by solving *block by block* the formal Abel equation. By *block*, we mean the (formal) sum of all the monomials of $\widehat{f}$ which share a common power of x. We get the precise form (7.2) of $\widehat{\Psi}$. Recall that the formal Fatou coordinate is unique by Proposition 4.3.

Simultaneously, we provide the "block by block" construction of the Fatou coordinate $\Psi\in\mathcal{G}_{AN}$, where the germ for each block is represented by an integral. We prove that each formal block is the asymptotic expansion of the appropriate corresponding integrally defined germ, or at least up to a constant term (Remark 3.12).

Additionally, we control the support of the formal Fatou coordinate, and prove that the powers of x it contains belong to a *finitely generated lattice*.

Step 2. Section 7.2. The control of the support obtained in the previous step allows to conclude that the principal part $\widehat{\Psi}_\infty$ of the Fatou coordinate is obtained after finitely many steps of the "block by block" algorithm. We prove also that the principal part $\widehat{\Psi}_\infty$ (with a specific choice of the constant term) is the sectional asymptotic expansion with respect to any integral section $\mathbf{s}$ in $\widehat{\mathcal{L}}_2^{\infty}$ of the principal part Ψ_∞ (with the choice of the constant terms in integrals depending on the constant term of $\widehat{\Psi}_\infty$ and on the choice of the integral section).

Step 3. In Section 7.3, we solve the modified Abel equation (7.1) for the remaining infinitesimal part of the Fatou coordinate $\widehat{R}$. The infinitesimal part R is obtained directly from the equation in the form of a *convergent* series, following the method explained in [11]. We prove in Section 7.4 that the formal infinitesimal part $\widehat{R}$ of $\widehat{\Psi}$ obtained blockwise (in countably many steps) is indeed the sectional asymptotic expansion in $\widehat{\mathcal{L}}$ with respect to any integral section $\mathbf{s}$ of the infinitesimal part R of Ψ.

Step 4. Finally, in Section 7.5, we prove the uniqueness of the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ up to an additive constant. Furthermore, we prove the uniqueness, up to an additive constant, of the Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ admitting a sectional asymptotic expansion in $\widehat{\mathfrak{L}}$. Moreover, for a fixed constant term in $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ and for a fixed integral section $\mathbf{s}$, we prove the uniqueness of the Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ admitting $\widehat{\Psi}$ as its sectional asymptotic expansion with respect to $\mathbf{s}$.

To conclude, if we change the integral section $\mathbf{s}$, the Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ admitting the same sectional asymptotic expansion $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ with respect to $\mathbf{s}$ changes only by an additive constant. □

7.1. **The Fatou coordinate and the control of the support.**

Let

$$f(x) = x - x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1}) - x^{\alpha_2} P_2(\boldsymbol{\ell}^{-1}) + o(x^{\alpha_2})$$

be a parabolic Dulac germ. Here, $1 < \alpha_1 < \alpha_2 < \ldots$ is a strictly increasing sequence tending to $+\infty$, whose elements belong to a finitely generated sub-semigroup of $\mathbb{R}^+_{>0}$, and the P_i are *polynomials.* Let $\widehat{f} \in \widehat{\mathcal{L}}$ be its Dulac expansion.

We construct the formal Fatou coordinate $\widehat{\Psi}$ satisfying the formal Abel equation

$$\widehat{\Psi}(\widehat{f}) - \widehat{\Psi} = 1 \tag{7.5}$$

block by block and we control its support. By one *block*, we mean the sum of all monomials sharing a common power of x. We call the power of x of a block the *order of the block.* In each step, we consider the construction from two sides:

1. the side of formal transseries, and
2. the side of analytic germs in $\mathcal{G}_{AN}$.

Let us now describe the induction step. Put

$$\widehat{\Psi} = \widehat{\Psi}_1 + \widehat{R}_1,$$

where $\widehat{\Psi}_1$ represents the lowest-order block of $\widehat{\Psi}$. Since we search for a solution $\widehat{\Psi}$ in $\widehat{\mathfrak{L}}$, and since $\widehat{f}$ is parabolic, we can expand the Abel equation (7.5) using Taylor expansions:

$$\widehat{\Psi}' \cdot \widehat{g} + \frac{1}{2!}\widehat{\Psi}'' \cdot \widehat{g}^2 + \cdots = 1,$$

where $\widehat{g} = \mathrm{id} - \widehat{f} = x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1}) + x^{\alpha_2} P_2(\boldsymbol{\ell}^{-1}) + \cdots$. The term $(\widehat{\Psi}_1)' \cdot x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1})$ is the block of the strictly lowest order on the left-hand side, so it should equal 1 on the right-hand side. Therefore,

$$(\widehat{\Psi}_1)' = \frac{x^{-\alpha_1}}{P_1(\boldsymbol{\ell}^{-1})} = x^{-\alpha_1} Q_1(\boldsymbol{\ell}).$$

Here, Q_1 is a rational function. We obtain the formal antiderivative $\widehat{\Psi}_1^\infty \in \widehat{\mathcal{L}}_2^\infty$ by expanding the integral *formally, using integration by parts*, see Proposition 10.3 in the Appendix.

On the other hand, we define:

$$\Psi_1(x) := -\int_x^d t^{-\alpha_1} Q_1(\boldsymbol{\ell})\, dt,\ d>0.$$

Obviouisly, $\Psi_1 \in \mathcal{G}_{AN}$. Note that $\alpha_1 > 1$ or $(\alpha_1 = 1,\ \mathrm{ord}(Q_1) \leq 1)$, so $\Psi_1(x) \to \infty$, as $x \to 0$. Note that Ψ_1 is unique only up to an additive constant (free choice of $d > 0$). In the sequel, this will be the case infinite blocks. On the other hand, infinitesimal blocks will give unique germs.

By Proposition 10.3 in the Appendix, $\widehat{\Psi}_1$ is the asymptotic expansion of $\Psi_1 \in \mathcal{G}_{AN}$ in $\widehat{\mathcal{L}}_2^\infty$, up to an additive constant.

The Abel equation for R_1 becomes:

$$R_1(f(x)) - R_1(x) = 1 - \big(\Psi_1(f(x)) - \Psi_1(x)\big) = 1 - \int_x^{f(x)} t^{-\alpha_1} Q_1(\boldsymbol{\ell})\, dt. \tag{7.6}$$

Let us denote by $\delta_1(x) := 1 - \int_x^{f(x)} t^{-\alpha_1} Q_1(\boldsymbol{\ell})\, dt$ the new right-hand side of the equation. Obviously, $\delta_1 \in \mathcal{G}_{AN}$, as a difference of analytic germs.

On the other hand, the *formal* Abel equation becomes, applying Taylor expansion:

$$\begin{aligned}
\widehat{R}_1(\widehat{f}(x)) - \widehat{R}_1(x) &= 1 - \big(\widehat{\Psi}_1(f(x)) - \widehat{\Psi}_1(x)\big),\\
&= 1 - (\widehat{\Psi}_1)'\widehat{g} - \frac{1}{2!}(\widehat{\Psi}_1)''\widehat{g}^2 + \cdots,\\
&= 1 - \frac{x^{-\alpha_1}}{P_1(\boldsymbol{\ell}^{-1})}\widehat{g} - \frac{1}{2!}\Big(\frac{x^{-\alpha_1}}{P_1(\boldsymbol{\ell}^{-1})}\Big)'\widehat{g}^2 + \cdots = \widehat{\delta}_1(x).
\end{aligned}$$

It can be checked from the above computation that the leading block in $\widehat{\delta}_1$ is of order $\min\{\alpha_1 - 1, \alpha_2 - \alpha_1\}$. Similarly, whe have $\delta_1 = O(x^{\min\{\alpha_2-\alpha_1,\alpha_1-1\}-\varepsilon})$, for every $\varepsilon > 0$.

Furthermore, it can be seen that $\widehat{\delta}_1$ consists of blocks of the type $x^\beta R(\boldsymbol{\ell}^{-1})$, where R is a rational function whose denominator can only be a *positive* integer power of the polynomial $P_1(\boldsymbol{\ell}^{-1})$, and β belongs to the set:

$$\mathcal{R}_1 := \{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - (\alpha_1 - 1),\ k \in \mathbb{N}\},$$

where $\alpha_{n_i} > 1$ are powers of x in $\widehat{f}$. Since the sequence $(\alpha_i)_i$ is finitely generated, the set $\mathcal{R}_1$ belongs to a finitely generated lattice. In particular, it is well-ordered.

Now, we repeat the same procedure of elimination (on one hand for germs, and on the other hand for formal series), but with the right-hand side δ_1 (that is, $\widehat{\delta}_1$) instead of 1. We put $\widehat{R}_1 = \widehat{\Psi}^\infty + \widehat{R}_2$. To eliminate the first block from $\widehat{\delta}_1$, say $x^\beta R(\boldsymbol{\ell}^{-1})$, we take:

$$(\widehat{\Psi}_2)' = \frac{x^\beta R(\boldsymbol{\ell}^{-1})}{x^{\alpha_1} P_1(\boldsymbol{\ell}^{-1})} = x^{\beta-\alpha_1} Q_2(\boldsymbol{\ell}), \tag{7.7}$$

where Q_2 is a rational function.

As a germ, we paralelly define

(7.8)

$$\Psi_2(x) := \begin{cases} (1)\ \ -\int_x^d t^{-(\alpha_1-\beta)} Q_2(\boldsymbol{\ell}(t))\, dt & \alpha_1 - \beta > 1 \text{ or } (\alpha_1 = \beta + 1,\ \mathrm{ord}(Q_2) \leq 1),\\ (2)\ \ \int_0^x t^{-(\alpha_1-\beta)} Q_2(\boldsymbol{\ell}(t))\, dt, & \alpha_1 - \beta < 1 \text{ or } (\alpha_1 = \beta + 1,\ \mathrm{ord}(Q_2) > 1). \end{cases}$$

Obviously, $\Psi_2^\infty \in \mathcal{G}_{AN}$. Note that there are no new singularities created in $Q_2(\boldsymbol{\ell})$, since the denominator of $\frac{R(\boldsymbol{\ell}^{-1})}{P_1(\boldsymbol{\ell}^{-1})}$ is just a positive integer power of $P_1(\boldsymbol{\ell}^{-1})$.

Note here that, depending on the order of the right-hand side in (7.7) (that is, on the step of the algorithm), in the case (1) in (7.8) we get $\Psi_2(x) \to \infty$ (infinite block), while in the case (2) we get $\Psi_2(x) = o(1)$ (infinitesimal block), as $x \to 0$. In (1), we may chose the constant in Ψ_2 arbitrarily (choosing any $d > 0$), while in (2) we get the unique germ Ψ_2. We will show below that there are only finitely many infinite steps, but at most countably many infinitesimal steps.

Repeating the same procedure, we conclude that the monomials from the support of $\widehat{\delta}_2$ are of the form $x^\beta R(\boldsymbol{\ell}^{-1})$, where R is a rational function with same properties as before, and

$$\beta \in \mathcal{R}_2 := \{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - 2(\alpha_1 - 1),\ k \in \mathbb{N},\ k \geq 2\} \cup \mathcal{R}_1.$$

At the r-th step of this procedure, the powers of x in $\widehat{\delta}_r$ are:

$$\beta \in \mathcal{R}_r := \cup_{p\in\mathbb{N},\, p\leq r}\{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - p(\alpha_1 - 1),\ k \in \mathbb{N},\ k \geq p\}.$$

Therefore, the monomials appearing in the algorithm on the right-hand side of the modified Abel equation are always of the form $x^\beta R(\boldsymbol{\ell}^{-1})$, where

$$\beta \in \mathcal{R} := \bigcup_{r\in\mathbb{N}}\{(\alpha_{n_1} + \cdots + \alpha_{n_k} - k) - r(\alpha_1 - 1) |\ k \in \mathbb{N},\ k \geq r\} =$$
$$=\{(\alpha_{n_1} - \alpha_1) + \cdots + (\alpha_{n_r} - \alpha_1) + (\alpha_{n_{r+1}} - 1) + \cdots + (\alpha_{n_k} - 1),\ r \leq k,\ r, k \in \mathbb{N}\}.$$

That is, the set $\mathcal{R}$ is the set of *all* finite sums of nonegative elements of the form $(\alpha_i - \alpha_1)$ and $(\alpha_i - 1)$, where $\alpha_i \geq 1$, $i \in \mathbb{N}$, is the sequence of powers of x in the Dulac expansion $\widehat{f}(x)$. Since $(\alpha_i)_i$ belong to a finitely generated lattice, it is the same for the elements of $\mathcal{R}$. In particular, $\mathcal{R}$ is well-ordered. Its order type is ω, and its elements form a sequence tending to $+\infty$. Since all the powers of x in the common support of all the right-hand sides δ in the course of the algorithm belong to $\mathcal{R}$, they can either be ordered in an *infinite strictly increasing sequence tending to* $+\infty$ or there are *only finitely many* of them. In the latter case, the block by block algorithm terminates in finitely many steps. Otherwise, it needs ω steps to terminate. In any case, the construction by blocks of the formal Fatou coordinate is not transfinite, as it terminates in at most ω steps.

Furthermore, thanks to the direct relation of $\widehat{\Psi}_r$ and the leading block of $\widehat{\delta}_r$ described in (7.7), and by Proposition 10.3, we also see that the support of $\widehat{\Psi}$ is well-ordered.

7.2. **The principal (*infinite*) part of the Fatou coordinate.** Let $\alpha_1 > 1$ be such that $\mathrm{ord}(\mathrm{id} - \widehat{f}) = (\alpha_1, m)$, $m \in \mathbb{N}_0^-$, as above. We have proved in Section 7.1 that the orders of the blocks on the right-hand sides of the Fatou equation in the course of eliminations belong to a finitely generated lattice. The order of the leading block on the right-hand side in every step strictly increases. Therefore, it follows that after *finitely many steps* of block by block eliminations, the order of the right-hand side $\widehat{\delta}$ becomes strictly bigger than $\alpha_1 - 1$.

We denote by $r_0 \in \mathbb{N}$ the smallest number such that, after r_0 steps, the Abel equation becomes:

$$R_{r_0}(f(x)) - R_{r_0}(x) = o(x^{\alpha_1 - 1}).$$

The r_0-th step is the *critical step* between the infinite and the infinitesimal part of the Fatou coordinate. That is:

$$R_{r_0-1}(x) \to \infty,\ R_{r_0}(x_0) \to 0,\ x \to 0.$$

This is a direct consequence of the following Proposition 7.5:

Proposition 7.5 (Order of the blocks in the algorithm)**.** *Let $\beta \in \mathcal{R}$ be the order of the first block of the right-hand side $\widehat{\delta}_k \in \widehat{\mathcal{L}}$ of the equation*

$$\widehat{R}(\widehat{f}) - \widehat{R} = \widehat{\delta}_k.$$

Then the first block $\widehat{\Psi}_k \in \widehat{\mathcal{L}}_2^\infty$ of $\widehat{R}$ is a block of order $\beta - (\alpha_1 - 1)$. Moreover,

a) if $\beta > \alpha_1 - 1$, then $\widehat{\Psi}_k \in \widehat{\mathcal{L}}$,
b) if $\beta < \alpha_1 - 1$, then $\widehat{\Psi}_k \in \widehat{\mathcal{L}}^\infty$,
c) if $\beta = \alpha_1 - 1$, then $\widehat{\Psi}_k \in \widehat{\mathcal{L}}_2^\infty$.

Proof. By Taylor expansion, as described in the algorithm in Section 7.1,

$$\widehat{\Psi}_k = \int x^{-(\alpha_1-\beta)} \widehat{Q}(\boldsymbol{\ell})\, dx,$$

where $\widehat{Q}$ is an asymptotic expansion of a rational function, and $\int$ denotes the formal integral. The result now follows by Proposition 10.3. $\square$

It follows that there exists an index $r_0 \in \mathbb{N}$, called the *critical index*, such that all the infinite blocks of Ψ or $\widehat{\Psi}$ are exactly those (finitely many) indexed by $r \leq r_0$. Hence we define *the principal part* of $\Psi \in \mathcal{G}_{AN}$ or $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ by:

$$\Psi^\infty = \Psi_1 + \cdots + \Psi_{r_0},$$
$$\widehat{\Psi}^\infty = \widehat{\Psi}_1 + \cdots + \widehat{\Psi}_{r_0}.$$

Note that, by the integral definition (7.8) (1) of infinite blocks Ψ_i, $i = 1 \ldots r_0$, due to arbitrary choices of $d > 0$, Ψ^∞ defined above is unique only up to an additive constant. Therefore, by Proposition 10.3, parallel construction and by the definition of integral sections and sectional asymptotic expansions in Section 3, the formal principal part $\widehat{\Psi}^\infty \in \widehat{\mathcal{L}}_2^\infty$ obtained by the blockwise integration by parts is the sectional asymptotic expansion of Ψ^∞ with respect to any integral section $\mathbf{s}$, up to appropriate choices of constant terms in both Ψ^∞ and $\widehat{\Psi}^\infty$. Also, the change of the integral section $\mathbf{s}$ leads to the change in constant terms in Ψ^∞ or in $\widehat{\Psi}^\infty$.

7.3. **The infinitesimal part of the Fatou coordinate.** Let r_0 be the *critical index* defined at the end of Section 7.2. The germ $R = \Psi - \Psi^\infty$ satisfies the difference equation:

$$R(f(x)) - R(x) = \delta(x), \tag{7.9}$$

where $\delta(x) = O(x^\gamma)$ with $\gamma > \alpha_1 - 1$, $\delta \in \mathcal{G}_{AN}$. Note that this is the first step of the block by block algorithm for which we obtain an *infinitesimal* solution. That is, $R = o(1)$, as $x \to 0$.

On one hand, we continue solving formally block by block (expanding integrals by integration by parts). We have already proved at the end of Section 7.1 that we terminate the formal block by block algorithm in countably many steps:

$$\widehat{R} = \sum_{i \in \mathbb{N}} \widehat{R}_{r_0+i}. \tag{7.10}$$

By Proposition 7.5, $\widehat{R}_{r_0+i} \in \widehat{\mathcal{L}}$, $i \in \mathbb{N}$, are blocks of strictly increasing orders, so $\widehat{R} \in \widehat{\mathcal{L}}$.

On the other hand, in each step we get a germ $R_{r_0+i} \in \mathcal{G}_{AN},\ i \in \mathbb{N}$, which is defined by an appropriate integral, see (7.8) (2). Note that $R_{r_0+i} = o(1)$, as $x \to 0$, as a consequence of the fact that we do not allow adding arbitrary constant terms in the definition (7.8) (2). Therefore, by Proposition 10.3, $\widehat{R}_{r_0+i}$ is exactly the asymptotic expansion of R_{r_0+i} in $\widehat{\mathcal{L}}$.

Instead of proving that the infinite series of analytic germs is an analytic germ, once that we have reached the equation for the infinitesimal part (7.9), we directly construct a germ $R \in \mathcal{G}_{AN},\ R(x) = o(1)$ and satisfying (7.9), by following the classical construction explained in [11]. We define:

$$R(x) := -\sum_{k=0}^{\infty} \delta\big(f^{\circ k}(x)\big) \tag{7.11}$$

We prove that the above sum converges *uniformly* on $(0, d)$ for sufficiently small $d > 0$. Take $\varepsilon > 0$ small enough such that $\alpha_1 + \varepsilon - 1 < \gamma$. By Proposition 7.7 below, we see that $0 < f^{\circ k}(x) < \big(\frac{k}{2}\big)^{-\frac{1}{\alpha_1+\varepsilon-1}}$, $x \in (0, d)$, for d sufficiently small. By *Weierstrass theorem*, $R \in \mathcal{G}_{AN}$. It is easy to check that R satisfies (7.9).

Now, $\Psi = \Psi^{\infty} + R$, $\Psi \in \mathcal{G}_{AN}$, is a Fatou coordinate for f. Directly by Proposition 7.6 below, we get that $R(x) = o(1)$, as $x \to 0$.

Proposition 7.6. *Let f be the Dulac germ as above. Let $\alpha_1 > 1$ be the order of the first block in the Dulac expansion of* $\mathrm{id} - f$. *Let δ be an analytic germ outside* 0*, satisfying $\delta(x) = O(x^{\gamma}),\ \gamma > \alpha_1 - 1$. Then h defined by the series*

$$h(x) := -\sum_{k=0}^{\infty} \delta\big(f^{\circ k}(x)\big) \tag{7.12}$$

is an analytic germ on $(0, d)$, $d > 0$. Moreover, for every $\varepsilon > 0$, $h(x) = O(x^{\gamma-(\alpha_1-1+\varepsilon)})$, as $x \to 0$.

Proof. Let $\varepsilon > 0$ such that $\gamma > \alpha_1 - 1 + \varepsilon$. By Proposition 7.7, we get that there exists $d > 0$ such that

$$f^{\circ k}(x) \le \big(\frac{k}{2}\big)^{-\frac{1}{\alpha_1-1+\varepsilon}},\ x \in (0, d).$$

By *Weierstrass theorem*, the function h defined on $(0, d)$ by (7.12) is analytic on $(0, d)$. The last point of the proposition follows from the following inequalities:

$$\begin{aligned}|h(x)| = \Big| -\sum_{k=0}^{\infty} \delta(f^{\circ k}(x))\Big| &\le C \sum_{k\ge 0} x^{\gamma}\big(1 + \frac{k}{2}x^{\alpha_1-1+\varepsilon}\big)^{-\frac{\gamma}{\alpha_1-1+\varepsilon}} \\ &\le C_1 x^{\gamma} \int_0^{\infty} \big(1 + \frac{t}{2}x^{\alpha_1+\varepsilon-1}\big)^{-\frac{\gamma}{\alpha_1-1+\varepsilon}}\, dt \le O(x^{\gamma-(\alpha_1-1+\varepsilon)}),\ x \to 0+,\end{aligned}$$

where $C > 0$ and $C_1 > 0$ are constants. $\square$

Proposition 7.7. *Let f be an analytic germ on $(0, d)$ satisfying $0 < f(x) < x$, $x \in (0, d)$. Let $\lim_{x\to 0} \frac{x^{\alpha+\varepsilon}}{x-f(x)} \to 0$, for some $\alpha \ge 1$ and for every $\varepsilon > 0$. Then for* *every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $d_1 > 0$, such that*

$$0 < f^{\circ n}(x) < x\Big(1 + \frac{n}{2}x^{\alpha+\varepsilon-1}\Big)^{-\frac{1}{\alpha+\varepsilon-1}},\ x \in (0, d_1),\ n \geq n_0. \tag{7.13}$$

Proof. Take $\varepsilon > 0$ and $f_1(x) = x - x^{\alpha+\varepsilon}$. We prove the proposition for f_1, and the statement for f follows. To prove the estimate, we use the change of variables $w = \frac{1}{(\alpha+\varepsilon-1)x^{\alpha+\varepsilon}}$, $w \in (M, \infty)$, by which f_1 becomes:

$$F_1(w) = w + 1 + O(w^{-1}).$$

It is now immediate to end the proof by working with F_1. □

7.4. **The sectional asymptotic expansions of the Fatou coordinate with respect to integral sections.**

We prove here that $R \in \mathcal{G}_{AN}$ defined in (7.11) admits the formal infinitesimal part of the Fatou coordinate $\widehat{R} \in \widehat{\mathcal{L}}$ constructed in (7.10) as its sectional asymptotic expansion with respect to *any* integral section $\mathbf{s}$. We have already proven in Subsection 7.2 that $\Psi^\infty \in \mathcal{G}_{AN}$ admits $\widehat{\Psi}^\infty \in \widehat{\mathcal{L}}_2^\infty$ as its sectional asymptotic expansion with respect to any integral section for an appropriate choice of constant terms in Ψ^∞, $\widehat{\Psi}^\infty$. Consequently, Fatou coordinate $\Psi = \Psi^\infty + R \in \mathcal{G}_{AN}$ will admit formal Fatou coordinate $\widehat{\Psi} = \widehat{\Psi}^\infty + \widehat{R} \in \widehat{\mathcal{L}}_2^\infty$ as its sectional asymptotic expansion with respect to any integral section, but only with appropriate choice of constant terms in Ψ^∞, $\widehat{\Psi}^\infty$. Finally, the different choice of integral section leads to different choice in constants in Ψ or in $\widehat{\Psi}$, if $\widehat{\Psi}$ is to be the sectional asymptotic expansion of Ψ with respect to the new integral section.

Put $h_n := R - \sum_{i=1}^n R_{r_0+i}$, $n \in \mathbb{N}$. Obviously, $h_n \in \mathcal{G}_{AN}$ and $h_n = o(1)$ (since $R = o(1)$ and $R_{r_0+i} = o(1)$, $i \in \mathbb{N}$). It can easily be checked that h_n satisfies the difference equation with the right-hand side $\delta_n(x) = O(x^{\gamma_n})$, where $\gamma_n \to \infty$ and $\delta_n \in \mathcal{G}_{AN}$. Iterating the equation for h_n and passing to limit as for $R(x)$ before, we get that h_n is necessarily given by the formula:

$$h_n(x) = -\sum_{k=0}^{\infty} \delta_n(f^{\circ k}(x)).$$

By Proposition 7.6, we get that $h_n = O(x^{\beta_n})$, where $\beta_n \to \infty$. Together with the fact that $\widehat{R}_{r_0+i} \in \widehat{\mathcal{L}}$ is the asymptotic expansion of $R_{r_0+i} \in \mathcal{G}_{AN}$, $i \in \mathbb{N}$ (see Proposition 10.3), this proves that $R \in \mathcal{G}_{AN}$, $R = o(1)$, admits $\widehat{R} \in \widehat{\mathcal{L}}$ as the sectional asymptotic expansion with respect to any integral section.

7.5. **Uniqueness of the Fatou coordinate.**

The formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ is unique in $\widehat{\mathfrak{L}}$, up to an additive constant, by Proposition 4.3.

Fix the integral section $\mathbf{s}$ and the formal coordinate $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$. In Subsections 7.1-7.4, we have constructed a Fatou coordinate $\Psi \in \mathcal{G}_{AN}$ which admits $\widehat{\Psi}$ as its sectional asymptotic expansion with respect to $\mathbf{s}$. We prove here the uniqueness (up to an additive constant) of the Fatou coordinate in $\mathcal{G}_{AN}$ which admits a sectional asymptotic expansion in the class $\widehat{\mathfrak{L}}$. Suppose that there exists some other $\widetilde{\Psi} \in \mathcal{G}_{AN}$ (differing from $\Psi \in \mathcal{G}_{AN}$ obtained in our construction by more than a constant) which satisfies the Abel equation and admits a sectional asymptotic expansion in $\widehat{\mathfrak{L}}$. Let Ψ^∞ be the infinite part of Ψ from the construction. Put $\widetilde{R} = \widetilde{\Psi} - \Psi^\infty \in \mathcal{G}$. The germ $\widetilde{R}$ satisfies (7.9) and has the power-iterated logarithm asymptotic behavior. Therefore, possibly after substracting an additive constant, $\widetilde{R} = o(1)$, as $x \to 0$. But under the assumption $\widetilde{R} = o(1)$, the solution $\widetilde{R}$ of the equation (7.9) is unique. Indeed, by summing the equation (7.9) for the iterates $f^{\circ k}$ and taking the limit as $k \to \infty$, we get that any solution $\widetilde{R} = o(1)$ is necessarily given by the series (7.11). Therefore, $\widetilde{R} \equiv R$, up to an additive constant and, consequently, $\widetilde{\Psi} \equiv \Psi$, up to an additive constant.

8. Proof of Theorem A

Proof of Theorem A. Let $\widehat{g} = \mathrm{id} - \widehat{f}$. The formal continuous time length of the ε-neighborhood of orbits for $\widehat{f}$ is given by the formula:

$$\widehat{A}^c_{\hat{f}}(\varepsilon) = \widehat{g}^{-1}(2\varepsilon) + 2\varepsilon \cdot \widehat{\Psi}\big(\widehat{g}^{-1}(2\varepsilon)\big). \tag{8.1}$$

By Proposition 6.1, we have that $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_2$ and does not contain a double logarithm in the leading term.

By Proposition 4.3, the formal Fatou coordinate $\widehat{\Psi}$ of $\widehat{f}$ exists in $\widehat{\mathcal{L}}_2^\infty$ and is unique (up to an additive constant) in $\widehat{\mathfrak{L}}$. Furthermore, by Remark 10.1 there exists $\rho \in \mathbb{R}$ such that

$$\widehat{\Psi} - \rho \boldsymbol{\ell}_2^{-1} \in \widehat{\mathcal{L}}^\infty.$$

By formal composition it now easily follows that $\widehat{\Psi} \circ \widehat{g}^{-1} \in \widehat{\mathcal{L}}_2^\infty$. The conclusion of Theorem A follows. $\square$

The following remark provides preciser information on $\widehat{A}^c_{\hat{f}}(\varepsilon)$:

Remark 8.1. Let $\widehat{f} \in \widehat{\mathcal{L}}$ parabolic, $\widehat{g} = \mathrm{id} - \widehat{f}$. Let

$$\widehat{g}(x) = a x^\alpha \boldsymbol{\ell}^m + \text{h.o.t.},\ a \neq 0,\ (\alpha, m) \succ (1, 0).$$

(1) (The residual term of $\widehat{A}^c_{\hat{f}}(\varepsilon)$) If $\widehat{g}$ does not contain a logarithm in the leading term, then by Proposition 6.1 $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_1$, without logarithm in the leading term. By Remark 10.1, there exists $\rho \in \mathbb{R}$ such that $\widehat{\Psi} - \rho \boldsymbol{\ell}_2^{-1} \in \widehat{\mathcal{L}}^\infty$. Consequently, by (8.1),

$$\widehat{A}^c_{\hat{f}}(\varepsilon) - \rho \cdot 2\varepsilon \boldsymbol{\ell}_2(\varepsilon)^{-1} \in \widehat{\mathcal{L}}_1.$$

We will call $\rho \cdot 2\varepsilon \boldsymbol{\ell}_2(\varepsilon)^{-1}$ *the residual term* of $\widehat{A}^c_{\hat{f}}(\varepsilon)$.

Notice that, in the general case where $\widehat{g}$ contains logarithm in the leading term, $\widehat{g}^{-1} \in \widehat{\mathcal{L}}_2$ and there may be other terms with double logarithm in $\widehat{A}^c_{\hat{f}}(\varepsilon)$. See also Example 3 in Section 5.

(2) (The leading term of $\widehat{A}^c_{\hat{f}}(\varepsilon)$) Inverting formally as in Proposition 6.1, we get

$$\widehat{g}^{-1}(2\varepsilon) = (2/a)^{1/\alpha} \alpha^{-m/\alpha} \varepsilon^{1/\alpha} \boldsymbol{\ell}(\varepsilon)^{-m/\alpha} + \text{h.o.t.}$$

By Remark 10.1 in the Appendix (or simply directly finding the first term of the Fatou coordinate $\widehat{\Psi}$ from the Abel equation):

$$\widehat{\Psi}(x) = \begin{cases} -\frac{1}{a}\frac{1}{1-\alpha} x^{-\alpha+1} \boldsymbol{\ell}^{-m} + \text{h.o.t.}, & \alpha \neq 1, \\ \frac{1}{a(m+1)} \boldsymbol{\ell}^{-m-1} + \text{h.o.t.}, & \alpha = 1,\ m \in \mathbb{N}. \end{cases}$$

Therefore, by (8.1),

$$\widehat{A}^c_{\widehat{f}}(\varepsilon)=\begin{cases}\big(1+\frac{1}{2(\alpha-1)}\big)2^{1/\alpha}\alpha^{-m/\alpha}a^{-1/\alpha}\cdot\varepsilon^{1/\alpha}\boldsymbol{\ell}(\varepsilon)^{-m/\alpha}+\text{h.o.t.}, & \alpha\neq 1,\\ \frac{1}{a(m+1)}\varepsilon\boldsymbol{\ell}(\varepsilon)^{-m-1}+\text{h.o.t.}, & \alpha=1,\ m\in\mathbb{N}.\end{cases}$$

9. Proof of Theorem B

In the proof we will use the following definition:

Definition 9.1. *(1) We say that a transseries* $\widehat{f}\in\widehat{\mathcal{L}}_1^\infty$ *is a* Laurent transseries in $\widehat{\mathcal{L}}_1^\infty$ *if its terms can be regrouped in the form:*

$$\widehat{f}(x):=x^\gamma\widehat{F}\Big(\boldsymbol{\ell}(x),\frac{x}{\boldsymbol{\ell}(x)}\Big),\tag{9.1}$$

where $\gamma\in\mathbb{R}$ *and* $\widehat{F}$ *is the Taylor expansion of an analytic germ* F *of two variables at* $(0,0)$.

(2) We say that a transseries $\widehat{f}\in\widehat{\mathcal{L}}_1^\infty$ *is a* generalized Laurent transseries in $\widehat{\mathcal{L}}_1^\infty$ *if it can be written as a* finite *sum of Laurent transseries.*

Note that generalized Laurent transseries are *convergent* transseries in $\widehat{\mathcal{L}}_1^\infty$ in the sense of Definition 3.7. The sum of Laurent transseries $\widehat{f}$ from (9.1) is thus unique and equal to $f(x)=x^\gamma F\Big(\boldsymbol{\ell}(x),\frac{x}{\boldsymbol{\ell}(x)}\Big)\in\mathcal{G}_{AN}$. The sum of a generalized Laurent transseries is analogously the sum of its Laurent summands, thus unique.

Note further that all derivatives of the generalized Laurent transseries are again generalized Laurent, therefore *convergent*. Moreover, the sums and the derivatives commute.

Proof of Theorem B.

(1) was already proven in even more generality (for all parabolic transseries in $\widehat{\mathcal{L}}$) in Theorem A.

(2) Recall that

$$\widehat{A}^c_{\widehat{f}}(\varepsilon)=\widehat{g}^{-1}(2\varepsilon)+2\varepsilon\cdot\widehat{\Psi}\big(\widehat{g}^{-1}(2\varepsilon)\big).$$

Here, $\widehat{\Psi}\in\widehat{\mathcal{L}}_2^\infty$ is the formal Fatou coordinate for the Dulac expansion $\widehat{f}$ of f, unique up to a constant term $K\in\mathbb{R}$ by Proposition 7.1. Therefore, $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ is uniquely defined up to εK, $K\in\mathbb{R}$, due to the choice of the constant in $\widehat{\Psi}$. By Theorem A, $\widehat{A}^c_{\widehat{f}}(\varepsilon)\in\widehat{\mathcal{L}}_2$, so it can be written in the form:

$$\widehat{A}^c_{\widehat{f}}(\varepsilon)=\sum_{j\in\mathbb{N}}\widehat{F}_j(\boldsymbol{\ell}(\varepsilon))\varepsilon^{\beta_j},\tag{9.2}$$

where $\beta_j>0,\ j\in\mathbb{N}$, form a strictly increasing sequence tending to $+\infty$, and $\widehat{F}_j\in\widehat{\mathcal{L}}_1^\infty$. The proof of Theorem B is organized as follows:

Step 1. We show that $\widehat{F}_j$ are *integrally summable* in the sense of Definition 3.14. That is, that $\widehat{F}_j\in\widehat{\mathcal{L}}_1^I\subset\widehat{\mathcal{L}}_1^\infty$.

Step 2. We show that every $A^c_f(x_0,\varepsilon)\in\mathcal{G}_{AN}$ defined by (2.7) satisfies, up to an additive term εK, $K\in\mathbb{R}$ (stemming from the choice of the additive constant in Ψ) the following asymptotics in powers of ε:

$$A^c_f(x_0,\varepsilon)=\sum_{j=1}^n F_j(\boldsymbol{\ell}(\varepsilon))\varepsilon^{\beta_j}+o(\varepsilon^{\beta_n}),\ n\in\mathbb{N},\ \varepsilon\to 0,\tag{9.3}$$

where $F_j \in \mathcal{G}_{AN}$ are any integral sums of $\widehat{F}_j$.

Step 3. Different choices of integral sums by Definition 3.14 correspond to different choices of the integral section $\mathbf{s}$. Thus, in (2) we have implicitely shown the following. Let (9.3) hold for a particular $A_f^c(x_0,\varepsilon)$ with the choice of the integral sums F_j of $\widehat{F}_j$ dictated by the integral section $\mathbf{s}$. Then (9.3) means that $A_f^c(x_0,\varepsilon)$ admits $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ from (9.2) as its $\mathbf{s}$-sectional asymptotic expansion. On the other hand, if we choose a different integral section $\mathbf{s}_1$, then there exists $K \in \mathbb{R}$ such that $A_f^c(x_0,\varepsilon)+K\varepsilon$ admits $\widehat{A}^c_{\widehat{f}}(\varepsilon)$ as its $\mathbf{s}_1$-sectional asymptotic expansion.

Proof of Step 1. We prove that $\widehat{F}_j \in \widehat{\mathcal{L}}_1^I$ in (9.2). We have shown in Subsection 6.1 in Proposition 6.6 that

$$\widehat{g}^{-1}(x)=\sum_{i\in\mathbb{N}}\widehat{g}_i(\boldsymbol{\ell})x^{\beta_i}, \tag{9.4}$$

with $\widehat{g}_i \in \widehat{\mathcal{L}}_1^\infty$ *Laurent* (see Definition 9.1) and the first coefficient $\widehat{g}_1(\boldsymbol{\ell})$ without double logarithm in the leading term. Put

$$\widehat{\Psi}_1 := \widehat{\Psi} - \rho\boldsymbol{\ell}_2^{-1}. \tag{9.5}$$

We have shown in Section 7, Proposition 7.1, that

$$\widehat{\Psi}_1(x)=\sum_j \widehat{f}_j(\boldsymbol{\ell})x^{\alpha_j},\ \widehat{f}_j \in \widehat{\mathcal{L}}_0^I. \tag{9.6}$$

From the proof of Theorem A,

$$\widehat{\Psi}\circ\widehat{g}^{-1}(2\varepsilon)=\sum_j \widehat{h}_j\big(\boldsymbol{\ell}(\varepsilon)\big)\varepsilon^{\gamma_j} \in \widehat{\mathcal{L}}_2^\infty,\ \varepsilon\approx 0. \tag{9.7}$$

We prove now that $\widehat{h}_j \in \widehat{\mathcal{L}}_1^I$ by analysing the form of these *coefficients*.

By Proposition 6.6, $\widehat{g}^{-1}(2\varepsilon)=\varepsilon^\beta\cdot\widehat{g}_1\big(\boldsymbol{\ell}(\varepsilon)\big)+O(\varepsilon^{\beta+\delta})$, $\beta>0$, $\delta>0$, $\widehat{g}_1 \in \widehat{\mathcal{L}}_1^\infty$ Laurent with no double logarithm in the first term:

$$\widehat{g}_1(\boldsymbol{\ell})=a\boldsymbol{\ell}^\gamma(1+\widehat{R}(\boldsymbol{\ell}_2,\frac{\boldsymbol{\ell}}{\boldsymbol{\ell}_2})),\ \gamma\in\mathbb{R}, \tag{9.8}$$

where $\widehat{R}$ is the Taylor expansion of an analytic germ R at $(0,0)$. Note that $O(\varepsilon^\delta) \in \widehat{\mathcal{L}}_2$ with Laurent coefficients in $\widehat{\mathcal{L}}_1^\infty$.

Put

$$\widehat{H}_i(x):=\widehat{f}_i(\boldsymbol{\ell})x^{\alpha_i},\ i\in\mathbb{N}.$$

Then $\widehat{H}_i'(x)=\widehat{R}_i(\boldsymbol{\ell})x^{\alpha_i-1}$, with $\widehat{R}_i$ convergent Laurent in $\widehat{\mathcal{L}}_0^\infty$ (i.e. $\widehat{R}_i(\boldsymbol{\ell})=\boldsymbol{\ell}^{-m}\cdot\widehat{T}(\boldsymbol{\ell})$, $\widehat{T}$ the Taylor expansion of an analytic germ of one variable). Also, all other (formal) derivatives are of the form:

$$\widehat{H}_i^{(k)}(x)=\widehat{R}_{i,k}(\boldsymbol{\ell})x^{\alpha_i-k},\ k\geq 2,\ \widehat{R}_{i,k} \text{ convergent Laurent in } \widehat{\mathcal{L}}_0^\infty. \tag{9.9}$$

Now,

$$\widehat{\Psi}_1\big(\widehat{g}^{-1}(2\varepsilon)\big)=\widehat{H}_1\big(\widehat{g}^{-1}(2\varepsilon)\big)+\widehat{H}_2\big(\widehat{g}^{-1}(2\varepsilon)\big)+\widehat{H}_3\big(\widehat{g}^{-1}(2\varepsilon)\big)+\ldots \tag{9.10}$$

By the Taylor expansion,

$$
\begin{aligned}
&\widehat{H}_i\big(\widehat{g}^{-1}(2\varepsilon)\big)=\widehat{H}_i\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\cdot(1+O(\varepsilon^{\delta+\beta}))\big)=\\
&=\widehat{H}_i\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)+\widehat{H}_i'\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)\cdot O(\varepsilon^{\delta+\beta})+\frac{1}{2!}\widehat{H}_i''\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)\cdot O(\varepsilon^{2(\delta+\beta)})+\dots
\end{aligned}
\tag{9.11}
$$

We have:

$$
\begin{aligned}
&\widehat{H}_i\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)=\widehat{f}_i\big(\boldsymbol{\ell}(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon)))\big)\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)^{\alpha_i},\\
&\widehat{H}_i^{(k)}\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)=\widehat{R}_{i,k}\big(\boldsymbol{\ell}(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon)))\big)\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)^{\alpha_i-k},\ k\in\mathbb{N},
\end{aligned}
\tag{9.12}
$$

$\widehat{R}_{i,k}$ convergent Laurent. Note using (9.8) that $\widehat{g}_1(\boldsymbol{\ell})^{\alpha_i-k}$ is a Laurent transseries in $\in\widehat{\mathcal{L}}_1^{\infty}$. By a similar argument that $\widehat{R}_{i,k}\big(\boldsymbol{\ell}(x^{\beta}\widehat{g}_1(\boldsymbol{\ell}))\big)$ is a Laurent transseries in $\widehat{\mathcal{L}}_1^{\infty}$.

By (9.5), since $\widehat{\Psi}=\widehat{\Psi}_1+\rho\boldsymbol{\ell}_2^{-1}$, it is left to analyse the formal composition $\boldsymbol{\ell}_2^{-1}\big(\widehat{g}^{-1}(2\varepsilon)\big)$. By the formal Taylor expansion, we have:

$$
\begin{aligned}
\boldsymbol{\ell}_2^{-1}(g^{-1}(2\varepsilon))=\boldsymbol{\ell}_2^{-1}\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)+\big(\boldsymbol{\ell}_2^{-1}\big)'\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)\cdot O(\varepsilon^{\beta+\delta})+&\\
+\frac{1}{2}\big(\boldsymbol{\ell}_2^{-1}\big)''\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)\cdot O(\varepsilon^{2\beta+2\delta})+\dots&
\end{aligned}
\tag{9.13}
$$

Using $(\boldsymbol{\ell}_2^{-1})^{(k)}(x)=x^{-k}P_k(\boldsymbol{\ell})$, where P_k is a polynomial of degree k, $k\in\mathbb{N}$, we get:

$$
\begin{aligned}
&\boldsymbol{\ell}_2^{-1}\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)=\boldsymbol{\ell}_2^{-1}(\varepsilon)+\widehat{R}_1\Big(\boldsymbol{\ell}(\varepsilon),\frac{\boldsymbol{\ell}(\varepsilon)}{\boldsymbol{\ell}_2(\varepsilon)}\Big)\\
&(\boldsymbol{\ell}_2^{-1})^{(k)}\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)=P_k\big(\boldsymbol{\ell}(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon)))\big)\cdot\big(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))\big)^{-k},\ k\in\mathbb{N}.
\end{aligned}
\tag{9.14}
$$

Here, $\widehat{R}_1$ is a Taylor expansion of an analytic germ of two variables at $(0,0)$. Note that $\boldsymbol{\ell}_2^{-1}\big(x^{\beta}\widehat{g}_1(\boldsymbol{\ell})\big)$ is also generalized Laurent in $\widehat{\mathcal{L}}_1^{\infty}$. Therefore, concluding as in (9.12) above, the coefficients in front of every power of ε in $\boldsymbol{\ell}_2^{-1}(g^{-1}(2\varepsilon))$ are generalized Laurent transseries in $\widehat{\mathcal{L}}_1^{\infty}$.

Finally, putting (9.12) in (9.11) and then in (9.10) and (9.14) in (9.13) and regrouping the terms in $\varepsilon\widehat{\Psi}\big(\widehat{g}^{-1}(\varepsilon)\big)$ with the same power of ε, we get the summands in $\widehat{A}^c_{\widehat{f}}\in\widehat{\mathcal{L}}_2$ of two possible types, with respect to the power of ε:

$$
\begin{cases}
\varepsilon^{1+\beta\alpha_i}\Big(\widehat{g}_1(\boldsymbol{\ell}(\varepsilon))^{\alpha_i}\cdot\widehat{f}_i\big(\boldsymbol{\ell}(\varepsilon^{\beta}\widehat{g}_1(\boldsymbol{\ell}(\varepsilon)))\big)+\widehat{H}(\boldsymbol{\ell}(\varepsilon))\Big), & \text{for } \alpha_i \text{ exponents of } \widehat{\Psi},\\
\varepsilon^{1+\gamma}\widehat{G}(\boldsymbol{\ell}(\varepsilon)), & \text{if } \gamma\neq\beta\alpha_i, \text{ for every } i\in\mathbb{N},
\end{cases}
\tag{9.15}
$$

where $\widehat{G},\ \widehat{H}\in\widehat{\mathcal{L}}_1^{\infty}$ are generalized Laurent transseries, α_i is the exponent of integration of $\widehat{f}_i\in\widehat{\mathcal{L}}_0^{I}$.

The 'coefficients' $\widehat{F}_j$ from (9.2) obviously belong to $\widehat{\mathcal{L}}_1^{I}$ as defined in Definition 3.13 and one decomposition (3.7) is given in (9.15).

Proof of Steps 2. and 3. We now show the asymptotics from (9.3).

The construction of $\Psi\in\mathcal{G}_{AN}$ from Proposition 7.1 follows the same term-by-term algorithm as formal construction of $\widehat{\Psi}$. Let $\Psi_1=\Psi-\rho\boldsymbol{\ell}_2^{-1}$. We have by the proof of Proposition 7.1 in Section 7 that:

$$\Psi_1(x) - \sum_{j=1}^{n} x^{\alpha_j} f_j(\boldsymbol{\ell}) = o(x^{\alpha_n}),\ n \in \mathbb{N}, \tag{9.16}$$

where $f_j \in \mathcal{G}_{AN}$ is an integral sum (in the sense of Definition 3.10, unique up to a constant) of $\widehat{f}_j \in \widehat{\mathcal{L}}_0^I$ from (9.6), $j \in \mathbb{N}$, and α_n strictly increasing to $+\infty$, the same as in (9.6). Put $H_i(x) := x^{\alpha_i} f_i(\boldsymbol{\ell}),\ i \in \mathbb{N}$. Then $H_i \in \mathcal{G}_{AN}$, $H_i'(x) = x^{\alpha_i - 1} R_i(\boldsymbol{\ell})$, and

$$H_i^{(k)}(x) = R_{i,k}(\boldsymbol{\ell}) x^{\alpha_i - k},\ k \geq 2,$$

where R, $R_{i,k}$ are the sums of convergent Laurent series $\widehat{R}$, $\widehat{R}_{i,k} \in \widehat{\mathcal{L}}_0^\infty$ from the formal construction (9.9).

On the other hand, we have proven in Proposition 6.7 in Section 6 the following expansion:

$$g^{-1}(2\varepsilon) - \sum_{i=1}^{k} g_i\big(\boldsymbol{\ell}(\varepsilon)\big)\varepsilon^{\beta_i} = o(\varepsilon^{\beta_k}),\ k \in \mathbb{N}, \tag{9.17}$$

where $g_i \in \mathcal{G}_{AN}$ are the sums of Laurent transseries $\widehat{g}_i \in \widehat{\mathcal{L}}_1^\infty$ from (9.4).

Use (9.16) and (9.17) and repeat, as in the formal counterpart above, the Taylor expansion of

$$\Psi_1(g^{-1}(2\varepsilon)) = \sum_{j=1}^{n} H_j(g^{-1}(\varepsilon)) + o\big((g^{-1}(2\varepsilon))^{\alpha_n}\big),$$

n sufficiently big. We repeat the Taylor expansion also for $\boldsymbol{\ell}_2(g^{-1}(2\varepsilon))$. By the correspondence of the procedure with the formal one above, we finally get:

$$\Psi(g^{-1}(2\varepsilon)) - \sum_{j=1}^{n} h_j\big(\boldsymbol{\ell}(\varepsilon)\big)\varepsilon^{\gamma_j} = o(\varepsilon^{\gamma_j}),\ n \in \mathbb{N}.$$

where γ_n are strictly increasing to $+\infty$, as in (9.7) and h_j are exactly the integral sums of $\widehat{h}_j \in \mathcal{L}_1^I$ from (9.7), as in Definition 3.14. The asymptotics (9.3) now follows.

By the precise form of the coefficients $\widehat{F}_j \in \widehat{\mathcal{L}}_1^I$ given in (9.15), we see that their integral sums F_j are non-unique by the following terms:

1. In front of the power $\varepsilon^{\beta\alpha_i} + 1$ for $\alpha_i < 0$, the integral sum $F_j \in \mathcal{G}_{AN}$ is unique only up to an additive term (see Definition 3.13 and Proposition 10.5):

$$c\,\big(g_1(\boldsymbol{\ell}(\varepsilon))\big)^{\alpha_i} e^{\overline{\frac{\alpha_i}{\boldsymbol{\ell}(\varepsilon^\beta g_1(\boldsymbol{\ell}(\varepsilon)))}}} = c\,\big(g_1(\boldsymbol{\ell}(\varepsilon))\big)^{\alpha_i} \cdot \big(\varepsilon^\beta g_1(\boldsymbol{\ell}(\varepsilon))\big)^{-\alpha_i} = c\varepsilon^{-\beta\alpha_i},\ c \in \mathbb{R}.$$

For $\alpha_i > 0$, the sum F_j in front of the power $\varepsilon^{\beta\alpha_i} + 1$ is unique.

2. In front of all other powers of ε, $\widehat{F}_j \in \widehat{\mathcal{L}}_1^I$ are generalized Laurent, therefore *convergent*, and have thus unique sums $F_j \in \mathcal{G}_{AN}$.

Note that in the formal Fatou coordinate there are only finitely many α_i such that $\alpha_i < 0$ (see (9.5) and construction of the formal Fatou coordinate in Section 7). Therefore, depending on the choice of the integral sums, that is, on the choice of the section $\mathbf{s}$, a fixed $A_f^c(x_0, \varepsilon)$ admits a fixed $\widehat{A}_{\widehat{f}}^c$ as its sectional asymptotic expansion with respect to $\mathbf{s}$ only up to a term $c\varepsilon$, $c \in \mathbb{R}$.

(3) Let $\mathcal{O}^f(x_0)$ be an orbit of f. Then:

$$A_f(\varepsilon, x_0) = f^{n_\varepsilon(x_0)}(x_0) + 2\varepsilon \cdot n_\varepsilon(x_0) + 2\varepsilon,$$
$$A_f^c(\varepsilon, x_0) = \widehat{g}^{-1}(2\varepsilon) + 2\varepsilon \cdot \tau_\varepsilon(x_0) + 2\varepsilon.$$

We now prove the following:
(a) $|f^{n_\varepsilon(x_0)} - g^{-1}(2\varepsilon)| = O(\varepsilon)$, $\varepsilon \to 0$,
(b) $|n_\varepsilon(x_0) - \tau_\varepsilon(x_0)| = O(1),\ \varepsilon \to 0$.

Proof of (a). We have by (2.1) that

$$g\big(f^{n_\varepsilon(x_0)-1}(x_0)\big) > 2\varepsilon, \quad g\big(f^{n_\varepsilon(x_0)}(x_0)\big) \le 2\varepsilon,$$
$$f \circ \Big/ f^{n_\varepsilon(x_0)-1}(x_0) > g^{-1}(2\varepsilon), \quad f^{n_\varepsilon(x_0)}(x_0) \le g^{-1}(2\varepsilon),$$
$$f^{n_\varepsilon(x_0)}(x_0) > f(g^{-1}(2\varepsilon)) = g^{-1}(2\varepsilon) - 2\varepsilon, \quad f^{n_\varepsilon(x_0)}(x_0) \le g^{-1}(2\varepsilon),$$
$$\Rightarrow\ 0 \le g^{-1}(2\varepsilon) - f^{n_\varepsilon(x_0)}(x_0) < 2\varepsilon.$$

We conclude that $|g^{-1}(2\varepsilon) - f^{n_\varepsilon(x_0)}(x_0)| = O(\varepsilon),\ \varepsilon \to 0$.

Proof of (b). Let $\Psi \in \mathcal{G}_{AN}$ be the Fatou coordinate for f (up to an additive constant) with the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ as its sectional asymptotic expansion with respect to an integral section. Then:

$$n_\varepsilon(x_0) = \Psi(f^{n_\varepsilon(x_0)}(x_0)) - \Psi(x_0),\ \tau_\varepsilon(x_0) = \Psi\big(g^{-1}(2\varepsilon)\big) - \Psi(x_0).$$

Now, by (a), we get

$$\begin{aligned}\tau_\varepsilon(x_0) - n_\varepsilon(x_0) &= \Psi\big(g^{-1}(2\varepsilon)\big) - \Psi\big(f^{n_\varepsilon(x_0)}(x_0)\big)\\ &= \Psi\big(g^{-1}(2\varepsilon)\big) - \Psi\big(g^{-1}(2\varepsilon) + O(\varepsilon)\big)\\ &\sim \widehat{\Psi}\big(\widehat{g}^{-1}(2\varepsilon)\big) - \widehat{\Psi}\big(\widehat{g}^{-1}(2\varepsilon) + O(\varepsilon)\big)\\ &\sim \widehat{\Psi}'(\widehat{g}^{-1}(2\varepsilon)) \cdot O(\varepsilon) = O(1),\ \varepsilon \to 0.\end{aligned}$$

The last line is deduced by the formal Taylor expansion of $\widehat{\Psi}(x + \widehat{g}) - \widehat{\Psi} = 1$ (providing $\widehat{\Psi}' \sim 1/\widehat{g}$), since $\mathrm{ord}(\widehat{g}) \succ (1, 0)$. □

9.1. **Application of Theorem B to recognizing a Dulac germ from fractal properties of its orbits.** We finish the section with one immediate application of Theorem B: to read the formal class of a Dulac germ $f \in \mathcal{G}_{AN}$ from an asymptotic expansion of the length of the epsilon-neighborhood of its one orbit. This is a generalization of the result from [15] for regular parabolic germs belonging to $\mathbb{R}\{x\}$.

It was proved in [14, Theorem A] that a formal normal form in $\widehat{\mathcal{L}}$ for a (normalized) parabolic Dulac germ $f \in \mathcal{G}_{AN}$ of the form

$$\widehat{f}(x) = x - x^\alpha + o(x^\alpha),\ \alpha \in \mathbb{R},\ \alpha > 1, \tag{9.18}$$

is given by

$$f_0(x) = x - x^\alpha + \rho x^{2\alpha-1}\boldsymbol{\ell}^{-1},\ \rho \in \mathbb{R}.$$

The formal invariants are (α, ρ).

Using Theorem B and Remark 8.1, we have proved the following corollary:

Corollary 9.2 (Formal normal form of Dulac germ from fractal properties of orbits)**.** *Let $f = x - x^\alpha + o(x^\alpha)$ be a normalized parabolic Dulac germ, without logarithm in the second term. Let* $\mathbf{s}$ *be an integral section. The function of the length of the ε-neighborhood of any orbit, $\varepsilon \mapsto A_f(\varepsilon, x_0),\ \varepsilon \in (0, d)$, has the sectional asymptotic expanson in $\widehat{\mathcal{L}}_2$ with respect to* $\mathbf{s}$*, up to the order $O(\varepsilon)$. Moreover, the formal invariants (α, ρ) of f can be read from two terms of the expansion: the first and the residual term (the only one containing the double logarithm). More precisely, α is the order of the first monomial in the expansion, and ρ is the coefficient of the monomial $\frac{\varepsilon}{\boldsymbol{\ell}_2(\varepsilon)}$.*

10. Appendix

The following remark is used in the proof of Theorem A in Section 8 and of Proposition 7.4 in Section 7.

Remark 10.1 (Description of the formal Fatou coordinate for parabolic transseries)**.** We explain here another way of deducing that the formal Fatou coordinate $\widehat{\Psi} \in \widehat{\mathfrak{L}}$ of a parabolic $\widehat{f} \in \widehat{\mathcal{L}}$ belongs to the class $\widehat{\mathcal{L}}_2^\infty$ and its precise form.

Recall the formal normal form $\widehat{f}_0$ of $\widehat{f}$ in $\widehat{\mathcal{L}}$ deduced in [14], given as the formal time-one map of a simple vector field:

$$\widehat{f} = x + ax^\alpha \boldsymbol{\ell}^m + \text{h.o.t.},\ \text{with } \alpha > 1,\ \text{or } \alpha = 1 \text{ and } m \in \mathbb{N},$$

$$\widehat{f}_0(x) = \operatorname{Exp}\big(\widehat{\xi}_0(x)\frac{\mathrm{d}}{\mathrm{d}x}\big) \cdot \mathrm{id},$$

$$\widehat{\xi}_0(x) = \frac{ax^\alpha \boldsymbol{\ell}^m}{1 + \frac{a\alpha}{2}x^{\alpha-1}\boldsymbol{\ell}^m - \big(\frac{am}{2} + \frac{b}{a}\big)x^{\alpha-1}\boldsymbol{\ell}^{m+1}}\frac{\mathrm{d}}{\mathrm{d}x},\ b \in \mathbb{R}.$$

By the existence part of Proposition 4.3, we search for a Fatou coordinate of $\widehat{f}_0$ as the formal antiderivative of $\frac{1}{\widehat{\xi}_0}$:

$$\begin{aligned}\widehat{\Psi}_0 =& \frac{1}{a}\int x^{-\alpha}\boldsymbol{\ell}^{-m}\,dt + \frac{\alpha}{2}\log x + (\frac{m}{2} + \frac{b}{a^2})\int \frac{dx}{x\log x} \\ =& \frac{1}{a}\widehat{h}(x) - \frac{\alpha}{2}\boldsymbol{\ell}^{-1} + (\frac{m}{2} + \frac{b}{a^2})\boldsymbol{\ell}_2^{-1} + C,\ C \in \mathbb{R}.\end{aligned}$$

Here, $\widehat{h} \in \widehat{\mathcal{L}}^\infty$ is obtained by repeated formal *integration by parts*:

$$\int x^{-\alpha}\boldsymbol{\ell}^{-m}\,dx = \begin{cases} \frac{1}{1-\alpha}x^{-\alpha+1}\boldsymbol{\ell}^{-m} + \frac{m}{1-\alpha}\int x^{-\alpha}\boldsymbol{\ell}^{-m+1}\,dx, & \alpha \neq 1, \\ -\frac{1}{m+1}\boldsymbol{\ell}^{-m-1} + C,\ C \in \mathbb{R}, & \alpha = 1,\ m \in \mathbb{N}. \end{cases} \tag{10.1}$$

In particular, in the case when $m \in \mathbb{N}$, $\widehat{h}$ contains only finitely many monomials.

Now put $\widehat{\Psi} := \widehat{\Psi}_0 \circ \widehat{\varphi}$, where $\widehat{\varphi} \in \widehat{\mathcal{L}}$ parabolic is the formal change of variables reducing $\widehat{f}$ to $\widehat{f}_0$. It is easy to check that the composition $\widehat{\Psi}$ belongs to $\widehat{\mathcal{L}}_2^\infty$. It is obviously a Fatou coordinate for $\widehat{f}$, since $\widehat{\Psi}_0$ satisfies the Abel equation for $\widehat{f}_0$. By Proposition 4.3, the formal Fatou coordinate of $\widehat{f}$ is unique (up to a constant term).

Therefore, at most one term with double logarithm $\boldsymbol{\ell}_2^{-1}$ appears in the formal Fatou coordinate $\widehat{\Psi}$ of $\widehat{f}$. It corresponds to the residual term $bx^{2\alpha-1}\boldsymbol{\ell}^{2m+1}$ in the normal form $\widehat{f}_0(x) = x + ax^\alpha\boldsymbol{\ell}^m + bx^{2\alpha-1}\boldsymbol{\ell}^{2m+1} + h.o.t.$

10.1. **Propositions for Section 3.** *Proof of Proposition 3.11.*

Suppose there are two such, $\alpha \neq \beta$. Then

$$\frac{d}{dx}\big(x^{\alpha}\widehat{f}(\boldsymbol{\ell})\big) = x^{\alpha-1}R_1(\boldsymbol{\ell}),\ \frac{d}{dx}\big(x^{\beta}\widehat{f}(\boldsymbol{\ell})\big) = x^{\beta-1}R_2(\boldsymbol{\ell}).$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\big(x^{\alpha}\widehat{f}(\boldsymbol{\ell})\big) &= \frac{d}{dx}\big(x^{\alpha-\beta}\cdot x^{\beta}\widehat{f}(\boldsymbol{\ell})\big) = (\alpha-\beta)x^{\alpha-1}\widehat{f}(\boldsymbol{\ell}) + x^{\alpha-\beta}\frac{d}{dx}\big(x^{\beta}\widehat{f}(\boldsymbol{\ell})\big) = \\ &= (\alpha-\beta)x^{\alpha-1}\widehat{f}(\boldsymbol{\ell}) + x^{\alpha-1}R_2(\boldsymbol{\ell}) \\ &\Rightarrow x^{\alpha-1}R_1(\boldsymbol{\ell}) = (\alpha-\beta)x^{\alpha-1}\widehat{f}(\boldsymbol{\ell}) + x^{\alpha-1}R_2(\boldsymbol{\ell}).\end{aligned}$$

Since $R_{1,2}(y)$ are both convergent Laurent series and $\alpha-\beta\neq 0$, this is a contradiction with divergence of $\widehat{f}(y)$. $\square$

Proposition 10.2. *Let $\alpha\in\mathbb{R}$, $m\in\mathbb{Z}$. Let*

$$a(x) := \begin{cases}\int_x^d t^{-\alpha}\boldsymbol{\ell}^m\,dt, & \alpha>1 \text{ or } (\alpha=1,\ m\le 1),\ d>0,\\ \int_0^x t^{-\alpha}\boldsymbol{\ell}^m\,dt, & \alpha<1 \text{ or } (\alpha=1,\ m>1).\end{cases}$$

Then $a\in\mathcal{G}_{AN}$. Additionally, in the second case, $a(0)=0$. Moreover,

$$a(x) = \begin{cases} O(x^{-\alpha+1}\boldsymbol{\ell}^m),\ x\to 0+, & \alpha\neq 1,\\ \frac{\boldsymbol{\ell}^{m-1}}{m-1}+C,\ C\in\mathbb{R}, & \alpha=1,\ m<1,\\ \boldsymbol{\ell}_2^{-1}+C,\ C\in\mathbb{R}, & \alpha=1,\ m=1,\\ \frac{\boldsymbol{\ell}^{m-1}}{1-m}, & \alpha=1,\ m>1.\end{cases}$$

The proof is based on basic calculus (see e.g. the chapter on the asymptotic expansion of a primitive in [2]).

The following proposition is an easy consequence of Proposition 10.2 and integration by parts:

Proposition 10.3. *Let $R\in\mathcal{G}_{AN}$ admit the integer power asymptotic expansion $\widehat{R}\in\widehat{\mathcal{L}}_0$. Let $n_0 := ord(\widehat{R})$, $n_0\in\mathbb{Z}$. Let $h\in\mathcal{G}_{AN}$ be the germ defined by:*

$$h(x) = \begin{cases} -\int_x^d t^{-\alpha}R(\boldsymbol{\ell}(t))\,dt, & \alpha>1 \text{ or } (\alpha=1,\ n_0\le 1),\\ \int_0^x t^{-\alpha}R(\boldsymbol{\ell}(t))\,dt, & \alpha<1 \text{ or } (\alpha=1,\ n_0>1),\end{cases} \tag{10.2}$$

Let $\widehat{h}\in\widehat{\mathcal{L}}_2^\infty$ be the formal primitive of $x^{-\alpha}\widehat{R}(\boldsymbol{\ell})$ in $\widehat{\mathcal{L}}_2^\infty$, defined up to an additive constant:

$$\widehat{h} = \int x^{-\alpha}\widehat{R}(\boldsymbol{\ell})\,dx.$$

Then:

a) If $\alpha\neq 1$, then $\widehat{h}\in\widehat{\mathcal{L}}^\infty$, of the form:

$$\widehat{h}(x) = x^{-\alpha+1}\sum_{n=0}^{\infty} a_n\boldsymbol{\ell}^{n_0+n},\ a_n\in\mathbb{R}.$$

b) If $\alpha=1$, then $\widehat{h}\in\widehat{\mathcal{L}}_2^\infty$ of the form:

$$\widehat{h}(x) = \begin{cases}\sum_{n=0}^{-n_0} a_n\boldsymbol{\ell}^{n_0-1+n} + b\boldsymbol{\ell}_2^{-1} + \sum_{n=0}^{\infty} a_n\boldsymbol{\ell}^n, & n_0\le 1,\\ \sum_{n=0}^{\infty} a_n\boldsymbol{\ell}^{n_0-1+n}, & n_0>1.\end{cases}$$

In both cases, the formal integral $\widehat{h}$ *is the asymptotic expansion of* $h \in \mathcal{G}_{AN}$ *defined in* (10.2).

The following proposition is necessary for Definition 3.14 of *integral sections.*

Proposition 10.4. *Let* $\mathbf{s}$ *be any section such that* $\mathbf{s}\Big|_{\widehat{\mathcal{S}}_0}$ *is coherent. Then* $\widehat{\mathcal{L}}_1^I \subset \widehat{\mathcal{S}}_1^{\mathbf{s}}$.

Proof. Take $\widehat{F} \in \widehat{\mathcal{L}}_1^I$ and let $F \in \mathcal{G}$ be its one integral sum, as in (3.8). It is sufficient to verify that the algorithm of Poincaré applied to F with respect to any section $\mathbf{s}$ coherent on $\widehat{\mathcal{S}}_0$ gives the asymptotic expansion $\widehat{F}$.

Due to coherence of $\mathbf{s}$ on $\widehat{\mathcal{S}}_0$, it is sufficient to prove the following:

1. That the terms of $\widehat{F} \in \widehat{\mathcal{L}}_1^I$ from (3.7) can be re-grouped as:

$$\widehat{F}(y) = \sum_{i=1}^{\infty} \widehat{g}_i\big(\boldsymbol{\ell}(y)\big) y^{\alpha_i}, \tag{10.3}$$

where (α_i) are strictly increasing real numbers and tending to $+\infty$ and $\widehat{g}_i$ are *convergent*,

2. At the same time, that the integral sum F of $\widehat{F}$ given in (3.8) satisfies:

$$F(y) - \sum_{i=1}^{n} g_i\big(\boldsymbol{\ell}(y)\big) y^{\alpha_i} = o(y^{\alpha_n}),\ y \to 0,\ n \in \mathbb{N}, \tag{10.4}$$

where g_i are the sums of the *convergent* $\widehat{g}_i$ and α_i the same as in 1.

First, by Fubini's theorem and absolute convergence for convergent $\widehat{G}_i$, $i = 0, 1$, and $\widehat{h}$ from (3.7), we have that:

$$\widehat{G}_i(y) = \sum_{j=1}^{\infty} \widehat{g}_j^i\big(\boldsymbol{\ell}(y)\big) y^{\beta_j^i},\ i = 0, 1,$$

$$\widehat{h}(y) = a y^{\gamma_1} + \sum_{j=2}^{\infty} \widehat{h}_j\big(\boldsymbol{\ell}(y)\big) y^{\gamma_j},\ a \in \mathbb{R}. \tag{10.5}$$

Here, $(\beta_j^i)_j$ and $(\gamma_j)_j$ are strictly increasing and tending to $+\infty$, and $\widehat{g}_j^i$, $i = 0, 1$, and $\widehat{h}_j$, $j \in \mathbb{N}$, are convergent with the sums g_i^j, h_j respectively, such that for the sums $G_i \in \mathcal{G}$, $i = 0, 1$, and $h \in \mathcal{G}$ from (3.8) it holds that:

$$G_i(y) - \sum_{j=1}^{n} g_j^i\big(\boldsymbol{\ell}(y)\big) y^{\beta_j^i} = o(x^{\beta_n^i}),\ n \in \mathbb{N},\ i = 0, 1,$$

$$h(y) - a y^{\gamma_1} - \sum_{j=2}^{n} h_j\big(\boldsymbol{\ell}(y)\big) y^{\gamma_j} = o(y^{\gamma_n}),\ y \to 0,\ n \in \mathbb{N}. \tag{10.6}$$

It is easy to see, with $\widehat{h}$ and h as above, that:

$$\boldsymbol{\ell}(e^{-\frac{\gamma}{y}} \widehat{h}(y)) = \frac{y}{\gamma} + \widehat{k}(y), \tag{10.7}$$

as well as that

$$\boldsymbol{\ell}(e^{-\frac{\gamma}{y}} h(y)) = \frac{y}{\gamma} + k(y),$$

where $\widehat{k} \in \widehat{\mathcal{L}}_1$ is a convergent transseries with the sum $k \in \mathcal{G}$. In particular,

$$\widehat{k}(y) = cy^{\delta_0} + \sum_{i=2}^{\infty} \widehat{k}_i(\boldsymbol{\ell}(y))y^{\delta_i},\ k(y) - cy^{\delta_0} - \sum_{i=2}^{n} k_i(\boldsymbol{\ell}(y))y^{\delta_i} = o(y^{\delta_n}),\ n \in \mathbb{N},$$

where $(\delta_i)_i$ are strictly increasing to $+\infty$, $\delta_0 > 1$, $c \in \mathbb{R}$, and $\widehat{k}_i$ are (convergent) power asymptotic expasions of $k_i \in \mathcal{G}$, $i \in \mathbb{N}$.

Now suppose that $\widehat{f}(y) = \sum_{k=N}^{\infty} a_k y^k \in \widehat{\mathcal{L}}_0^I$ from (3.7) with integral factor α and $\widehat{R} \in \widehat{\mathcal{L}}_0^{\infty}$ convergent Laurent series, see (3.5):

$$\frac{d}{dy}(y^{\alpha}\widehat{f}\big(\boldsymbol{\ell}(y)\big) = y^{\alpha-1}\widehat{R}\big(\boldsymbol{\ell}(y)\big).$$

Let f be its sum as in (3.6). Then, by Proposition 10.3, f admits $\widehat{f}$ as its power asympotic expansion. Moreover, differentiating (3.5) and (3.6) and since $\widehat{R}$ is a convergent Laurent series, inductively it follows that $f^{(k)}$ admits the formal derivative $\widehat{f^{(k)}}$, $k \in \mathbb{N}$, as its power asymptotic expansion. Indeed, inductively, $\widehat{f}^{(k)}$ is a finite combination of $\widehat{f}$, $\widehat{R}$, and the formal derivatves $\widehat{R}'$, …, $\widehat{R}^{(k-1)}$, $k \in \mathbb{N}$, and the same combination holds for the formal counterparts.

Therefore, using (10.7), we have the following Taylor expansions (formal and on germs):

$$\widehat{f}\big(\boldsymbol{\ell}(e^{-\frac{\gamma}{y}}\widehat{h}(y))\big) = \widehat{f}\big(\frac{y}{\gamma}\big) + \widehat{f}'\big(\frac{y}{\gamma}\big)\widehat{k}(y) + \frac{1}{2!}\widehat{f}''\big(\frac{y}{\gamma}\big)\widehat{k}(y)^2 + \ldots, \tag{10.8}$$

$$f\big(\boldsymbol{\ell}(e^{-\frac{\gamma}{y}}h(y))\big) = f\big(\frac{y}{\gamma}\big) + f'\big(\frac{y}{\gamma}\big)k(y) + \frac{1}{2!}f''\big(\frac{y}{\gamma}\big)k(y)^2 + \ldots. \tag{10.9}$$

Combining (10.5) with (10.8), as well as on the other hand (10.6) with (10.9), we conclude (10.3) formally for $\widehat{F} \in \widehat{\mathcal{L}}^I$ and analogously (10.4) for its sum F. Here, $g_i \in \mathcal{G}$ are exactly the sums of convergent $\widehat{g}_i$, $i \in \mathbb{N}$, since they are given as the same finite combinations of convergent series, respectively their sums.

The integral sections are coherent and the convergent $\widehat{g}_i$ are trivially the power asymptotic expansions of g_i. Therefore, using the Poincaré algorithm and the coherence of integral sections to uniquely surpass the limit ordinals, it easily follows that $\widehat{F}$ is the asymptotic expansion of F with respect to any integral section. □

Proposition 10.5 (Uniqueness of the integral sum of integrally summable transseries in $\mathcal{L}_1^I$)**.** *Let $\widehat{F} \in \widehat{\mathcal{L}}_1^I$. Let*

$$\widehat{F}(y) = \widehat{G}_1(y) \cdot \widehat{f}\big(\boldsymbol{\ell}(e^{-\frac{\gamma}{y}}\widehat{h}_1(y))\big) + \widehat{G}_0(y) \tag{10.10}$$

be its one decomposition of the form (3.7) *(not necessarily unique). Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, be the exponent of integration of $\widehat{f}$. If $\alpha < 0$, then the integral sum F corresponding to this decomposition is unique up to an additive term $cG_1(y) \cdot \big(e^{-\frac{\gamma}{y}}h_1(y)\big)^{-\alpha}$, $c \in \mathbb{R}$. Here, h_1 is the sum of $\widehat{h}_1$. If $\alpha > 0$, the integral sum F is unique.*

Morever, let

$$\widehat{F}(y) = \widehat{H}_1(y) \cdot \widehat{g}\big(\boldsymbol{\ell}(e^{-\frac{\delta}{y}}\widehat{h}_2(y))\big) + \widehat{H}_0(y) \tag{10.11}$$

be another decomposition (3.7) *of the same $\widehat{F}$, with $\beta \neq 0$ the exponent of integration of $\widehat{g}$ (not necessarily equal to α). Then its sum is again equal to F, up to the same additive term $cG_1(y) \cdot \big(e^{-\frac{\gamma}{y}}h_1(y)\big)^{-\alpha}$, $c \in \mathbb{R}$.*

Proof of Proposition 10.5.

The first statement of the proposition follows directly from Definition 3.13 of the integral sum and Remark 3.12. Therefore, the integral sum corresponding to decomposition (10.10) is unique up to $cG_1(y)\cdot\big(e^{-\frac{\gamma}{y}}h_1(y)\big)^{-\alpha}$, $c\in\mathbb{R}$ (supposing $\alpha<0$), and the integral sum corresponding to decomposition (10.11) is unique up to $dH_1(y)\cdot\big(e^{-\frac{\delta}{y}}h_2(y)\big)^{-\beta}$, $d\in\mathbb{R}$ (supposing $\beta<0$). We show below that if (10.10) and (10.11) are two decompositions of the same $\widehat F$, then formally in $\widehat{\mathcal{L}}_2^\infty$:

$$\frac{\widehat H_1(\boldsymbol{\ell})}{\widehat G_1(\boldsymbol{\ell})}\cdot\frac{(x^\gamma\widehat h_1(\boldsymbol{\ell}))^\alpha}{(x^\delta\widehat h_2(\boldsymbol{\ell}))^\beta}=C,\ C\in\mathbb{R}, \tag{10.12}$$

$$\frac{\widehat H_1(\boldsymbol{\ell})}{\widehat G_1(\boldsymbol{\ell})}\cdot\frac{(\widehat h_1(\boldsymbol{\ell}))^\alpha}{(\widehat h_2(\boldsymbol{\ell}))^\beta}=Cx^{\delta\beta-\gamma\alpha},\ C\in\mathbb{R}.$$

It follows that $\gamma\alpha=\delta\beta$. Moreover, since $\widehat H_1,\ \widehat G_1,\ \widehat h_1,\ \widehat h_2\in\widehat{\mathcal{L}}_1^\infty$ are convergent, their sums are unique, and it follows that:

$$\frac{H_1(y)}{G_1(y)}\cdot\frac{(h_1(y))^\alpha}{(h_2(y))^\beta}=C,\ C\in\mathbb{R}. \tag{10.13}$$

Consequently, the additive term in the integral sum of $\widehat F\in\widehat{\mathcal{L}}_1^I$ is the same for all decompositions of $\widehat F$.

Now, since both decompositions (10.10) and (10.11) represent the same $\widehat F$ and since $\widehat H_1(\boldsymbol{\ell})=C\widehat G_1(\boldsymbol{\ell})\frac{(x^\delta\widehat h_2(\boldsymbol{\ell}))^\beta}{(x^\gamma\widehat h_1(\boldsymbol{\ell}))^\alpha}$, we get:

$$\widehat f\big(\boldsymbol{\ell}(x^\gamma\widehat h_1(\boldsymbol{\ell}))\big)-C\frac{(x^\delta\widehat h_2(\boldsymbol{\ell}))^\beta}{(x^\gamma\widehat h_1(\boldsymbol{\ell}))^\alpha}\widehat g\big(\boldsymbol{\ell}(x^\delta\widehat h_2(\boldsymbol{\ell}))\big)=\frac{\widehat H_0(\boldsymbol{\ell})-\widehat G_0(\boldsymbol{\ell})}{\widehat G_1(\boldsymbol{\ell})}. \tag{10.14}$$

Multiply (10.14) by $\big(x^\gamma\widehat h_1(\boldsymbol{\ell})\big)^\alpha\in\widehat{\mathcal{L}}_2^\infty$ and differentiate formally by $\frac{d}{dx}$ (in $\widehat{\mathcal{L}}_2^\infty$). By Definition 3.10, $\frac{d}{dx}\big(x^\alpha\widehat f(\boldsymbol{\ell})\big)=x^{\alpha-1}\widehat R_1(\boldsymbol{\ell})$, and $\frac{d}{dx}\big(x^\beta\widehat g(\boldsymbol{\ell})\big)=x^{\beta-1}\widehat R_2(\boldsymbol{\ell})$, where $\widehat R_1,\ \widehat R_2\in\widehat{\mathcal{L}}_0^\infty$ are convergent Laurent series. We get formally in $\widehat{\mathcal{L}}_2^\infty$:

$$\begin{aligned}&\big(x^\gamma\widehat h_1(\boldsymbol{\ell})\big)^{\alpha-1}\widehat R_1\big(\boldsymbol{\ell}(x^\gamma\widehat h_1(\boldsymbol{\ell}))\big)\cdot\frac{d}{dx}\boldsymbol{\ell}(x^\gamma\widehat h_1(\boldsymbol{\ell}))-\\&-C\big(x^\delta\widehat h_2(\boldsymbol{\ell})\big)^{\beta-1}\widehat R_2\big(\boldsymbol{\ell}(x^\delta\widehat h_2(\boldsymbol{\ell}))\big)\cdot\frac{d}{dx}\boldsymbol{\ell}(x^\delta\widehat h_2(\boldsymbol{\ell}))=\frac{d}{dx}\Big(\frac{\widehat H_0(\boldsymbol{\ell})-\widehat G_0(\boldsymbol{\ell})}{\widehat G_1(\boldsymbol{\ell})}\cdot\big(x^\gamma\widehat h_1(\boldsymbol{\ell})\big)^\alpha\Big)\end{aligned}$$

That is, differentiating and grouping the terms and using the fact that $\gamma\alpha=\delta\beta$:

$$\begin{aligned}&x^{\gamma\alpha-1}\Big(\widehat h_1(\boldsymbol{\ell})^{\alpha-1}\cdot\widehat R_1\big(\boldsymbol{\ell}(x^\gamma\widehat h_1(\boldsymbol{\ell}))\big)\cdot\boldsymbol{\ell}^2(x^\gamma\widehat h_1(\boldsymbol{\ell}))\cdot\big(\gamma\widehat h_1(\boldsymbol{\ell})+\widehat h_1'(\boldsymbol{\ell})\boldsymbol{\ell}^2\big)-\\&-\widehat h_2(\boldsymbol{\ell})^{\beta-1}\cdot\widehat R_2\big(\boldsymbol{\ell}(x^\beta\widehat h_2(\boldsymbol{\ell}))\big)\cdot\boldsymbol{\ell}^2(x^\beta\widehat h_2(\boldsymbol{\ell}))\cdot\big(\beta\widehat h_2(\boldsymbol{\ell})+\widehat h_2'(\boldsymbol{\ell})\boldsymbol{\ell}^2\big)\Big)=\\&=x^{\gamma\alpha-1}\Big(\widehat h_1(\boldsymbol{\ell})^\alpha\cdot\boldsymbol{\ell}^2\cdot\frac{d}{d\boldsymbol{\ell}}\big(\frac{\widehat H_0(\boldsymbol{\ell})-\widehat G_0(\boldsymbol{\ell})}{\widehat G_1(\boldsymbol{\ell})}\big)+\\&\qquad+\alpha\widehat h_1(\boldsymbol{\ell})^{\alpha-1}\cdot\big(\gamma\widehat h_1(\boldsymbol{\ell})+\widehat h_1'(\boldsymbol{\ell})\cdot\boldsymbol{\ell}^2\big)\cdot\frac{\widehat H_0(\boldsymbol{\ell})-\widehat G_0(\boldsymbol{\ell})}{\widehat G_1(\boldsymbol{\ell})}\Big).\end{aligned}$$

Since $\widehat h_{1,2},\ \widehat G_{0,1},\ \widehat H_0$ and their first derivatives are *convergent* in $\widehat{\mathcal{L}}_1^\infty$ and their derivatives commute with the sums, the terms in brackets on both sides are *convergent transseries* in $\widehat{\mathcal{L}}_1^\infty$. We may *remove the hats* and get the following analogon of (10.14) in $\mathcal{G}_{AN}$:

(10.15)

$$\big(x^\gamma h_1(\boldsymbol{\ell})\big)^{\alpha-1} R_1\big(\boldsymbol{\ell}(x^\gamma h_1(\boldsymbol{\ell}))\big)\cdot\frac{d}{dx}\boldsymbol{\ell}(x^\gamma h_1(\boldsymbol{\ell}))-$$
$$-\,C\big(x^\delta h_2(\boldsymbol{\ell})\big)^{\beta-1}R_2\big(\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))\big)\cdot\frac{d}{dx}\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))=\frac{d}{dx}\Big(\frac{H_0(\boldsymbol{\ell})-\widehat{G}_0(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}\cdot\big(x^\gamma h_1(\boldsymbol{\ell})\big)^\alpha\Big).$$

Now integrate (10.15) in $\mathcal{G}_{AN}$ with respect to $\int d(x^\gamma h_1(\boldsymbol{\ell}))$. Note that $\frac{d(x^\gamma h_1(\boldsymbol{\ell}))}{dx}=x^{\gamma-1}\big(\gamma h_1(\boldsymbol{\ell})+h_1(\boldsymbol{\ell})\boldsymbol{\ell}^2\big)$. By Definition 3.10,

$$f(\boldsymbol{\ell}(x^\gamma h_1))(x^\gamma h_1)^\alpha=\Big(\int\frac{d}{dx}\big(\widehat{f}(\boldsymbol{\ell})x^\alpha\big)\,dx\Big)\Big|_{x=x^\gamma h_1(\boldsymbol{\ell})},$$
$$g(\boldsymbol{\ell}(x^\delta h_2))(x^\delta h_2)^\beta=\Big(\int\frac{d}{dx}\big(\widehat{g}(\boldsymbol{\ell})x^\beta\big)\,dx\Big)\Big|_{x=x^\delta h_2(\boldsymbol{\ell})},$$

and the integral sums f and g are unique up to an additive constant of integration. Since $DG_1(x^\gamma h_1)^\alpha=H_1(x^\delta h_2)^\beta$, $D\in\mathbb{R}$, for any two integral sums f and g of $\widehat{f}$ resp. $\widehat{g}$ it holds that:

(10.16)
$$f(\boldsymbol{\ell}(x^\gamma h_1))-C\frac{(x^\delta h_2)^\beta}{(x^\gamma h_1)^\alpha}g(\boldsymbol{\ell}(x^\delta h_2))=\frac{H_0(\boldsymbol{\ell})-G_0(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}+D\big(x^\gamma h_1(\boldsymbol{\ell})\big)^\alpha,\ D\in\mathbb{R}.$$

By (10.10), (10.11) and (10.16), the integral sum of $\widehat{F}$ is obviously unique up to $DG_1(\boldsymbol{\ell})(x^\gamma h_1(\boldsymbol{\ell}))^\alpha$, $D\in\mathbb{R}$. Note that this term is not dependent on decomposition, and that $G_1,\ \alpha,\ \gamma,\ h_1$ are elements of an *arbitrary chosen* decomposition. Q.E.D.

It is only left to prove (10.12). We have (formally in $\widehat{\mathcal{L}}_1^\infty$):

$$\widehat{F}(\boldsymbol{\ell})=\widehat{G}_1(\boldsymbol{\ell})\cdot\widehat{f}\big(\boldsymbol{\ell}(x^\gamma\widehat{h}_1(\boldsymbol{\ell})))\big)+\widehat{G}_0(\boldsymbol{\ell})=\widehat{H}_1(\boldsymbol{\ell})\cdot\widehat{g}\big(\boldsymbol{\ell}(x^\delta\widehat{h}_2(\boldsymbol{\ell})))\big)+\widehat{H}_0(\boldsymbol{\ell}).$$

First we prove that it is not possible that $\widehat{G}_1=0$ and $\widehat{H}_1\neq 0$. In that case, we would have that $\widehat{g}\big(\boldsymbol{\ell}(x^\delta\widehat{h}_2(\boldsymbol{\ell}))\big)$ is convergent in $\widehat{\mathcal{L}}_1^\infty$ as a quotient of convergent transseries in $\widehat{\mathcal{L}}_1^\infty$ (where we denote the convergent transseries without hats):

$$\widehat{g}\big(\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))\big)=\frac{G_0(\boldsymbol{\ell})-H_0(\boldsymbol{\ell})}{H_1(\boldsymbol{\ell})}. \tag{10.17}$$

Note that $\widehat{g}$ is divergent close to 0. Take $\boldsymbol{\ell}$ sufficiently small so that the convergent transseries on the right-hand side, as well as $\widehat{h}_2(\boldsymbol{\ell})$, evaluated at $\boldsymbol{\ell}$, converge. On the other hand, since $\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))=\boldsymbol{\ell}(1+o(1))$, $\boldsymbol{\ell}\to 0$, by taking $\boldsymbol{\ell}$ sufficiently small, we may ensure that $\widehat{g}$ evaluated at $\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))$ diverges. This is a contradiction with the equality (10.17). Therefore, either both $\widehat{G}_1$ and $\widehat{H}_1$ are zero or both are different from zero. In the first case it trivially follows that $G_0\equiv H_0$ (both are convergent) and the decompositions (10.10) and (10.11) are exactly the same.

Suppose now without loss of generality that $\widehat{G}_1\neq 0$. In the rest of the proof, without hats we will denote *convergent* transseries in $\widehat{\mathcal{L}}_1^\infty$. Dividing both sides of the equality by G_1, we get (formally in $\widehat{\mathcal{L}}_2^\infty$):

$$\widehat{f}(\boldsymbol{\ell}(x^\gamma h_1(\boldsymbol{\ell})))-\frac{H_1(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}\widehat{g}(\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell})))=\frac{H_0(\boldsymbol{\ell})-G_0(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}.$$

Multiplying by $(x^\gamma h_1(\boldsymbol{\ell}))^\alpha \in \widehat{\mathcal{L}}_2^\infty$ and differentiating formally by $\frac{d}{dx}$, we get (in $\widehat{\mathcal{L}}_2^\infty$):

$$\frac{d}{dx}\Big((x^\gamma h_1(\boldsymbol{\ell}))^\alpha \widehat{f}\big(\boldsymbol{\ell}(x^\gamma h_1(\boldsymbol{\ell}))\big)\Big) - \frac{d}{dx}\Big(\widehat{g}\big(\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))\big)(x^\delta h_2(\boldsymbol{\ell}))^\beta \cdot \frac{H_1(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}\frac{(x^\gamma h_1(\boldsymbol{\ell}))^\alpha}{(x^\delta h_2(\boldsymbol{\ell}))^\beta}\Big)$$
$$= \frac{d}{dx}\Big(\frac{H_0(\boldsymbol{\ell}) - G_0(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}(x^\gamma h_1(\boldsymbol{\ell}))^\alpha\Big).$$

Differentiating and dividing by $x^{\alpha\gamma-1}$ and re-grouping the convergent transseries, similarly as above, we get that

$$\widehat{g}\big(\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell}))\big)h_2(\boldsymbol{\ell})^\beta \cdot \frac{\frac{d}{dx}\Big(\frac{H_1(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}\frac{(x^\gamma h_1(\boldsymbol{\ell}))^\alpha}{(x^\delta h_2(\boldsymbol{\ell}))^\beta}\Big)}{x^{\gamma\alpha-1-\delta\beta}}$$

is a convergent transseries in $\widehat{\mathcal{L}}_1^\infty$. If the derivative in the brackets is different from 0, it follows that $\widehat{g}(\boldsymbol{\ell}(x^\delta h_2(\boldsymbol{\ell})))$ is a convergent transseries in $\widehat{\mathcal{L}}_1^\infty$. This is a contradiction, as already explained in detail above. Therefore, it necessarily holds that the derivative is 0, that is, that:

$$\frac{H_1(\boldsymbol{\ell})}{G_1(\boldsymbol{\ell})}\frac{(x^\gamma h_1(\boldsymbol{\ell}))^\alpha}{(x^\delta h_2(\boldsymbol{\ell}))^\beta} = C,\ C \in \mathbb{R}.$$

□

10.2. **Proofs of propositions from Section 4.**

Proof of Proposition 4.3.

The chain rule. As a prerequisite for the proof, we prove that the chain rule is valid in our formal setting. That is, if $\widehat{\Psi} \in \widehat{\mathcal{L}}_2^\infty$ and $\{\widehat{f}^t\}$, $\widehat{f}^t \in \widehat{\mathcal{L}}$ is a C^1-flow as defined in [14, Def.1.2], then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\widehat{\Psi}(\widehat{f}^t(x))\big) = \widehat{\Psi}'\big(\widehat{f}^t(x)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x)$$

holds formally in $\widehat{\mathcal{L}}_2^\infty$. Here, $\frac{\mathrm{d}}{\mathrm{d}t}$ applied to a transseries means the derivation coefficient by coefficient. Since $\widehat{f}^t(x) = x + \text{h.o.t.}$ with coefficients in $C^1(\mathbb{R})$, it stems from *Neuman's Lemma* (see [1]) that the coefficients of $\widehat{\Psi}(\widehat{f}^t(x))$ also belong to $\mathcal{C}^1(\mathbb{R})$.

It is sufficient to prove the equality in $\widehat{\mathcal{L}}_2^\infty$ for a single monomial $m(x)$ from $\mathcal{S}(\widehat{\Psi})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m(\widehat{f}^t(x))) = m'(\widehat{f}^t(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x).$$

The both sides share the common well-ordered support. Take any monomial from this support. By *Neumann's lemma*, on both sides only finitely many monomials from $\widehat{f}^t$ contribute to it. Now the equality holds if we replace $\widehat{f}^t$ by the finite sum of its terms corresponding to these first monomials. Therefore, the coefficients of every monomial on both sides coincide.

The existence. Take $\widehat{\Psi}$ to be the formal antiderivative of $1/\widehat{\xi}$, where

$$\widehat{\xi} := \frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t\Big|_{t=0}.$$

We prove that $\widehat{\Psi}$ satisfies the equation (2.6) for the formal Fatou coordinate. Integrating formally $1/\widehat{\xi}$ (every monomial is formally integrated by parts), we conclude that $\widehat{\Psi}\in\widehat{\mathcal{L}}_2^\infty$. Indeed, since $\widehat{\xi}\in\widehat{\mathcal{L}}$, we get that $1/\widehat{\xi}\in\widehat{\mathcal{L}}^\infty$. In the integration process, the double logarithm $\boldsymbol{\ell}_2^{-1}$ is generated integrating the monomial $x^{-1}\boldsymbol{\ell}$.

Using $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t=\widehat{\xi}(\widehat{f}^t)$, by the chain rule proved above, we get that

$$\frac{d}{dt}\big(\widehat{\Psi}(\widehat{f}_t(x))\big)=\widehat{\Psi}'(\widehat{f}^t(x))\cdot\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x)=\frac{1}{\widehat{\xi}(\widehat{f}^t(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}^t(x)=1.$$

Integrating this equality from 0 to t gives the equality (4.1). In particular, $\widehat{\Psi}(\widehat{f}(x))-\widehat{\Psi}(x)=1$.

The uniqueness. Suppose that there exist two formal Fatou coordinates $\widehat{\Psi}_1,\ \widehat{\Psi}_2\in\widehat{\mathfrak{L}}$. Let $\widehat{\Psi}:=\widehat{\Psi}_1-\widehat{\Psi}_2$. Then $\widehat{\Psi}\in\widehat{\mathfrak{L}}$, that is, $\widehat{\Psi}\in\widehat{\mathcal{L}}_j^\infty$ for some $j\in\mathbb{N}_0$, and it satisfies:

$$\widehat{\Psi}(\widehat{f}(x))-\widehat{\Psi}(x)=0.$$

Since $\widehat{f}\in\widehat{\mathcal{L}}$, by Taylor expansion in $\widehat{\mathcal{L}}_j^\infty$, we get:

$$\widehat{\Psi}'\cdot\widehat{g}+\frac{1}{2!}\widehat{\Psi}''\cdot\widehat{g}^2+\cdots=0.$$

If $\widehat{\Psi}'\neq 0$ in $\widehat{\mathcal{L}}_j^\infty$, since $\operatorname{ord}(\widehat{g})\succ(1,0)$, the leading term of the left-hand side is the non-zero leading term of $\widehat{\Psi}'\cdot\widehat{g}$, which is a contradiction. Therefore, $\widehat{\Psi}'=0$, so $\widehat{\Psi}=C,\ C\in\mathbb{R}$. $\square$

Proof of Proposition 4.4. We prove both directions.

1. Suppose that an analytic Fatou coordinate Ψ for f exists and is strictly monotone on $(0,d)$. Then one easily sees that the family $\{f^t\}$ of analytic germs on $(0,d)$, $d>0$, defined by:

$$f^t(x):=\Psi^{-1}\big(\Psi(x)+t\big),\ t\in\mathbb{R}, \tag{10.18}$$

form a C^1-flow in which f embeds as the time-one map.

Furthermore,

$$\xi:=\frac{\mathrm{d}}{\mathrm{d}t}f^t\Big|_{t=0}=\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{-1}\big(\Psi(x)+t\big)\Big|_{t=0}=\frac{1}{\Psi'}.$$

Since Ψ is a strictly monotone germ, either $\Psi'>0$ or $\Psi'<0$ in some interval $(0,d)$, so ξ is non-oscillatory in $(0,d)$.

2. The vector field whose flow is given by the C^1-family $\{f^t\}$, $t\in\mathbb{R}$, of analytic germs on $(0,d)$, is given by the formula:

$$X=\xi(x)\frac{\mathrm{d}}{\mathrm{d}x},\ \text{where }\xi:=\frac{\mathrm{d}}{\mathrm{d}t}f^t\Big|_{t=0}.$$

Obviously, ξ is also analytic on $(0,d)$. Take Ψ to be the antiderivative of $1/\xi$. That is, $\Psi'=\frac{1}{\xi}$. We prove that Ψ is a Fatou coordinate for f, that is, satisfies (2.5). We solve the differential equation for the flow:

$$\begin{aligned}\dot{x}&=\xi(x)\\ \frac{\mathrm{d}x}{\xi(x)}&=\mathrm{d}t,\\ t&=\int_x^{f^t(x)}\frac{ds}{\xi(s)}.\end{aligned}$$

We get that

$$\Psi(f^t(x)) - \Psi(x) = t,\ t \in \mathbb{R}.$$

In particular, $\Psi(f(x)) - \Psi(x) = 1$.

Moreover, since $\Psi' = \frac{1}{\xi}$, and ξ does not change sign in some interval $(0, d)$, Ψ is strictly monotone in the same interval. □

Remark 10.6 (The importance of *non-oscillatority* in Proposition 4.4)**.** Consider the flow $\{f^t\}_t$ of an analytic vector field $X = \xi \frac{\mathrm{d}}{\mathrm{dx}}$ on $(0, d)$. Take a non-singular point $x_0 > 0$ of the vector field ($\xi(x_0) \neq 0$). Then, by (4.2), the Fatou coordinate Ψ_{x_0} is defined at a point x as the time $t \in \mathbb{R}$ such that $f^t(x_0) = x$. In particular, $\Psi_{x_0}(x_0) = 0$. Obviously, Ψ_{x_0} cannot be defined at any singular (equilibrium) point of vector field ξ.

For example, the flow $\{f^t\}_{t\in\mathbb{R}}$ of the analytic vector field $\xi = x^2 \sin(1/x) \frac{\mathrm{d}}{\mathrm{d}x}$ on $(0, d)$ consists of analytic maps on $(0, d)$. But, as the vector field ξ admits infinitely many singular points which accumulate at the origin, we cannot define a Fatou coordinate on any interval $(0, d_1)$.

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