MINKOWSKI MEASURABILITY CRITERIA FOR COMPACT SETS AND RELATIVE FRACTAL DRUMS IN EUCLIDEAN SPACES

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Abstract. We establish a Minkowski measurability criterion for a large class of relative fractal drums (or, in short, RFDs), in Euclidean spaces of arbitrary dimension in terms of their complex dimensions, which are defined as the poles of their associated fractal zeta functions. Relative fractal drums represent a far-reaching generalization of bounded subsets of Euclidean spaces as well as of fractal strings studied extensively by the first author and his collaborators. In fact, the Minkowski measurability criterion established here is a generalization of the corresponding one obtained for fractal strings by the first author and M. van Frankenhuijsen. Similarly as in the case of fractal strings, the criterion established here is formulated in terms of the locations of the principal complex dimensions associated with the relative drum under consideration. The criterion itself is obtained by an application of the Wiener–Pitt Tauberian theorem to the corresponding fractal zeta function on one side and by an application of the Uniqueness theorem for almost periodic distributions on the other side. These complex dimensions are defined as poles or, more generally, singularities of the corresponding distance (or tube) zeta function. We also reflect on the notion of gauge-Minkowski measurability of RFDs and establish several results connecting it to the nature and location of the complex dimensions. (This is especially useful when the underlying scaling does not follow a classic power law.) We illustrate our results and their applications by means of a number of interesting examples.

Contents

2010 *Mathematics Subject Classification.* Primary: 11M41, 28A12, 28A75, 28A80, 28B15, 42B20, 44A05. Secondary: 35P20, 40A10, 42B35, 44A10, 45Q05.

Key words and phrases. Mellin transform, complex dimensions of a relative fractal drum, relative fractal drum, fractal set, box dimension, fractal zeta functions, distance zeta function, tube zeta function, fractal string, Minkowski content, Minkowski measurability criterion, Minkowski measurable set, residue, meromorphic extension, gauge-Minkowski measurability, singularities of fractal zeta functions.

The research of Michel L. Lapidus was partially supported by the National Science Foundation under grants DMS-0707524 and DMS-1107750, as well as by the Institut des Hautes Études Scientifiques (IHÉS) where the first author was a visiting professor in the Spring of 2012 while part of this research was completed. The research of Goran Radunović and Darko Žubrinić was supported in part by the Croatian Science Foundation under the project IP-2014-09-2285 and by the Franco-Croatian PHC-COGITO project.

1. Introduction

For a nonempty bounded subset A of the N-dimensional Euclidean space $\mathbb{R}^N$, with $N \geq 1$, the attribute 'fractal' is most readily associated with the notion of fractal dimension. As is well known, there are several different notions of fractal dimension, e.g., Hausdorff dimension, packing dimension, etc. (see [Fal1]) but not any single one of them encompasses all of the sets that we would like to call fractal. One only has to recall the example of the famous 'devil's staircase'; namely, the graph of the Cantor function, for which all of the (known) fractal dimensions are trivial (more specifically, are equal to 1). Nevertheless, one glance at the Cantor graph suffices to make us want to call it "fractal".

The development of the higher-dimensional theory of complex dimensions in [LapRaŽu1–8] provides us with a new tool which can be used, among many other things, to reveal some of the elusive 'fractality' of sets for which the other well known fractal dimensions fail. The complex dimensions of a relative fractal drum (and, in particular, of a bounded subset) are defined as the multiset of poles (or more general singularities) of the associated fractal distance (or tube) zeta function. These complex dimensions also generalize the notion of the Minkowski (or box) dimension of a given relative fractal drum. Actually, even more importantly, they play a major role in determining the asymptotics of the volume of the t-neighborhoods of the given RFD (or bounded set) as $t \to 0^+$ and therefore can be considered as a footprint of its inner geometry. (See [LapRaŽu8] and [LapRaŽu1, Chapter 5].)

Of course, being a generalization of the Minkowski dimension, the complex dimensions should also be connected to the property of Minkowski measurability. Showing this in an explicit manner is one of the main goals of the present paper. Indeed, the main result of this paper states that under suitable hypotheses, A is Minkowski measurable if and only if the subset of complex dimensions of A which have real part equal to D, the Minkowski dimension of A, consists only of D itself, and D is simple. Furthermore, in that case, the value of the Minkowski content of A is directly connected to the residue of its distance (or tube) zeta function at D. (See Theorem 3.1 for a precise statement of this result, along with Theorems 4.14 and 4.16 for a detailed proof.)

Recall that the value of the Minkowski content of a bounded subset A of $\mathbb{R}^N$ can be used as one of the equivalent ways of defining the Minkowski dimension. More precisely, for a bounded subset A of $\mathbb{R}^N$ and $0 \leq r \leq N$, we denote the

r-dimensional Minkowski content of A by

$$\mathcal{M}^r(A) := \lim_{\delta\to 0^+} \frac{|A_\delta|}{\delta^{N-r}}, \tag{1.1}$$

whenever this limit exists as a value in $[0,+\infty]$. Here, $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$ and

$$A_\delta := \{x \in \mathbb{R}^N : d(x,A) < \delta\} \tag{1.2}$$

is the *δ-neighborhood* (or the *δ-parallel set*) of A, with $d(x,A) := \inf\{|x-a| : a \in A\}$ denoting the Euclidean distance from $x \in \mathbb{R}^N$ to A. The set A is said to be Minkowski measurable (of dimension r) if $\mathcal{M}^r(A)$ exists and satisfies $0 < \mathcal{M}^r(A) < \infty$. Then, necessarily, r coincides with the Minkowski dimension of A.

It has been of considerable interest in the past to determine whether or not a set A is Minkowski measurable. One of the motivations can be found in Mandelbrot's suggestion in [Man2] to use the Minkowski content as a characteristic for the texture of sets (see also [Man1, §X]). Mandelbrot called the quantity $1/\mathcal{M}^r(A)$ the *lacunarity* of the set A and made the heuristic observation that for subsets of $\mathbb{R}^N$, small lacunarity corresponds to the spatial homogeneity of the set, which means that the set has small, uniformly distributed holes. On the other hand, large lacunarity corresponds to the clustering of the set and to large holes between different clusters. More information on this subject can be found in [BedFi, Fr, Lap-vFr1] and in [Lap-vFr2, §12.1.3].

More directly relevant to our present work, prior to that, a lot of attention was devoted to the notion of Minkowski content in connection to the (modified) Weyl–Berry conjecture [Lap1] which relates the spectral asymptotics of the Laplacian on a bounded open set and the Minkowski content of its boundary. In dimension one, that is, for fractal strings (i.e., for one-dimensional drums with fractal boundary), this conjecture was resolved affirmatively in [LapPo1].[1] A crucial part of this result was the characterization of Minkowski measurability of bounded subsets of $\mathbb{R}$ obtained in [LapPo1].[2] In particular, this led to a useful reformulation of the Riemann hypothesis in terms of an inverse spectral problem for fractal strings; see [LapMa]. See the formulation given in [Lap1] and which was proved for subsets of $\mathbb{R}$ in 1993 by the first author and C. Pomerance in [LapPo1].

Our motivations for introducing relative fractal drums (in [LapRaŽu4] and [LapRaŽu1, Ch. 4]) and stating the results of this paper in such a generality are two-fold. They are partly of a practical nature and are also partly due to the emergence of new and interesting phenomena arising in this general context, such as the possibility of negative Minkowski dimensions and of studying the local properties of fractals via the associated fractal zeta functions and their poles, called complex dimensions. The practicality of working with relative fractal drums (RFDs, in short) reflects itself in the fact that it provides us with a unified approach to both the higher-dimensional theory of complex dimensions and to the well known theory of complex dimensions

[1] For the original Weyl–Berry conjecture and its physical applications see Berry's papers [Berr1–2]. Furthermore, early mathematical work on this conjecture and its applications can be found in [BroCar, Lap1–2, Lap4, LapPo1–2, FlVa] For a more extensive list of later work, see [Lap-vFr2, §12.5].

[2] A new proof of a part of this result was given in [Fal2] and more recently, in [RatWi].

for fractal strings. Furthermore, RFDs provide us with a more elegant way of computing some of the fractal zeta functions for bounded sets by subdividing them into appropriate relative fractal subdrums.

In closing the main part of this introduction, we mention that works involving the notion of Minkowski measurability (or of Minkowski content) include [BroCar, CaLapPe-vFr, Fal1–2, Fed2, FlVa, Gat, HamLap, HeLap, HerLap, KeKom, Kom, KomPeWi, Lap1–4, LapLu, LapLu-vFr1–2, LapMa, LapPe1–2, LapPeWi1–2, LapPo1–2, LapRaŽu1–8, Lap-vFr1–2, Man2, Pe, PeWi, Ra1–2, RatWi, Res, Sta, Tri1–2, Wi, WiZä, Zä3, Žu] and the many relevant references therein.

Furthermore, we refer to the end of §2 for references on tube formulas (including, especially, fractal tube formulas) which play a key role in the proofs of our main results.

The plan of this paper is as follows:

In §2, we recall some of the main definitions, including the notion of RFD, as well as of the associated fractal zeta functions (i.e., the distance and tube zeta functions) and their poles in a suitable region (i.e., the visible complex dimensions). We also recall the notion of (relative) Minkowski dimension and content.

In §3, we state our main result (Theorem 3.1) according to which, under suitable assumptions, an RFD is Minkowski measurable if and only if the only complex dimension with real part D (the Minkowski dimension of the RFD) is D itself, and it is simple. We also recover earlier results about fractal strings obtained in [Lap-vFr2] (Corollary 3.3 and §3.2) and discuss a variety of higher-dimensional examples illustrating our main results; see §3.3, §3.4 and §3.5, respectively about the classic Sierpiński gasket and the 3-dimensional carpet, families of fractal nests and unbounded geometric chirps, and self-similar sprays (higher-dimensional analogs of self-similar strings, but of a more general nature than those considered in [LapPe2, LapPeWi1–2]).

In §4, we give a detailed proof of our main result (Theorem 3.1, restated more specifically in Theorems 4.14 and 4.16), the Minkowski measurability criterion for RFDs (and, in particular, for bounded sets) in $\mathbb{R}^N$, with $N \geq 1$ arbitrary. In the process, we state and establish new related results, including a sufficient condition for Minkowski measurability (Theorem 4.2, whose proof makes use of the Wiener–Pitt Tauberian theorem) and a necessary condition for Minkowski measurability (Theorem 4.11, whose proof makes use, in particular, of the distributional tube formula for RFDs obtained in [LapRaŽu8] and the Uniqueness theorem for almost periodic distributions).

Finally, in §5, we construct and study a family of h-Minkowski measurable RFDs with respect to a suitable (nontrivial) gauge function h and therefore not obeying the standard power scaling law. (See, especially, Theorems 5.4, 5.7 and 5.9.) In the process, we show, in particular, the hypotheses of some of our earlier results in [LapRaŽu3–4] about the existence of meromorphic continuations of fractal zeta functions (of bounded sets and RFDs in $\mathbb{R}^N$) are optimal.

2. Preliminaries

We begin by introducing some necessary definitions and results from [LapRaŽu4] (see also [LapRaŽu1]) that are needed in order to establish the main results of this paper. We will always assume throughout this article that all the sets A and Ω are nonempty, in order to avoid dealing with trivial cases.

Definition 2.1. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^N$, not necessarily bounded, but of finite N-dimensional Lebesgue measure (or "volume"). Furthermore, let $A \subseteq \mathbb{R}^N$, also possibly unbounded, be such that Ω is contained in A_δ for some $\delta > 0$. Then, the *distance zeta function* $\zeta_{A,\Omega}$ *of* A *relative to* Ω (or the *relative distance zeta function* of (A, Ω)) is defined by the following Lebesgue integral:

$$\zeta_{A,\Omega}(s) := \int_\Omega d(x, A)^{s-N} \mathrm{d}x, \tag{2.1}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

The ordered pair (A, Ω), appearing in Definition 2.1 is called a *relative fractal drum* or RFD, in short. We will also use the phrase *zeta functions of relative fractal drums* instead of relative zeta functions.

Remark 2.2. In the above definition, we may replace the domain of integration Ω in (2.1) with $A_\delta \cap \Omega$ for some fixed $\delta > 0$; that is, we may let

$$\zeta_{A,\Omega}(s; \delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N} \mathrm{d}x. \tag{2.2}$$

Indeed, the difference $\zeta_{A,\Omega}(s) - \zeta_{A,\Omega}(s; \delta)$ is then an entire function (see [LapRaŽu4]) so that the above change of the domain of integration does not affect the principal part of the distance zeta function in any way. Therefore, we can alternatively define the relative distance zeta function of (A, Ω) by (2.2), since in the theory of complex dimensions we are mostly interested in poles (or, more generally, in singularities) of meromorphic extensions of (various) fractal zeta functions. Then, in light of the principle of analytic continuation, the dependence of $\zeta_{A,\Omega}(\,\cdot\,; \delta)$ on δ is inessential.

The condition that $\Omega \subseteq A_\delta$ for some $\delta > 0$ is of a technical nature and ensures that the function $x \mapsto d(x, A)$ is bounded for all $x \in \Omega$. If Ω does not satisfy this condition, we can still use the alternative definition given by Equation (2.2).[3]

Remark 2.3. As was already pointed out in the introduction, the notion of a relative fractal drum generalizes the notion of a bounded subset. Indeed, any bounded subset A of $\mathbb{R}^N$ may be identified with the relative fractal drum (A, A_{δ_0}), for some fixed $\delta_0 > 0$, or even more practically, with (A, Ω), where Ω is any bounded open set containing A_δ for some $\delta > 0$. Of course, in light of Equation (2.1), one can then choose the most convenient Ω.

Entirely analogous remarks can be made about the tube zeta function of a relative fractal drum, which we now introduce.

Definition 2.4. Let (A, Ω) be an RFD in $\mathbb{R}^N$ and fix $\delta > 0$. We define the *tube zeta function* $\widetilde{\zeta}_{A,\Omega}(s; \delta)$ *of* A *relative to* Ω (or the *relative tube zeta function*) by the Lebesgue integral

$$\widetilde{\zeta}_{A,\Omega}(s; \delta) := \int_0^\delta t^{s-N-1} |A_t \cap \Omega| \, \mathrm{d}t, \tag{2.3}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. Here, $|A_t \cap \Omega| := |A_t \cap \Omega|_N$ denotes the N-dimensional volume of $A_t \cap \Omega \subseteq \mathbb{R}^N$.

[3] Since then, $\Omega \setminus A_\delta$ and A are a positive distance apart, this replacement will not affect the relative box dimension of (A, Ω) introduced in Equation (2.6) or any other fractal properties of (A, Ω) that will be introduced later on.

The distance and tube zeta functions of relative fractal drums belong to the class of Dirichlet-type integrals (or, in short, DTIs), and as such, have a well-defined *abscissa of* (*absolute*) *convergence*. The abscissa of convergence of a DTI $f\colon E \to \mathbb{C}$, where $E \subseteq \mathbb{C}$ is a domain, is defined as the infimum of all real numbers α for which the integral $f(\alpha)$ is absolutely convergent and we denote it by $D(f)$.[4] A basic result about a DTI f is the fact that it is a holomorphic function in the open half-plane to the right of its abscissa of convergence; that is, on the *half-plane of* (*absolute*) *convergence* $\Pi(f) := \{\operatorname{Re} s > D(f)\}$.[5] Furthermore, the relative distance and tube zeta functions are connected by the functional equation

$$\zeta_{A,\Omega}(s;\delta) = \delta^{s-N}|A_\delta \cap \Omega| + (N-s)\widetilde{\zeta}_{A,\Omega}(s;\delta), \tag{2.4}$$

which is valid on any open connected subset U of $\mathbb{C}$ to which any of these two zeta functions has a meromorphic continuation (see [LapRaŽu4]). This result is very important since the distance zeta function is much more practical to calculate in concrete examples, as opposed to the tube zeta function for which we need information about the *tube function* $t \mapsto |A_t \cap \Omega|$ itself. On the other hand, the tube zeta function has an important theoretical value and gives us a way to obtain the information about the asymptotics of the tube function via an application of the inverse Mellin transform directly from the location of the complex dimensions of the relative fractal drum under consideration. This connection enabled us to obtain our *pointwise and distributional fractal tube formulas* in [LapRaŽu8] which will play a central part in obtaining the aforementioned Minkowski measurability criterion of the present paper. These fractal tube formulas extend to any dimension $N \geq 1$ the corresponding pointwise and distributional fractal tube formulas for fractal strings (i.e., when $N = 1$) obtained in [Lap-vFr1–2]; see [Lap-vFr2, Ch. 8, esp., §8.1].

We now introduce the notions of Minkowski content and Minkowski (or box) dimension of a relative fractal drum and relate them to its distance (and tube) zeta functions. For any *real* number r, we define the *upper r-dimensional Minkowski content of A relative to* Ω (or *the upper relative Minkowski content*, or *the upper Minkowski content of the relative fractal drum* (A,Ω)) by

$$\mathcal{M}^{*r}(A,\Omega) := \limsup_{t\to 0^+} \frac{|A_t \cap \Omega|}{t^{N-r}}, \tag{2.5}$$

and we then proceed in the usual way:

$$\begin{aligned} \overline{\dim}_B(A,\Omega) &= \inf\{r \in \mathbb{R} : \mathcal{M}^{*r}(A,\Omega) = 0\} \\ &= \sup\{r \in \mathbb{R} : \mathcal{M}^{*r}(A,\Omega) = +\infty\}. \end{aligned} \tag{2.6}$$

We call $\overline{\dim}_B(A,\Omega)$ the *relative upper box dimension* of (A,Ω) (or *relative Minkowski dimension* of A with respect to Ω). Note that $\overline{\dim}_B(A,\Omega) \in [-\infty, N]$, and that the values can indeed be negative, even equal to $-\infty$; see [LapRaŽu4]. Also note that for these definitions to make sense, it is sufficient that $|A_\delta \cap \Omega| < \infty$ for some $\delta > 0$.

The value, $\mathcal{M}_*^r(A,\Omega)$, of the *lower r-dimensional Minkowski content* of (A,Ω) is defined as in (2.5), except for a lower instead of an upper limit. Analogously as in (2.6), we define the *relative lower box* (or *Minkowski*) *dimension* of (A,Ω) by

[4]For more information about DTIs, as well as for the results mentioned here concerning them (and their generalization) we refer the interested reader to [LapRaŽu1, App. A].

[5]Here, and in the rest of this paper we will abbreviate in the following way open half-planes and vertical lines; that is, given $\alpha \in \mathbb{R}$, subsets of $\mathbb{C}$ of the type $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ and $\{s \in \mathbb{C} : \operatorname{Re} s = \alpha\}$ are denoted by $\{\operatorname{Re} s > \alpha\}$ and $\{\operatorname{Re} s = \alpha\}$, respectively.

using the lower instead of the upper r-dimensional Minkowski content of (A,Ω). Furthermore, in the case when $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega)$, we denote by $\dim_B(A,\Omega)$ this common value and call it the *relative box* (or *Minkowski*) *dimension.* (We then say that $\dim_B(A,\Omega)$ exists.) If $0 < \mathcal{M}_*^D(A,\Omega) \leq \mathcal{M}^{*D}(A,\Omega) < \infty$ for some real number D, we say that the relative fractal drum (A,Ω) is *Minkowski nondegenerate.* It then follows that $\dim_B(A,\Omega)$ exists and is equal to D.

If $\mathcal{M}_*^D(A,\Omega) = \mathcal{M}^{*D}(A,\Omega)$, we denote this common value by $\mathcal{M}^D(A,\Omega)$ and call it the *relative Minkowski content* of (A,Ω). If $\mathcal{M}^D(A,\Omega)$ exists and is different from 0 and $+\infty$ (in which case $\dim_B(A,\Omega)$ exists and then, necessarily, $D = \dim_B(A,\Omega)$), we say that the relative fractal drum (A,Ω) is *Minkowski measurable.* Various examples and properties of the relative box dimension can be found in [Lap1–4], [LapPo1–2], [HeLap], [Lap-vFr1–2], [Žu], [LapPe1–2], [LapPeWi1–2] and, in full generality, [LapRaŽu1–8]. For the standard Minkowski dimension and content, many further references were given towards the end of §1 above.

In the following three theorems (namely, Theorems 2.5, 2.7 and 2.8) we recall some basic results from [LapRaŽu1,4] about the zeta functions of relative fractal drums.

Theorem 2.5. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then the following properties hold*:

(*a*) *The relative distance zeta function $\zeta_{A,\Omega}(s)$ is holomorphic in the half-plane $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$. More precisely, $D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$.*

(*b*) *If the relative box* (*or Minkowski*) *dimension $D := \dim_B(A,\Omega)$ exists, $D < N$, and $\mathcal{M}_*^D(A,\Omega) > 0$, then $\zeta_{A,\Omega}(s) \to +\infty$ as $s \to D^+$, $s \in \mathbb{R}$.*

Remark 2.6. If $\overline{\dim}_B(A,\Omega) < N$, then in light of the functional equation (2.4), Theorem 2.5 is also valid if we replace the relative distance zeta function with the relative tube zeta function in its statement. Moreover, it can be shown directly (i.e., without the use of the functional equation), that in the case of the tube zeta function, Theorem 2.5 is also valid when $\overline{\dim}_B(A,\Omega) = N$; see [LapRaŽu1].

Theorem 2.7. *Assume that (A,Ω) is a nondegenerate RFD in $\mathbb{R}^N$, that is, $0 < \mathcal{M}_*^D(A,\Omega) \leq \mathcal{M}^{*D}(A,\Omega) < \infty$* (*in particular, $\dim_B(A,\Omega) = D$*)*, and $D < N$. If $\zeta_{A,\Omega}(s)$ can be meromorphically extended to a connected open neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta_{A,\Omega}(s)$, the residue $\mathrm{res}(\zeta_{A,\Omega}, D)$ of $\zeta_{A,\Omega}$ at $s = D$ is independent of the choice of $\delta > 0$ and satisfies the inequalities*

$$(N-D)\mathcal{M}_*^D(A,\Omega) \leq \mathrm{res}(\zeta_{A,\Omega}, D) \leq (N-D)\mathcal{M}^{*D}(A,\Omega). \tag{2.7}$$

Furthermore, if (A,Ω) is Minkowski measurable, then

$$\mathrm{res}(\zeta_{A,\Omega}, D) = (N-D)\mathcal{M}^D(A,\Omega). \tag{2.8}$$

Also, one can reformulate the above theorem in terms of the relative tube zeta function and in that case, we can remove the condition $\dim_B(A,\Omega) < N$.

Theorem 2.8. *Assume that (A,Ω) is a nondegenerate RFD in $\mathbb{R}^N$* (*so that $D := \dim_B(A,\Omega)$ exists*)*, and that for some $\delta > 0$ there exists a meromorphic extension* *of $\widetilde{\zeta}_{A,\Omega} := \widetilde{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ to a connected open neighborhood of D. Then, D is a simple pole, and $\operatorname{res}(\widetilde{\zeta}_{A,\Omega}, D)$ is independent of δ. Furthermore, we have*

$$\mathcal{M}_*^D(A,\Omega) \le \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, D) \le \mathcal{M}^{*D}(A,\Omega). \tag{2.9}$$

In particular, if (A,Ω) is Minkowski measurable, then

$$\operatorname{res}(\widetilde{\zeta}_{A,\Omega}, D) = \mathcal{M}^D(A,\Omega). \tag{2.10}$$

Let us now introduce the notion of complex dimensions of an RFD.

Definition 2.9 (Complex dimensions of an RFD [LapRaŽu1,4])**.** Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Assume that $\zeta_{A,\Omega}$ has a meromorphic extension to a connected open neighborhood U of the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$. Then, the set of *visible complex dimensions of* (A,Ω) (*with respect to* U) is the set of poles of $\zeta_{A,\Omega}$ that are contained in U and we denote it by

$$\mathcal{P}(\zeta_{A,\Omega}, U) := \{\omega \in U : \omega \text{ is a pole of } \zeta_{A,\Omega}\}. \tag{2.11}$$

If $U = \mathbb{C}$, we say that $\mathcal{P}(\zeta_{A,\Omega}, \mathbb{C})$ is the set of *complex dimensions* of (A,Ω) and denote it by $\dim_{\mathbb{C}}(A,\Omega)$.

Furthermore, we call the set of poles located on the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$ the set of *principal complex dimensions of* (A,Ω) and denote it by

$$\dim_{PC}(A,\Omega) := \{\omega \in \mathcal{P}(\zeta_{A,\Omega}, U) : \operatorname{Re}\omega = \overline{\dim}_B(A,\Omega)\}. \tag{2.12}$$

Finally, for each (visible or principal) complex dimension ω, we define its *multiplicity* as the order of the pole ω of $\zeta_{A,\Omega}$. In light of this, we will sometimes refer to the set of (visible or principal) complex dimensions as a multiset (that is, a set with integer multiplicities). It is clearly independent of the domain U.

Remark 2.10. We may also define the complex dimensions in terms of the tube zeta function instead of the distance zeta function; that is, analogously as in the above definition but with $\zeta_{A,\Omega}$ replaced by $\widetilde{\zeta}_{A,\Omega} := \widetilde{\zeta}_{A,\Omega}(\,\cdot\,;\delta)$ for some $\delta > 0$.[6] Furthermore, if $\overline{\dim}_B(A,\Omega) < N$, then in light of the functional equation (2.4), the complex dimensions do not depend on the choice of the fractal zeta function. This is also true in the special case when $\overline{\dim}_B(A,\Omega) = N$ except that care must be taken in this situation since then, it may happen that the tube zeta function has a pole at $s = N$ while the distance function does not because it is canceled by the factor $(N - s)$. In order to avoid this problem, we will always assume that $\overline{\dim}_B(A,\Omega) < N$ when working with the distance zeta function.

In order to be able to formulate our main result, we will need to introduce some definitions connected to the growth properties of the distance (or tube) zeta functions. These definitions are introduced in [LapRaŽu8] in order to obtain fractal tube formulas for general RFDs and are adapted from [Lap-vFr2] where they were used in the setting of generalized fractal strings and their geometric zeta functions.

Definition 2.11. The *screen* $\boldsymbol{S}$ is the graph of a bounded, real-valued Lipschitz continuous function $S(\tau)$, with the horizontal and vertical axes interchanged:

$$\boldsymbol{S} := \{S(\tau) + \mathbb{i}\tau \,:\, \tau \in \mathbb{R}\}, \tag{2.13}$$

[6]It does not matter which $\delta > 0$ we choose since it does not affect the singularities of the tube zeta function.

where $\mathbb{i} := \sqrt{-1}$. The Lipschitz constant of S is denoted by $\|S\|_{\mathrm{Lip}}$; so that

$$|S(x) - S(y)| \leq \|S\|_{\mathrm{Lip}}|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

Furthermore, the following quantities are associated to the screen:

$$\inf S := \inf_{\tau \in \mathbb{R}} S(\tau) \quad \text{and} \quad \sup S := \sup_{\tau \in \mathbb{R}} S(\tau).$$

From now on, given an RFD (A, Ω) in $\mathbb{R}^N$, we denote its upper relative box dimension by $\overline{D} := \overline{\dim}_B(A, \Omega)$; recall that $\overline{D} \leq N$. We always assume, additionally, that $\overline{D} > -\infty$ and that the screen $\boldsymbol{S}$ lies to the left of the *critical line* $\{\mathrm{Re}\, s = \overline{D}\}$, i.e., that $\sup S \leq \overline{D}$. Also, in the sequel, we assume that $\inf S > -\infty$; hence, we have that $-\infty < \inf S \leq \sup S \leq \overline{D}$. Moreover, the *window* $\boldsymbol{W}$ is defined as the part of the complex plane to the right of $\boldsymbol{S}$; that is,

$$\boldsymbol{W} := \{s \in \mathbb{C} : \mathrm{Re}\, s \geq S(\mathrm{Im}\, s)\}. \tag{2.14}$$

(Note that $\boldsymbol{W}$ is a closed subset of $\mathbb{C}$.) We say that the relative fractal drum (A, Ω) is *admissible* if its relative tube (or distance) zeta function can be meromorphically extended (necessarily uniquely) to an open connected neighborhood of some window $\boldsymbol{W}$, defined as above.

The next definition adapts [Lap-vFr2, Def. 5.2] to the case of relative fractal drums in $\mathbb{R}^N$ (and, in particular, to the case of bounded subsets of $\mathbb{R}^N$).

Definition 2.12 (d-languidity)**.** An admissible relative fractal drum (A, Ω) in $\mathbb{R}^N$ is said to be *d-languid* if for some fixed $\delta > 0$, its distance zeta function $\zeta_{A,\Omega} := \zeta_{A,\Omega}(\,\cdot\,; \delta)$ satisfies the following growth conditions:

There exists a real constant $\kappa_d = \kappa_d(A, \Omega, \delta)$ and a two-sided sequence $(T_n)_{n \in \mathbb{Z}}$ of real numbers such that $T_{-n} < 0 < T_n$ for all $n \geq 1$ and $T_n \to \pm\infty$ as $n \to \pm\infty$ satisfying the following two hypotheses, **L1** and **L2**:

L1 For a fixed real constant $c > N$, there exists a positive constant $C > 0$ such that for all $n \in \mathbb{Z}$ and all $\sigma \in (S(T_n), c)$,

$$|\zeta_{A,\Omega}(\sigma + \mathbb{i}T_n; \delta)| \leq C(|T_n| + 1)^{\kappa_d}. \tag{2.15}$$

L2 For all $\tau \in \mathbb{R}$, $|\tau| \geq 1$,

$$|\zeta_{A,\Omega}(S(\tau) + \mathbb{i}\tau; \delta)| \leq C|\tau|^{\kappa_d}. \tag{2.16}$$

Note that these are (at most) polynomial growth conditions on $\zeta_{A,\Omega}(\,\cdot\,; \delta)$ along a sequence of horizontal segments and along the vertical direction of the screen. We call the exponent $\kappa_d \in \mathbb{R}$ appearing in the above definition the *d-languidity exponent* of (A, Ω) (or of $\zeta_{A,\Omega}$).

Definition 2.13. We will also use the notion of *languid* relative tube zeta function (or *languid* RFD) if the analogous conditions as in Definition 2.12 are satisfied for the tube zeta function. (See [LapRaŽu8, Defs. 2.12 and 2.13].) In that case, we denote the *languidity exponent* by $\kappa \in \mathbb{R}$.

In light of the functional equation (2.4), the notions of languidity and d-languidity are 'essentially (if not strictly) equivalent', but possibly with different corresponding exponents. There also exist conditions of *strong d-languidity* and *strong languidity* in which we assume that the screen may be "pushed" to $-\infty$; these conditions were needed to obtain the exact fractal tube formulas in [LapRaŽu7–8]. However, since we will not need them explicitly in this paper (except, especially, in the statement of Theorems 5.4 and 5.9), we refrain from defining them rigorously here and refer the interested reader to [LapRaŽu8] instead.

We will see that all of the interesting examples of RFDs (and, in particular, of bounded sets) in $\mathbb{R}^N$ considered here are d-languid (relative to a suitable screen). This illustrates the fact that the results of this paper can be applied to a large class of RFDs. For instance, the so-called "self-similar" RFDs with generators that are nice "enough" (including, but not limited to, monophase and pluriphase generators in the sense of [LapPe2, LapPeWi1–2]) belong to this class.

Although, as was already explained, the dependence of the distance zeta function $\zeta_{A,\Omega}(\,\cdot\,;\delta)$ on $\delta > 0$ is inessential, it is not clear if this is also true for the d-languidity conditions. More precisely, it is true that changing the parameter $\delta > 0$ will preserve d-languidity, but possibly with a different exponent $\kappa_d = k_d(\delta)$; namely, $\kappa_d(\delta_1) := \max\{\kappa_d(\delta), 0\}$ for any $\delta, \delta_1 > 0$. (See [LapRaŽu8, Prop. 5.27].)

We note that tube formulas in convex, 'smooth' and 'fractal' geometry, as well as in integral geometry, have a long history, going back to the work of Steiner [Stein], Minkowski [Mink], Weyl [Wey], Federer [Fed1]. Works on these topics include [BergGos, Bla, DemDenKoÜ, DemKoÖÜ, DeKÖÜ, Gra, HamLap, HuLaWe, KlRot, LapLu, LapLu-vFr1–2, LapPe1–2, LapPeWi1–2, LapRaŽu1,7–8, Lap-vFr1–2, Ol1–2, Ra1, Schn, Wi, WiZä, Zä1–3] and the many relevant references therein. We point out, in particular, the books [BergGos, esp., §§6.6–6.9], [Gra], [KlRot], [Schn], [Lap-vFr2, esp., Ch. 8 and §§13.1–13.2] and [LapRaŽu1, esp., Ch. 5] for a discussion of these topics in a variety of contexts.

3. Statement of the Main Result and Applications

In this section, we state our main result in Theorem 3.1 and illustrate its applications in a variety of examples and situations, including the recovery of the corresponding result for self-similar fractal strings in Corollary 3.3, the Cantor set in Example 3.5, the a-string in Example 3.7, as well as the higher-dimensional examples of the Sierpiński gasket and 3-carpet in §3.3, the fractal nests and unbounded geometric chirps in §3.4, and self-similar sprays in §3.5. We will give the detailed proof of Theorem 3.1 in §4.

3.1. **The Minkowski measurability criterion.**

We begin by stating the following theorem, in which we give a characterization of the Minkowski measurability of a large class of relative fractal drums in terms of the location of their complex dimensions with maximal real part (i.e., of their principal complex dimensions).

Theorem 3.1 (Minkowski measurability criterion)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D := \dim_B(A,\Omega)$ exists and $D < N$. Furthermore, assume that (A,Ω) is d-languid for a screen passing* strictly *between the critical line $\{\mathrm{Re}\, s = D\}$ and all the complex dimensions of (A,Ω) with real part strictly less than D. Then the following statements are equivalent*:

(*a*) *The RFD (A,Ω) is Minkowski measurable.*

(*b*) *D is the only pole of the relative distance zeta function $\zeta_{A,\Omega}$ located on the critical line $\{\mathrm{Re}\, s = D\}$ and it is simple.*

Furthermore, if (a) *or* (b) *is satisfied, then the Minkowski content of* (A, Ω) *is given by*

$$\mathcal{M}^D(A, \Omega) = \frac{\operatorname{res}(\zeta_{A,\Omega}, D)}{N - D}. \tag{3.1}$$

Remark 3.2. Theorem 3.1 extends to RFDs in $\mathbb{R}^N$, with $N \geq 1$ arbitrary, the Minkowski measurability criterion for fractal strings obtained in [Lap-vFr2, Thm. 8.15 of §8.3]. More specifically, the latter criterion corresponds to the $N = 1$ case of Theorem 4.14.

In the next result (Corollary 3.3), which follows from a combination of Theorem 4.2 and Theorem 4.14, we recover the aforementioned characterization of the Minkowski measurability of self-similar fractal strings (with possibly multiple gaps, in the sense of [Lap-vFr2, Chs. 2 and 3]), obtained in [Lap-vFr2, §8.4, esp., Thms. 8.23, 8.25 and 8.36, along with Cor. 8.27].

In the statement of Corollary 3.3, Ω denotes an arbitrary geometric realization of a (nontrivial) bounded self-similar fractal string $\mathcal{L} := (\ell_j)_{j=1}^{\infty}$, as a bounded open subset of $\mathbb{R}$. Furthermore, $\partial\Omega$ denotes its boundary (in $\mathbb{R}$) and $(\partial\Omega, \Omega)$ is the associated relative fractal drum (or RFD). We note that in [Lap-vFr2], the term RFD (or 'relative fractal drum') was not used but that an equivalent notion was used instead in the present situation of fractal strings. Finally, recall that a self-similar string is said to be *lattice* if the multiplicative group generated by the distinct values of its underlying scaling ratios is of rank 1, and *nonlattice* otherwise.[7]

Corollary 3.3 (Characterization of the Minkowski measurability of self-similar strings, [Lap-vFr2, §8.4])**.** *Let* $(\partial\Omega, \Omega)$ *(or* $\mathcal{L}$*) be a (nontrivial, bounded) self-similar fractal string, with (upper) Minkowski dimension* $D < 1$*; so that* $D = \sigma_0$*, its similarity dimension. Then the following statements are equivalent:*

(i) The RFD $(\partial\Omega, \Omega)$ *is Minkowski measurable.*

(ii) The self-similar string $\mathcal{L}$ *(or, equivalently, the self-similar RFD* $(\partial\Omega, \Omega)$*) is nonlattice.*

(iii) The only principal scaling complex dimension of $(\partial\Omega, \Omega)$ *is* $D = \sigma_0$*.*

Proof. The fact that *(ii)* and *(iii)* are equivalent follows from [Lap-vFr2, Thms. 8.23 and 8.36].

Next, we show that *(i)* and *(iii)* are equivalent. Note that since $D = \sigma_0$ and $\sigma_0 > 0$, we have that $D \in (0, 1)$.

First, observe that (since σ_0 is always simple, cf. [Lap-vFr2, Chs. 2,3]) the fact that *(iii)* implies *(i)* follows from Theorem 4.2, the sufficient condition for the Minkowski measurability of an RFD. Observe that in order to verify that the hypotheses of Theorem 4.2 are satisfied by $(\partial\Omega, \Omega)$ and its distance zeta function $\zeta_{\partial\Omega,\Omega}$, we use the fact that the scaling (and therefore, the geometric) zeta function $\zeta_{\mathfrak{S}}$ of a self-similar string is strongly languid with exponent $\kappa := 0$ (and hence, also

[7] Besides [LapRaŽu1–8], references on (or significantly involving) fractal strings include [DubSep, Fal2, Fr, HamLap, HeLap HerLap, Kom, Lap1–4, LapLéRo, LapLu, LapLu-vFr1–2, LapMa, LapPe1–2, LapPeWi1–2, LapPo1–2, LéMen, Ol1–2, MorSep, MorSepVi, Pe, PeWi, Ra1–2, RatWi, Tep1–2, Žu].

for any $\kappa \geq 0$), as is shown in [Lap-vFr2, §6.4, just above Rem. 6.12], combined with the functional equation

$$\zeta_{\partial\Omega,\Omega}(s) = \frac{2^{1-s}\zeta_{\mathcal{L}}(s)}{s} \tag{3.2}$$

which connects the distance zeta function of $(\partial\Omega, \Omega)$ and the geometric zeta function $\zeta_{\mathcal{L}}$ of the fractal string $\mathcal{L}$. For the proof of the functional equation (3.2), we refer to [LapRaŽu8, Prop. 6.3 and Rem. 6.4]. Thus, the distance zeta function $\zeta_{\partial\Omega,\Omega}$ is strongly d-languid (and hence, d-languid) with exponent $\kappa_d := -1$. Consequently (and assuming that *(iii)* holds), the hypotheses of Theorem 4.2 are satisfied and so, it follows that $(\partial\Omega, \Omega)$ is Minkowski measurable; i.e., *(i)* holds, as desired.

Now, all that remains to show is that *(i)* implies *(iii)*. More explicitly, we need to show that the fact that $(\partial\Omega, \Omega)$ is Minkowski measurable, implies that $\mathcal{L}$ (or, equivalently, $(\partial\Omega, \Omega)$) is nonlattice. For this purpose, we reason by contradiction. Namely, we assume that *(i)* holds (i.e., $(\partial\Omega, \Omega)$ is Minkowski measurable) but that $\mathcal{L}$ is a *lattice* (self-similar) string. Since $\mathcal{L}$ is lattice, its scaling complex dimensions are located (and periodically distributed) on finitely many vertical lines (possibly on a single such line), the right most of which is the vertical line $\{\operatorname{Re} s = \sigma_0\}$, the critical line (since $\sigma_0 = D$; see [Lap-vFr2, Ch. 3]). Therefore, we can obviously choose, as is required by the hypotheses of Theorem 4.14 (the Minkowski measurability criterion), a screen $\boldsymbol{S}$ passing strictly between the vertical line $\{\operatorname{Re} s = \sigma_0 = D\}$ and all the complex dimensions (i.e., the poles of $\zeta_{\partial\Omega,\Omega}$) with real part strictly less than $D = \sigma_0$. In light of Equation (3.2), it suffices to let $\boldsymbol{S}$ be any vertical line $\{\operatorname{Re} s = \Theta\}$, where $\Theta \in (\max\{0, \sigma_1\}, \sigma_0)$ and σ_1 is the abscissa of the second to last (right most) vertical line on which the scaling complex dimensions of $\mathcal{L}$ (or of $(\partial\Omega, \Omega)$) are located. (If σ_1 does not exist, then we can choose any $\Theta \in (0, \sigma_0)$.)

We may therefore apply Theorem 4.14 and deduce from the fact that the RFD $(\partial\Omega, \Omega)$ is Minkowski measurable that $D = \sigma_0$ must be its only complex dimension of real part $D = \sigma_0$.[8] This contradicts the fact that $\mathcal{L}$ is a lattice string, and hence has infinitely many (and thus at least two complex conjugate) nonreal principal scaling complex dimensions. We deduce from this contradiction that $\mathcal{L}$ must be a nonlattice string (i.e., *(ii)* holds) and hence (since *(ii)* and *(iii)* are equivalent), that *(iii)* holds, as desired. □

Remark 3.4. In §3.5, by an analogous method, we will extend Corollary 3.3 to higher dimensions, that is, to a large class of self-similar sprays in $\mathbb{R}^N$, with $N \geq 1$ arbitrary. In the general case (and under some mild assumptions), Minkowski measurability will have to be replaced by 'possibly subcritical Minkowski measurability', in a sense to be explained there.

In the rest of this section, we illustrate our main results by applying them to several examples of bounded (fractal) sets and relative fractal drums. These examples include the line segment, the Cantor string, the a-string and general fractal strings (§3.2), the Sierpiński gasket and the 3-dimensional Sierpiński carpet (§3.3), fractal nests and (unbounded) geometric chirps (§3.4), as well as, finally, the recovery and significant extensions of the known fractal tube formulas (from [LapPe2, LapPeWi1–2]) for self-similar sprays (§3.5).

[8]Note that, in light of (3.2) and since $\sigma_0 > 0$, it follows that the principal complex dimensions of $(\partial\Omega, \Omega)$ and the principal scaling complex dimensions coincide: $\mathcal{P}_c(\zeta_{\partial\Omega,\Omega}) = \mathcal{P}_c(\zeta_{\mathcal{L}})$.

3.2. **Application to Fractal Strings.** In the present subsection, we illustrate how we can apply the Minkowski measurability criterion stated in Theorem 4.14 to the case of fractal strings.

We begin by discussing the prototypical example of the Cantor string (viewed as an RFD), in Example 3.5, and further illustrate our results by means of the well known example of the a-string (in Example 3.7). Along the way, we discuss the case of general fractal strings as well as the associated fractal tube formulas.

Example 3.5. (*The standard ternary Cantor set and string*). Let C be the standard ternary Cantor set in $[0,1]$ and fix $\delta \geq 1/6$. Then, the 'absolute' distance zeta function of C is meromorphic in all of $\mathbb{C}$ and given by

$$\zeta_{C,C_\delta}(s) = \frac{2^{1-s}}{s(3^s-2)} + \frac{2\delta^s}{s}, \quad \text{for all } s \in \mathbb{C} \tag{3.3}$$

(see [LapRaŽu2, Exple. 3.4]). Similarly, the relative distance zeta function of the relative fractal drum $(C,(0,1))$ is also meromorphic on all of $\mathbb{C}$ and given by

$$\zeta_{C,(0,1)}(s) = \frac{2^{1-s}}{s(3^s-2)}, \quad \text{for all } s \in \mathbb{C}. \tag{3.4}$$

Hence, in light of the functional equation (3.2), the sets of complex dimensions of the Cantor set C and of the Cantor string $(C,(0,1))$, viewed as an RFD, coincide:

$$\mathcal{P}(\zeta_C) = \mathcal{P}(\zeta_{C,(0,1)}) = \{0\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right), \tag{3.5}$$

where as before, $\mathbb{i} := \sqrt{-1}$. In Equation (3.5), each of the complex dimensions is simple. Furthermore, $D := \dim_B(C,(0,1))$, the Minkowski dimension of the Cantor string, exists and $D = \log_3 2$, the Minkowski dimension of the Cantor set, which also exists. Furthermore, $\mathbf{p} := \frac{2\pi}{\log 3}$ is the oscillatory period of the Cantor set (or string), viewed as a *lattice* self-similar set (or string); see [Lap-vFr2, Ch. 2, esp., §2.3.1 and §2.4].

It is easy to check that $(C,(0,1))$ is d-languid, where the screen may be chosen to coincide with any vertical line $\{\operatorname{Re} s = \sigma\}$ for $\sigma \in (0, \log_3 2)$. Therefore, the assumptions of Theorem 4.14 below are satisfied but since D is not the only pole on the critical line $\{\operatorname{Re} s = D\}$, we conclude that, as is well known, the relative fractal drum $(C,(0,1))$ (or, equivalently, the Cantor string) is not Minkowski measurable.

In fact, from [LapRaŽu8, Exple. 6.2] we have the following exact pointwise fractal tube formula for the 'inner' t-neighborhood of C, valid for all $t \in (0,1/2)$:

$$|C_t \cap (0,1)| = t^{1-D} G\left(\log_3 (2t)^{-1}\right) - 2t, \tag{3.6}$$

where, G is the positive, nonconstant 1-periodic function, which is bounded away from zero and infinity and given by the following Fourier series expansion:

$$G(x) := \sum_{k \in \mathbb{Z}} \frac{2^{-D} \mathrm{e}^{2\pi \mathbb{i} k x}}{\omega_k (1-\omega_k) \log 3}, \tag{3.7}$$

where $\omega_k := D + \mathbf{p}\mathbb{i}k$ and $\mathbf{p} = \frac{2\pi}{\log 3}$ is the oscillatory period of the Cantor string.

As expected, the above exact pointwise fractal tube formula (3.6) coincides with the one obtained by direct computation for the Cantor string (see [Lap-vFr2, §1.1.2]) or from the general theory of fractal tube formulas for fractal strings (see [Lap-vFr2, Ch. 8, esp., §8.1 and §8.2]) and, in particular, for self-similar strings (see, especially, [Lap-vFr2, §8.4.1, Exple. 8.2.2]).[9]

Finally, observe that, in agreement with the lattice case of the general theory of self-similar strings developed in [Lap-vFr2, Chs. 2–3, and §8.4], we can rewrite the pointwise fractal tube formula (3.6) as follows (with $D := \dim_B C = \log_2 3$):

$$t^{-(1-D)}V_{C,(0,1)}(t) = t^{-(1-D)}|C_t \cap (0,1)| = G\left(\log_3(2t)^{-1}\right) + o(1), \tag{3.8}$$

where G is given by (3.7). Therefore, since G is periodic and nonconstant, it is clear that $t^{-(1-D)}V_{C,(0,1)}(t)$ cannot have a limit as $t \to 0^+$. It also follows from the above discussion that the Cantor string RFD $(C,(0,1))$ (or, equivalently, the Cantor string $\mathcal{L}_{CS}$) is *not* Minkowski measurable but (since G is also bounded away from zero and infinity) is Minkowski nondegenerate. This was first proved in [LapPo1] via a direct computation, leading to the precise values of $\mathcal{M}_*$ and $\mathcal{M}^*$, and reproved in [Lap-vFr2, §8.4.2] by using either the pointwise fractal tube formulas or a self-similar fractal string analog of the Minkowski measurability criterion; i.e., of the $N = 1$ case of Theorem 4.14; see Remark 3.2 and, especially, Corollary 3.3.

The above example demonstrates how the theory developed in this chapter generalizes (to arbitrary dimensions $N \geq 1$) the corresponding one for fractal strings developed in [Lap-vFr2, Ch. 8].[10] More generally, the following result gives a general connection between the geometric zeta function of a nontrivial fractal string $\mathcal{L} = (\ell_j)_{j\geq 1}$ and the (relative) distance zeta function of the bounded subset of $\mathbb{R}$ given by

$$A_{\mathcal{L}} := \Big\{a_k := \sum_{j\geq k} \ell_j : k \geq 1\Big\} \tag{3.9}$$

or, more specifically, of the RFD $(A_{\mathcal{L}},(0,\ell))$. For the proof of this proposition, we refer the reader to [LapRaŽu8, Prop. 6.3].

Proposition 3.6. *Let $\mathcal{L} = (\ell_j)_{j\geq 1}$ be a bounded nontrivial fractal string and let $\ell := \zeta_{\mathcal{L}}(1) = \sum_{j=1}^{\infty} \ell_j$ denote its total length. Then, for every $\delta \geq \ell_1/2$, we have the following functional equation for the distance zeta function of the relative fractal drum $(A_{\mathcal{L}},(0,\ell))$:*

$$\zeta_{A_{\mathcal{L}},(0,\ell)}(s;\delta) = \frac{2^{1-s}\zeta_{\mathcal{L}}(s)}{s}, \tag{3.10}$$

valid on any connected open neighborhood $U \subseteq \mathbb{C}$ of the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A_{\mathcal{L}},(0,\ell))\}$ to which any (and hence, each) of the two fractal zeta functions $\zeta_{A_{\mathcal{L}},(0,\ell)}$ and $\zeta_{\mathcal{L}}$ possesses a meromorphic continuation.[11]

Furthermore, if $\zeta_{\mathcal{L}}$ is languid for some $\kappa_{\mathcal{L}} \in \mathbb{R}$, then $\zeta_{A_{\mathcal{L}},(0,\ell)}(\,\cdot\,;\delta)$ is d-languid for $\kappa_d := \kappa_{\mathcal{L}} - 1$, with any $\delta \geq \ell_1/2$.

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[9] *Caution*: in [Lap-vFr2, §8.4], the Cantor string is defined slightly differently, and hence, C is replaced by $3^{-1}C$.

[10] One should slightly qualify this statement, however, because the higher-dimensional counterpart of the theory of fractal tube formulas for self-similar strings developed in [Lap-vFr2, §8.4] is not developed in [LapRaŽu1, 7–8] in the general case of self-similar RFDs (and, for example, of self-similar sets satisfying the open set condition), except in the special case of self-similar sprays discussed in §3.5 below.

[11] If we do not require that $\delta \geq \ell_1/2$, then we have that $\zeta_{A_{\mathcal{L}}}(s;\delta) = 2^{1-s}s^{-1}\zeta_{\mathcal{L}}(s) + v(s)$, where v is holomorphic on $\{\operatorname{Re} s > 0\}$. On the other hand, for applying the theory, we may restrict ourselves to the case when $\delta \geq \ell_1/2$.

Example 3.7. (*The a-string*). For a given $a>0$, the a-string $\mathcal{L}_a$ can be realized as the bounded open set $\Omega_a\subset\mathbb{R}$ obtained by removing the points j^{-a} for $j\in\mathbb{N}$ from the interval $(0,1)$; that is,

$$\Omega_a=\bigcup_{j=1}^{\infty}\big((j+1)^{-a},j^{-a}\big), \tag{3.11}$$

so that the sequence of lengths of $\mathcal{L}_a$ is defined by

$$\ell_j=j^{-a}-(j+1)^{-a},\quad\text{for}\quad j=1,2,\cdots, \tag{3.12}$$

and $\partial\Omega_a=\{j^{-a}\,:\,j\geq 1\}\cup\{0\}=A_{\mathcal{L}_a}\cup\{0\}$. Hence, its geometric zeta function is given (for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>\dim_B\mathcal{L}_a$) by

$$\zeta_{\mathcal{L}_a}(s)=\sum_{j=1}^{\infty}\ell_j^s=\sum_{j=1}^{\infty}\big(j^{-a}-(j+1)^{-a}\big)^s.$$

It then follows from Proposition 3.6 that for $\delta>(1-2^{-a})/2$, its distance zeta function is given by

$$\zeta_{A_{\mathcal{L}_a},(0,1)}(s;\delta)=\frac{\zeta_{\mathcal{L}_a}(s)}{2^{s-1}s}=\frac{1}{2^{s-1}s}\sum_{j=1}^{\infty}\big(j^{-a}-(j+1)^{-a}\big)^s, \tag{3.13}$$

where the second equality holds for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>\dim_B\mathcal{L}_a$ while the first equality holds for all $s\in\mathbb{C}$ (since, as will be recalled just below, $\zeta_{\mathcal{L}_a}$ and hence also $\zeta_{A_{\mathcal{L}_a},(0,1)}$, admits a meromorphic extension to all of $\mathbb{C}$).

Furthermore, the properties of the geometric zeta function $\zeta_{\mathcal{L}_a}$ of the a-string are well known (see [Lap-vFr2, Thm. 6.21]). Namely, $\zeta_{\mathcal{L}_a}$ has a meromorphic continuation to the whole of $\mathbb{C}$ and its poles in $\mathbb{C}$ are located at $D:=\dim_B\mathcal{L}_a=\dim_B A_{\mathcal{L}_a}=(a+1)^{-1}$ and at (a subset of) $\{-\frac{m}{a+1}\,:\,m\in\mathbb{N}\}$. Moreover, all of its poles are simple and $\operatorname{res}(\zeta_{\mathcal{L}_a},D)=Da^D$. In addition, for any screen $\boldsymbol{S}$ not passing through a pole, the function $\zeta_{\mathcal{L}}$ satisfies **L1** and **L2** with $\kappa:=\frac{1}{2}-(a+1)\inf S$, if $\inf S\leq 0$ and $\kappa:=\frac{1}{2}$ if $\inf S\geq 0$. From these facts and Equation (3.13), we conclude that the set $A_{\mathcal{L}_a}$ is d-languid with $\kappa_d:=-\frac{1}{2}-(a+1)\inf S$ if $\inf S\leq 0$ and with $\kappa_d:=-\frac{1}{2}$ if $\inf S\geq 0$.

By choosing the screen $\boldsymbol{S}$ to be some vertical line $\{\operatorname{Re}s=\sigma\}$ for any $\sigma\in(-1/(a+1),1/(a+1))$, we conclude that the assumptions of Theorem 4.14 below are satisfied. Therefore, since $D=1/(a+1)$ is the only pole on the critical line $\{\operatorname{Re}s=D\}$ and is simple, we conclude that the a-string is Minkowski measurable and that its Minkowski content is given by $\mathcal{M}^D(A_{\mathcal{L}_a})=2^{1-D}a^D/(1-D)$, as was first established in [Lap1, Exple. 5.1] and later reproved in [LapPo1] via a general Minkowski measurability criterion for fractal strings (expressed in terms of the asymptotic behavior of $(\ell_j)_{j=1}^{\infty}$, here, $\ell_j\sim aj^{-1/D}$ as $j\to\infty$) and then, in [Lap-vFr1–2] (via the the theory of complex dimensions of fractal strings, specifically, via the special case of Theorem 4.14 when $N=1$).

3.3. **The Sierpiński Gasket and 3-Carpet.** In this subsection, we provide an exact, pointwise fractal tube formula for the Sierpiński gasket (Example 3.8) and for a three-dimensional analog of the Sierpiński carpet (Example 3.9).

The example of the Sierpiński gasket and the 3-carpet discussed in Examples 3.8 and 3.9 below should give a good idea as to how to proceed in other, related situations, including especially, for the higher-dimensional inhomogeneous N-gasket

RFDs (with $N \geq 4$) and for other self-similar RFDs which can also be dealt with within the general theory of fractal tube formulas and their applications developed in [LapRaŽu8] and in the present paper. (See also [LapRaŽu1].)

Example 3.8. (*The Sierpiński gasket*). Let A be the Sierpiński gasket in $\mathbb{R}^2$, constructed in the usual way inside the unit triangle. Furthermore, we assume without loss of generality that $\delta > 1/4\sqrt{3}$, so that A_δ be simply connected. Then, the distance zeta function ζ_A of the Sierpiński gasket is meromorphic on the whole complex plane and is given by

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}, \tag{3.14}$$

for all $s \in \mathbb{C}$ (see [LapRaŽu3, Exple. 4.12]). In particular, the set of complex dimensions of the Sierpiński gasket is given by

$$\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0,1\} \cup \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right), \tag{3.15}$$

with each complex dimension being simple. It is easy to check now that ζ_A is d-languid for any screen $\boldsymbol{S}$ chosen to be some vertical line $\operatorname{Re} s = \sigma$ where $\sigma \in (1, \log_2 3)$. Therefore, the assumptions of Theorem 4.14 are satisfied and since, clearly, D is not the only pole on the critical line $\{\operatorname{Re} s = D\}$, we conclude that A is not Minkowski measurable, a fact which is well known. Furthermore, it follows from the pointwise fractal tube formulas obtained in [LapRaŽu8] (or from those in [LapPe2], [LapPeWi1] or [DeKÖÜ]) that A is actually Minkowski nondegenerate; see also §3.5 below.

Example 3.9. (*The* 3*-carpet*). Let A be the three-dimensional analog of the Sierpiński carpet. More specifically, we construct A by dividing the closed unit cube of $\mathbb{R}^3$ into 27 congruent cubes and remove the open middle cube. Then, we iterate this step with each of the 26 remaining smaller closed cubes; and so on, ad infinitum. By choosing $\delta > 1/6$, we have that A_δ is simply connected.

It is shown in [LapRaŽu8, Exple. 6.9], that ζ_A is meromorphic on all of $\mathbb{C}$ and given for every $s \in \mathbb{C}$ by

$$\zeta_A(s) := \zeta_A(s,\delta) = \frac{48 \cdot 2^{-s}}{s(s-1)(s-2)(3^s-26)} + \frac{4\pi\delta^s}{s} + \frac{6\pi\delta^{s-1}}{s-1} + \frac{6\delta^{s-2}}{s-2}. \tag{3.16}$$

It follows that the set of complex dimensions of the 3-carpet A is given by

$$\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0,1,2\} \cup \left(\log_3 26 + \mathbf{p}\mathbb{i}\mathbb{Z}\right), \tag{3.17}$$

where $D := \log_3 26$ $(= D(\zeta_A))$ is the Minkowski (or box) dimension of the 3-carpet A and $\mathbf{p} := 2\pi/\log 3$ is the oscillatory period of A (viewed as a lattice self-similar set). In (3.17), each of the complex dimensions is simple.

In particular, we conclude that $D := \dim_B A = \log_3 26$ (as was noted before) and, by Theorem 4.14, that the three-dimensional Sierpiński carpet is not Minkowski measurable.

3.4. **Fractal Nests and Unbounded Geometric Chirps.** In this subsection, we apply our Minkowski measurability criterion to two families of examples, namely, fractal nests (in Example 3.10) and unbounded geometric chirps (in Example 3.11). Both of these families of fractal sets or RFDs provide natural examples which are *not* self-similar or, more generally, 'self-alike' in any sense. By carefully examining the example of the fractal nest, we get new insight into the situation when the fractal zeta function has a pole of second order at $s = D$, where D is the Minkowski dimension of the set or RFD under consideration. This situation will be further investigated in §5 below, where we will give some general results about the gauge-Minkowski measurability (for a specific family of gauge functions) in terms of the presence of complex dimensions of higher order on the critical line.

Example 3.10. (*Fractal nests*). We let $\mathcal{L} = (\ell_j)_{j\geq 1}$ be a bounded fractal string and, as before, let $A_{\mathcal{L}} = \{a_k : k \in \mathbb{N}\} \subset \mathbb{R}$, with $a_k := \sum_{j\geq k} \ell_j$ for each $k \geq 1$. (See Equation (3.9) and the text surrounding it.) Furthermore, consider now $A_{\mathcal{L}}$ as a subset of the x_1-axis in $\mathbb{R}^2$ and let A be the planar set obtained by rotating $A_{\mathcal{L}}$ around the origin; i.e., A is a union of concentric circles of radii a_k and center at the origin. For $\delta > \ell_1/2$, the distance zeta function of A is given (for $\operatorname{Re} s$ large enough) by

$$\zeta_A(s) = \frac{2^{2-s}\pi}{s-1}\sum_{j=1}^{\infty}\ell_j^{s-1}(a_j + a_{j+1}) + \frac{2\pi\delta^s}{s} + \frac{2\pi a_1\delta^{s-1}}{s-1}; \tag{3.18}$$

see [LapRaŽu8, Exple. 6.12]. The last two terms in the above formula correspond to the annulus $a_1 < r < a_1 + \delta$ and we will neglect them; that is, without affecting the final outcome, we will consider only the relative distance zeta function $\zeta_{A,\Omega}$, with $\Omega := B_{a_1}(0)$. Here, for $r > 0$, $B_r(x)$ denotes the open ball of radius r with center at x.

We next consider a special case of the fractal nest above; that is, the relative fractal drum (A_a, Ω) corresponding to the a-string $\mathcal{L} := \mathcal{L}_a$, with $a > 0$; so that $\ell_j := j^{-a} - (j+1)^{-a}$ for all $j \geq 1$ and hence, $a_j = j^{-a}$ for every $j \geq 1$. In this case, it is not difficult to see that

$$\zeta_{A_a,\Omega}(s) = \frac{2^{3-s}\pi}{s-1}\sum_{j=1}^{\infty} j^{-a}\ell_j^{s-1} - \frac{2^{2-s}\pi}{s-1}\zeta_{\mathcal{L}}(s), \tag{3.19}$$

for $\operatorname{Re} s$ large enough and where $\zeta_{\mathcal{L}}$ is the geometric zeta function of the a-string.

The above zeta function given by (3.19) is analyzed in detail in [LapRaŽu8, Exple. 6.12] and is shown to possess a meromorphic continuation to all of $\mathbb{C}$. It follows, in particular, that for $a \neq 1$, the set of complex dimensions of (A_a, Ω) satisfies the following inclusion:

$$\mathcal{P}(\zeta_{A_a,\Omega}) := \mathcal{P}(\zeta_{A_a,\Omega}, \mathbb{C}) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}. \tag{3.20}$$

Furthermore, all of the above (potential) complex dimension are simple and we are certain that $\frac{2}{a+1}$ is a complex dimension of (A_a, Ω); i.e., by letting $D := \frac{2}{a+1}$, we have for all positive $a \neq 1$ that $\operatorname{res}(\zeta_{A_a,\Omega}, D) = 2^{2-D}D\pi a^{D-1}/(D-1)$. It is also shown in [LapRaŽu8] that (A_a, Ω) is d-languid for any screen chosen to be a vertical line not passing through the (potential) poles of $\zeta_{A_a,\Omega}$.

It is not difficult to show directly that for every $a > 0$, the Minkowski dimension $\dim_B(A_a, \Omega)$ exists and is equal to $\max\{1, 2/(a+1)\}$. We now conclude from Theorem 4.14 that if $a \in (0, 1)$, $\dim_B(A_a, \Omega) = D(\zeta_{A_a,\Omega}) = D$ and (A_a, Ω) is Minkowski measurable with Minkowski content given by

$$\mathcal{M}^D(A_a, \Omega) = \frac{\operatorname{res}(\zeta_{A_a,\Omega}, D)}{2-D} = \frac{2^{2-D}D\pi}{(2-D)(D-1)}a^{D-1}. \tag{3.21}$$

Moreover, if $a > 1$, we have that $\dim_B(A_a, \Omega) = 1$ and it follows from Theorem 4.14 that (A_a, Ω) is Minkowski measurable with Minkowski content given by

$$\mathcal{M}^1(A_a, \Omega) = \frac{\operatorname{res}(\zeta_{A_a,\Omega}, 1)}{2-1} = 4\pi\zeta(a) - 2\pi; \tag{3.22}$$

where ζ is the Riemann zeta function. Note also that since $\zeta(a) > 1$ for $a > 1$, we have $2\pi < \mathcal{M}^1(A_a, \Omega) < \infty$. We stress that the computation of $\operatorname{res}(\zeta_{A_a,\Omega}, 1)$ above is not trivial and refer to [LapRaŽu8, Exple. 6.12] for the details.

We now turn our attention to the critical case when $a = 1$. In this case we have that $s = 1$ is a pole of second order (i.e., of multiplicity two) of $\zeta_{A_1,\Omega}$ since it is a simple pole (of the meromorphic continuation) of the Dirichlet series $\zeta_1(s) := \sum_{j=1}^{\infty} j^{-a} \ell_j^{s-1}$ appearing in (3.19).

According to [LapRaŽu8, Exple. 6.12] and still when $a = 1$, we obtain the following fractal tube formula for (A_1, Ω):

$$|(A_1)_t \cap \Omega| = 2\pi t \log t^{-1} + \text{const.} \cdot t + o(t) \quad \text{as } t \to 0^+. \tag{3.23}$$

It clearly follows from (3.23) that even though $\dim_B(A_1, \Omega) = 1$, the RFD (A_1, Ω) is Minkowski degenerate with $\mathcal{M}^1(A_1, \Omega) = +\infty$. On the other hand, if we use instead the notion of gauge-Minkowski measurability with the gauge function $h(t) := \log t^{-1}$ (for all $t \in (0, 1)$), it turns out that (A_1, Ω) is gauge-Minkowski measurable with h-Minkowski content 2π; i.e.,

$$\mathcal{M}^1(A_1, \Omega, h) := \lim_{t \to 0^+} \frac{|(A_1)_t \cap \Omega|}{t^{2-1} h(t)} = 2\pi. \tag{3.24}$$

In §5, we will precisely define the notion of gauge-Minkowski measurability and h-Minkowski content (see Equation (5.4) and the text surrounding it) and obtain a result showing how this notion is connected to the presence of complex dimensions of higher order on the critical line (see Theorem 5.4 for a suitable class of gauge functions, including the present one). Here, we have considered the motivating example of (A_1, Ω) in which even though the asymptotics of the tube function of this RFD do not obey a standard power law, nevertheless, the underlying scaling law is reflected in the nature of its distance zeta function; i.e., in the fact that it possesses a (unique) pole of order two on the critical line. One may also view this situation as a kind of "merging" of two simple complex dimensions of (A_a, Ω) when $a \neq 1$, namely, 1 and $2/(a + 1)$, into a single complex dimension of order two as $a \to 1$.

In Example 3.11 just below, we consider the example of an unbounded geometric chirp, viewed as a relative fractal drum (A, Ω) where both of the sets (A and Ω) are unbounded. A standard *geometric* (α, β)*-chirp*, with positive parameters α and β, is a simple geometric approximation of the graph of the chirp function $f(x) = x^\alpha \sin(\pi x^{-\beta})$, for all $x \in (0, 1)$.

If the parameters α and β satisfy the inequality $-1 < \alpha < 0 < \beta$, we obtain an example of an unbounded chirp function f which we can approximate by the unbounded geometric (α, β)-chirp.

Example 3.11. (*Unbounded geometric chirps*). Let $A_{\alpha,\beta}$ be the union of vertical segments with abscissae $x = j^{-1/\beta}$ and of lengths $j^{-\alpha/\beta}$, for every $j \in \mathbb{N}$. Furthermore, define Ω as a union of the rectangles R_j for $j \in \mathbb{N}$, where R_j has a base of length $j^{-1/\beta} - (j + 1)^{-1/\beta}$ and height $j^{-\alpha/\beta}$. The relative distance zeta function of (A,Ω) is computed in [LapRaŽu8, Exple. 6.15] and is given (for $\operatorname{Re} s$ sufficiently large) by

$$\zeta_{A_{\alpha,\beta},\Omega}(s)=\frac{2^{2-s}}{(s-1)}\sum_{j=1}^{\infty}j^{-\alpha/\beta}\left(j^{-1/\beta}-(j+1)^{-1/\beta}\right)^{s-1}. \tag{3.25}$$

In particular, we know from that reference that $\zeta_{A_{\alpha,\beta},\Omega}(s)$ has a meromorphic continuation to all of $\mathbb{C}$ and

$$\mathcal{P}(\zeta_{A_{\alpha,\beta},\Omega})\subseteq\left\{1,2-\frac{1+\alpha}{1+\beta}\right\}\cup\{D_m:m\in\mathbb{N}\}, \tag{3.26}$$

where $D_m:=2-\frac{1+\alpha+m\beta}{1+\beta}$ and all of the (potential) complex dimensions are simple. Furthermore, by letting $D:=2-\frac{1+\alpha}{1+\beta}>1$ it can be shown directly that $\dim_B(A,\Omega)=D$ and also, that 1 and D are always complex dimensions of (A,Ω); i.e., $1,D\in\mathcal{P}(\zeta_{A_{\alpha,\beta},\Omega})$. It also follows from [LapRaŽu8] that $\zeta_{A_{\alpha,\beta},\Omega}$ is d-languid for any vertical line $\{\operatorname{Re} s=\sigma\}$, with $\sigma\in(1,D)$. Therefore, the assumptions of Theorem 4.14 are satisfied and since D is the only pole on the critical line $\{\operatorname{Re} s=D\}$ and is simple, we conclude that $A_{\alpha,\beta}$ is Minkowski measurable.

Moreover, it follows from the same results and from (3.25) that both 1 and D are simple poles of $\zeta_{A_{\alpha,\beta}}$. In addition, we have that $D>1$; consequently, $\dim_B(A_{\alpha,\beta},\Omega)=D$ and the Minkowski content of $(A_{\alpha,\beta},\Omega)$ is given by

$$\mathcal{M}^D(A_{\alpha,\beta},\Omega)=\frac{\operatorname{res}(\zeta_{A_{\alpha,\beta},\Omega},D)}{2-D}=\frac{(2\beta)^{2-D}}{(2-D)(D-1)(1+\beta)}.$$

3.5. **Minkowski Measurability Criteria for Self-Similar Sprays.** We conclude this section by explaining how the results of this chapter can also be applied to recover and significantly extend, as well as place within a general conceptual framework, the tube formulas for self-similar sprays generated by an arbitrary open set $G\subset\mathbb{R}^N$ of finite N-dimensional Lebesgue measure. (See, especially, [LapPe2] extended to a significantly more general setting in [LapPeWi1], along with the exposition of those results given in [Lap-vFr2, §13.1]; see also [DeKÖÜ] for another, but related, proof of some of those results in a less general context.)

Recall that a self-similar spray (with a single generator G, assumed to be bounded and open) is defined as a collection $(G_k)_{k\in\mathbb{N}}$ of pairwise disjoint (bounded) open sets $G_k\subset\mathbb{R}^N$, with $G_0:=G$ and such that for each $k\in\mathbb{N}$, G_k is a scaled copy of G by some factor $\lambda_k>0$. (We let $\lambda_0:=1$.) The associated scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ is obtained from a ratio list $\{r_1,r_2,\dots,r_J\}$, with $0<r_j<1$ for each $j=1,\dots,J$ and such that $\sum_{j=1}^J r_j^N<1$, by considering all possible words built out of the scaling ratios r_j. Here, $J\geq 2$ and the scaling ratios $r_1,\dots,r_J$ are repeated according to their multiplicities.

Let us next assume that (A,Ω) is the self-similar spray considered as a relative fractal drum and defined as $A:=\partial(\sqcup_{k=0}^{\infty}G_k)$ and $\Omega:=\sqcup_{k=0}^{\infty}G_k$, with $\overline{\dim}_B(\partial G,G)<N$. For such relative fractal drums, a very useful *factorization formula* has been established in [LapRaŽu4, Thm. 3.36] for its associated distance zeta function $\zeta_{A,\Omega}$; it is expressed as follows in terms of the distance zeta function of the boundary of the generator (relative to the generator), $\zeta_{\partial G,G}$, and the scaling

ratios $\{r_j\}_{j=1}^J$:

$$\zeta_{A,\Omega}(s)=\frac{\zeta_{\partial G,G}(s)}{1-\sum_{j=1}^J r_j^s}. \tag{3.27}$$

This identity is a direct consequence of the scaling property of the distance zeta function (see [LapRaŽu4,8]) and generalizes to any $N\geq 1$ the one obtained in [Lap-vFr2, Thm. 2.3].

Remark 3.12. Observe that we can rewrite Equation (3.27) as follows:

$$\zeta_{A,\Omega}(s)=\zeta_{\mathfrak{S}}(s)\cdot\zeta_{\partial G,G}(s), \tag{3.28}$$

where the geometric zeta function $\zeta_{\mathfrak{S}}$ of the associated self-similar string (with scaling ratios $\{r_j\}_{j=1}^J$ and a single gap length, equal to one, in the terminology of [Lap-vFr2, Chs. 2 and 3]) is meromorphic in all of $\mathbb{C}$ and given for all $s\in\mathbb{C}$ by

$$\zeta_{\mathfrak{S}}(s)=\frac{1}{1-\sum_{j=1}^J r_j^s}. \tag{3.29}$$

In general, given a connected open set $U\subseteq\mathbb{C}$ (containing the vertical line $\{\operatorname{Re}s=\overline{\dim}_B(\partial G,G)\}$), $\zeta_{A,\Omega}$ is meromorphic in U if and only if $\zeta_{\partial G,G}$ is; furthermore, in that case, the *factorization formula* (3.28) then holds for all $s\in U$. We note that in the sequel and following [LapPe2] and [LapPeWi1–2], we will often refer to $\zeta_{\mathfrak{S}}$ as the *scaling zeta function* of the self-similar spray (A,Ω) and to its poles in $\mathbb{C}$ (composing the multiset $\mathfrak{D}$) as the *scaling complex dimensions* of (A,Ω). In the present case, they are the solutions (counting multiplicities) of the *complexified Moran equation* $\sum_{j=1}^J r_j^s=1$.

Let us denote by σ_0 the unique real solution of the Moran equation $\sum_{j=1}^J r_j^s=1$ and call it the *similarity dimension* of (A,Ω). Furthermore, let us assume that $\dim_B(A,\Omega)$ exists, which is the case in a large number of situations as we will now explain. Namely, we always have that

$$\overline{\dim}_B(A,\Omega)=D(\zeta_{A,\Omega})=\max\{\sigma_0,D(\zeta_{\partial G,G})\}, \tag{3.30}$$

where the last equality follows from the factorization formula (3.27) or, equivalently, (3.28). Next, assume that the generator G is such that $D_G:=\dim_B(\partial G,G)=D(\zeta_{\partial G,G})$ exists and is a simple pole of $\zeta_{\partial G,G}$ and, additionally, that $\zeta_{\partial G,G}$ is d-languid for some window $\boldsymbol{W}$ (containing the critical line $\{\operatorname{Re}s=\overline{\dim}_B(\partial G,G)\}$). For instance, this is the case when G is monophase or, more generally, pluriphase in the sense of [LapPe2] and [LapPeWi1–2]. A large class of examples of pluriphase generators is provided by convex polytopes, which covers most classical examples of self-similar sprays; this latter result was conjectured in [LapPe2] and proved in [KoRati].

Also note that (as was already observed in [Lap1]) $D_G\in[N-1,N]$ since ∂G is the boundary of a bounded open set in $\mathbb{R}^N$. To see this, consider the orthogonal projection $\pi(\partial G)$ of ∂G to $\mathbb{R}^{N-1}$. Since G is bounded and open, then $\pi(G)\subset\pi(\partial G)$ and hence, $\pi(\partial G)_\delta\cap\pi(G)=\pi(G)$ for every $\delta>0$. Since $\pi(G)$ is bounded and open in $\mathbb{R}^{N-1}$, we conclude that $\dim_B(\partial G,G)\geq\dim_B(\pi(\partial G),\pi(G))=N-1$.

We can now distinguish the following three cases: (i) $D_G<\sigma_0$; (ii) $D_G=\sigma_0$; and (iii) $D_G>\sigma_0$.

Case (i): $D_G < \sigma_0$. Then, by (3.30), $D = \sigma_0$ and all of the poles of $\zeta_{\partial G,G}$ are located strictly to the left of D. Therefore, in light of the factorization formula (3.27), it follows that the principal complex dimensions of (A,Ω) coincide with the complex solutions of the complexified Moran equation $\sum_{j=1}^{J} r_j^s = 1$ (i.e., with the scaling complex dimensions of (A,Ω)). The complexified Moran equation was extensively studied in [Lap-vFr2, Ch. 3] and here we recall from [Lap-vFr2, Thm. 3.6] that σ_0 is always a simple pole of $\zeta_{\mathfrak{S}}$ and that, in the nonlattice case, it is the only pole of $\zeta_{\mathfrak{S}}$ on the vertical line $\{\operatorname{Re} s = \sigma_0\}$, while in the lattice case, the poles of $\zeta_{\mathfrak{S}}$ form an infinite subset of $\sigma_0 + \mathbf{p}\mathrm{i}\mathbb{Z}$, where $\mathbf{p} = 2\pi/\log(r^{-1})$ is the *oscillatory period*, with $r \in (0,1)$ being the single generator of the multiplicative group (of rank 1) generated by the distinct values of the scaling ratios $r_1,\dots,r_J$.[12] (See also the comments preceding Corollary 3.3 above.)

In particular, since D is simple, it follows that the RFD (A,Ω) has a nonreal complex dimension with real part D $(=\sigma_0)$ if and only if we are in the lattice case, and hence, in light of Theorems 4.2 and 4.14, if and only if (A,Ω) is Minkowski measurable. More specifically, we reason exactly as in the proof of Corollary 3.3 (which corresponds to the case when $N = 1$). Namely, if (A,Ω) is lattice, then it satisfies the hypotheses of Theorem 4.11 concerning the languidity and the screen. Indeed, in this case we can choose the screen to be strictly to the right of all of the poles of $\zeta_{\partial G,G}$ and also strictly to the right of all of the poles of $\zeta_{\mathfrak{S}}$ having real part strictly less than σ_0. It is known from [Lap-vFr2] and can be also checked directly that $\zeta_{\mathfrak{S}}$ satisfies even the strong d-languidity conditions (after a possible scaling) and hence, by the factorization formula (3.28), this is also true for $\zeta_{A,\Omega}$ and our chosen screen (under the assumption of d-languidity we made upon $\zeta_{\partial G,G}$ above).

Therefore, since in the lattice case, in addition to D there are other poles with real part D, we conclude that (A,Ω) cannot be Minkowski measurable. On the other hand, if (A,Ω) is nonlattice, then the only pole with real part D is D itself and it is simple. Consequently, (A,Ω) satisfies the hypotheses of Theorem 4.2 and hence, is Minkowski measurable.[13]

Therefore, in case (i)*,* (A,Ω) *is Minkowski measurable if and only if* D *is its only principal complex dimension and also, if and only if the self-similar spray* (A,Ω) *is nonlattice.*

This proves (for the case of self-similar sprays) the geometric part of a conjecture of the first author in [Lap2, Conj. 3, pp. 163–164], in case (i). This result also goes beyond that conjecture since the latter is stated in terms of the oscillations (both in the geometry and in the spectrum) of the corresponding self-similar fractal drums but is not expressed explicitly in terms of the underlying complex dimensions. (It also very significantly extends the main results of [LapPeWi2].) Note that for a self-similar string (i.e., when $N = 1$), we are always in case (i) and therefore,

[12] Recall that the lattice-nonlattice dichotomy of self-similar sprays or, more generally, self-similar sets, is defined in terms of the nature of the multiplicative group generated by the distinct values of the scaling ratios $r_1,\dots,r_J$, the lattice case being described here and the nonlattice case being its complement, i.e., when the generating set has more than one element (or, equivalently, the rank of the group is strictly greater than 1).

[13] See [Lap-vFr2, Chs. 2 and 3, esp., Thms. 2.16 and 3.6] for a detailed analysis of the structure of the scaling complex dimensions, in the lattice and nonlattice cases. Observe that in [Lap-vFr2, Ch. 3], no assumption is made about the underlying scaling ratios (and gaps), so that the corresponding results (and hence also, [Lap-vFr2, Thms. 2.16 and 3.6]) can be applied to self-similar sprays in $\mathbb{R}^N$, for an arbitrary dimension $N \geq 1$.

we have reproved (via a different method) the characterization of the Minkowski measurability for self-similar strings obtained in [Lap-vFr2, §8.4, esp., Thms. 8.23 and 8.36], recovered in Corollary 3.3 above.

Case (ii): $D_G = \sigma_0$. Since D_G and σ_0 are simple poles of $\zeta_{\partial G,G}$ and $\zeta_{\mathfrak{S}}$, respectively, it follows from the factorization formula (3.27), $\zeta_{A,\Omega} = \zeta_{\mathfrak{S}} \cdot \zeta_{\partial G,G}$, that D is a double (and hence, a multiple) pole of $\zeta_{A,\Omega}$. Therefore, (A,Ω) *is not Minkowski measurable*, independently of whether or not the self-similar spray is lattice or nonlattice. This is a direct consequence of the first part of Theorem 2.7.

Case (iii): $D_G > \sigma_0$. Then, in light of (3.30), we must then have $D = D_G$. Furthermore, according to [Lap-vFr2, Thm. 1.10], all of the poles of $\zeta_{\mathfrak{S}}$ have real part $\leq \sigma_0$ and thus, have real part $< D$. In light of the factorization formula (3.27), it then follows that the principal complex dimension of (A,Ω) depend only on the generating relative fractal drum $(\partial G, G)$. In most cases, we have that $\dim_{PC}(A,\Omega) = \{D\}$ and that $D = D_G = N-1$ is simple (since D_G is a simple pole of $\zeta_{\partial G,G}$). However, we point out that under more general hypotheses on the generator $(\partial G, G)$, other cases are theoretically possible, depending on the fractal properties of the boundary ∂G. On the other hand, in our present case when $D = D_G$ is the only principal complex dimension and is simple, we conclude from Theorem 4.14 (the Minkowski measurability criterion) that the RFD (A,Ω) is Minkowski measurable, independently of whether or not the self-similar spray is lattice or nonlattice.

In summary, in case (i), which is the case encountered for essentially all clasical self-similar sprays, (A,Ω) is Minkowski measurable if and only if it is nonlattice. In particular, the geometric part of Conjecture 3 of [Lap2, pp. 163–164] is true in this case. (See also the comments below about the recent results of [KomPeWi].)

If we are in case (ii) and D_G is a simple (or possibly even multiple) pole of $\zeta_{\partial G,G}$, then $\sigma_0 = D_G$ $(= D)$ and hence, (A,Ω) is not Minkowski measurable since D is then a pole of $\zeta_{A,\Omega}$ which is (at least) of second order.

In case (iii), i.e., when $\sigma_0 < D_G$ $(= D)$, generally, the Minkowski measurability of (A,Ω) will depend only on the Minkowski measurability of the generating RFD $(\partial G, G)$. In light of this, clearly, the conclusion of the geometric part of [Lap2, Conj. 3] fails when we are not in case (i). Note, however, that it does not contradict [Lap2, Conj. 3] because case (iii) cannot occur in the usual case of self-similar sets satisfying the open set condition, which was the only situation considered in that conjecture.[14] Indeed, in that situation, we have (for any $\delta > 0$)

$$\dim_B(A,\Omega) \leq \dim_B(A, A_\delta) = \dim_B A = \sigma_0,$$

where the last equality is well known and follows from [Hut] (as described in [Fal1, Thm. 9.3]).

Each of the cases (i)–(iii) can be naturally realized for general self-similar sprays (or RFDs), as we will illustrate in Examples 3.13 and 3.14 below. On the other hand, for self similar sets F (satisfying the open set condition), only case (i) or case (ii) can occur because it is well known that then, $\dim_B F = \sigma_0$.

[14]It may be helpful to observe that self-similar sprays (and RFDs), in the generalized sense considered in this paper and in [LapRaŽu1,3,8] can correspond to inhomogeneous self-similar sets (in the sense of [BarDem]) that are not (strict) self-similar sets (in the usual sense of [Hut, Fal1]); see *ibid.*

We point out that the geometric part of [Lap2, Conj. 3, pp. 163–164] has been proved for self-similar sets (rather than for general self-similar sprays or RFDs) satisfying the open set condition, first when $N = 1$ in [Lap-vFr1–2] (by using the fractal tube formulas for fractal strings; see [Lap-vFr2, §8.4]) and then, after the completion of our work, when $N \geq 1$ in [KomPeWi] (by using the renewal theorem, in particular). The aforementioned works extend a variety of results previously obtained in [Lap2] (when $N = 1$ and by using the renewal theorem), then in [Fal2] (still when $N = 1$, and also by using this same theorem) and later, in [Gat] (when $N \geq 1$, and also based on the renewal theorem) and in [Lap-vFr1–2] (as mentioned above), as well as more recently, under some relatively restrictive hypotheses, for self-similar sprays in [LapPeWi2] (when $N \geq 1$ and by using the fractal tube formulas for self-similar sprays of [LapPeWi1], along with techniques from [Lap-vFr2, §8.4]).

What was missing in the results of [Lap2, Fal2, Gat] (but not of [Lap-vFr1–2] and of [LapPeWi2]) was to show that lattice self-similar sets are not Minkowski measurable (as was the content of the geometric part of the conjecture of [Lap2]), which is now known to be true when D is not an integer. We note that case (ii) was also considered in [KomPeWi], with the same conclusion as above. In our setting, however, and still in case (ii), by using the results and methods of §5 below, we can also (under appropriate assumptions) obtain definite conclusions about the h-Minkowski measurability of (A, Ω), for some suitable gauge function h. An explicit example of this situation and of such a conclusion is provided in Example 3.13 just below, which is, however, an inhomogeneous self-similar set rather than a strictly (or homogeneous) self-similar set. (See also the general result given in Theorem 5.4 of §5.)

Finally, we close this discussion by mentioning that we expect that a counterpart of the above results about self-similar sprays in cases (i) and (ii) can also be eventually proved by analogous methods for self-similar sets in $\mathbb{R}^N$ (satisfying the open set condition), via the fractal tube formulas obtained in [LapRaŽu8].

The next three examples illustrate interesting phenomena that may occur in the setting of self-similar sprays (or RFDs), which can also be viewed as corresponding to an inhomogeneous self-similar setting, in the sense of [BarDem]; see [LapRaŽu4]. We will also discuss some of the consequences of the associated fractal tube formulas obtained in [LapRaŽu8].

Example 3.13. (*The* $1/2$*-square fractal*). Let us consider the $1/2$-square fractal A from [LapRaŽu4, Exple. 3.38] and depicted in Figure 1, left. This example corresponds to case (ii) about self-similar sprays discussed above. Its distance zeta function was obtained in [LapRaŽu4], where it was shown to be meromorphic on all of $\mathbb{C}$ and given by

$$\zeta_A(s) = \frac{2^{-s}}{s(s-1)(2^s-2)} + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{3.31}$$

for every $s \in \mathbb{C}$. Furthermore, as was discussed in [LapRaŽu4], it follows at once from (3.31) that

$$D(\zeta_A) = 1, \quad \mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup (1 + \mathbf{p}\mathbb{i}\mathbb{Z}) \tag{3.32}$$

and

$$\dim_{PC} A := \mathcal{P}_c(\zeta_A) = \{1\}, \tag{3.33}$$

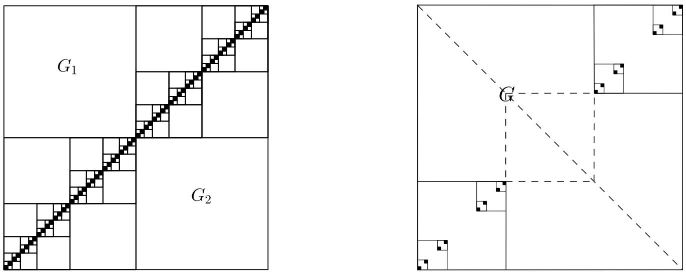

FIGURE 1. **Left:** The 1/2-square fractal A from Example 3.13. We start with a unit square $[0,1]^2$ and in the first step remove the open squares G_1 and G_2. In the next step, we repeat this procedure with the remaining squares $[1/2,1]^2$ and $[0,1/2]^2$; we continue this process ad infinitum and A is then the set which remains behind. The first 6 iterations are depicted in this figure. Here, $G := G_1 \cup G_2$ is the single generator of the corresponding self-similar spray or RFD (A,Ω), where $\Omega = (0,1)^2$. **Right:** The 1/3-square fractal A from Example 3.14. We start with a unit square $[0,1]^2$ and, in the first step, remove the open polygon G. In the next step, we repeat this procedure with the remaining squares $[1/3,1]^2$ and $[0,1/3]^2$; we continue this process ad infinitum and A is then the set which remains behind. The first 4 iterations are depicted in this figure. Here, G is the single generator of the corresponding self-similar spray or RFD (A,Ω), where $\Omega := (0,1)^2$.

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := \frac{2\pi}{\log 2}$ and all of the complex dimensions in $\mathcal{P}(\zeta_A)$ are simple, except for $\omega_0 := 1$ which is a double pole of ζ_A. We now conclude immediately from Theorem 2.7 that A cannot be Minkowski measurable.

By applying the theory developed in [LapRaŽu8], we can obtain the fractal tube formula for A directly from its distance zeta function. More precisely, by [LapRaŽu8, Exple. 6.18], we have the following pointwise tube formula, valid for all $t \in (0,1/2)$:

$$
\begin{aligned}
|A_t| &= \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s),\omega\right) \\
&= \frac{1}{4\log 2} t\log t^{-1} + t\,G\left(\log_2(4t)^{-1}\right) + \frac{1+2\pi}{2}t^2.
\end{aligned} \tag{3.34}
$$

Here, G is a nonconstant 1-periodic function on $\mathbb{R}$, which is bounded away from zero and infinity. It is given by the following pointwise convergent (and even, absolutely convergent) Fourier series:

$$
G(x) := \frac{29\log 2 - 4}{8\log 2} + \frac{1}{4}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\mathrm{e}^{2\pi\mathbb{i}kx}}{(2-\omega_k)(\omega_k-1)\omega_k}, \quad \text{for all } x \in \mathbb{R}, \tag{3.35}
$$

where we have let $\omega_k := 1 + \mathbb{i}\mathbf{p}k$ for each $k \in \mathbb{Z}$.

In conclusion, we note that it is now also clear from the fractal tube formula (3.34) for the 1/2-square fractal that $\dim_B A = 1$ and that A is actually Minkowski degenerate with $\mathcal{M}^1(A) = +\infty$. On the other hand, as we can see via a direct computation (and with the notations of §5 below), A is h-Minkowski measurable with $h(t) := \log t^{-1}$ (for all $t \in (0,1)$) and with h-Minkowski content given by $\mathcal{M}^1(A,h) = (4\log 2)^{-1}$. We point out here that, although $D := \dim_B A = 1$ (with 1 coinciding with the topological dimensions of A) and hence, A would not be considered fractal in the classical sense (of [Man1]), we also see from (3.34) that the nonreal complex dimensions of A with real part equal to D give rise to (intrinsic) geometric oscillations of order t^{2-D} in the fractal tube formula for A. Therefore, according to our proposed definition of fractality given in [LapRaŽu1,4,8], A is *critically fractal* in dimension $d := D = \dim_B A = 1$ (which means that it has nonreal complex dimensions of maximal real part D).

Example 3.14. (The 1/3-square fractal). Let us now consider the 1/3-square fractal A from [LapRaŽu4, Exple. 3.39] and depicted in Figure 1, right. This example corresponds to case (*iii*) for self-similar sprays discussed above; indeed, here, the similarity dimension $\sigma_0 = \log_3 2$ is strictly less than the dimension $D_G = 1$ of the generating relative fractal drum $(\partial G, G)$. Its distance zeta function was obtained in [LapRaŽu4], where it was shown to be meromorphic on all of $\mathbb{C}$ and given for all $s \in \mathbb{C}$ by

$$\zeta_A(s) = \frac{2}{s(3^s-2)}\left(\frac{6}{s-1} + Z(s)\right) + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{3.36}$$

where Z is an entire function given by

$$Z(s) := \int_0^{\pi/2} (\cos\varphi + \sin\varphi)^{-s}\,\mathrm{d}\varphi.$$

Furthermore, we have that $D(\zeta_A) = 1$ and

$$\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) \subseteq \{0\} \cup (\log_3 2 + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{1\}, \tag{3.37}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := \frac{2\pi}{\log 3}$ and all of the complex dimensions in $\mathcal{P}(\zeta_A)$ are simple. In Equation (3.37), we have an inclusion since, theoretically, some of the complex dimensions with real part $\log_3 2$ may be canceled by the zeros of $6/(s-1) + Z(s)$. However, there exist nonreal complex dimensions with real part $\log_3 2$ in $\mathcal{P}(\zeta_A)$, a fact which can be checked numerically. Moreover, it now follows from Theorem 4.2 below that A is Minkowski measurable with $\mathcal{M}^1(A) = \operatorname{res}(\zeta_A, 1) = 16$.

We next recall the following fractal tube formula for A, also obtained in [LapRaŽu8] and valid pointwise for every $t \in (0, 1/\sqrt{2})$:

$$\begin{aligned} |A_t| &= \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s), \omega\right) \\ &= 16t + t^{2-\log_3 2} G\left(\log_3 (3t)^{-1}\right) + \frac{12+\pi}{2}t^2. \end{aligned} \tag{3.38}$$

Here, the function G is a *nonconstant* 1-periodic function on $\mathbb{R}$, which is bounded away from zero and from infinity and is given by the following pointwise convergent

Fourier series:

$$G(x) := \frac{1}{\log 3}\sum_{k=-\infty}^{+\infty}\frac{\mathrm{e}^{2\pi\mathbbm{i}kx}}{(2-\omega_k)\omega_k}\left(\frac{6}{\omega_k-1}+Z(\omega_k)\right),\quad \text{for all } x\in\mathbb{R}, \tag{3.39}$$

where we have let $\omega_k := \log_3 2 + \mathbbm{i}\mathbf{p}k$ for each $k \in \mathbb{Z}$.

In conclusion, we observe that it is clear from the fractal tube formula (3.38) that $\dim_B A = 1$ and A is Minkowski measurable, with Minkowski content given by $\mathcal{M}^1(A) = 16$.

Finally, we point out that since $D := \dim_B A = 1$ (and 1 is the topological dimension of A), the set A would not be considered fractal in the classical sense (of [Man1]). On the other hand, we also see from (3.38) that the nonreal complex dimensions of A with real part equal to $\log_3 2$ give rise to (intrinsic) geometric oscillations of order $O(t^{2-\log_3 2})$ in its fractal tube formula. According to our proposed definition of fractality given in [LapRaŽu1,4,8], the 1/3-square fractal A is therefore fractal; more precisely, it is *strictly subcritically fractal* in dimension $d := \log_3 2$ (which means that it has nonreal complex dimensions with real part $< D$, but none with real part equal to D).

4. Proof of the Main Result

In this section, we obtain, in particular, a necessary and sufficient condition for the Minkowski measurability of a large class of RFDs (A, Ω) in $\mathbb{R}^N$, expressed in terms of the principal poles of their fractal zeta functions. More specifically, under suitable hypotheses, an RFD with Minkowski dimension D is shown to be Minkowski measurable if and only if its only complex dimension with real part D is equal to D itself, and D is simple. (See Theorems 4.14 and 4.16, along with Remark 4.15.) We also obtain a sufficient condition (with weaker hypotheses imposed on the RFD in comparison to the Minkowski measurability criterion stated in Theorem 4.14) for the Minkowski measurability of a relative fractal drum; see Theorem 4.2. Furthermore, we establish an upper bound for the upper Minkowski content of an RFD in terms of the residue at $s = \overline{D}$ of its fractal zeta function, where $\overline{D}$ denotes the upper Minkowski dimension of the RFD; see Theorem 4.4. Naturally, all of theses results apply, in particular, to bounded subsets A of $\mathbb{R}^N$, with $N \geq 1$ arbitrary, by simply considering the associated RFD (A, A_δ), for any $\delta > 0$.

4.1. A Sufficient Condition for Minkowski Measurability.

In this subsection, we show that a sufficient condition for the Minkowski measurability of a relative fractal drum (A, Ω) can be given in terms of its relative tube (or distance) zeta function. This will be a consequence of a well-known Tauberian theorem due to Wiener and Pitt (see [PittWie]) and which generalizes the famous Ikehara Tauberian theorem. The proof of the Wiener–Pitt Tauberian theorem can also be found in [Kor, Ch. III, Lem. 9.1 and Prop. 4.3] or in [Pitt, §6.1] and in [Dia], where a different proof using a technique due to Bochner is given. We next state this theorem, for the sake of completeness.[15]

Theorem 4.1 (The Wiener–Pitt Tauberian theorem, cited from [Kor])**.** *Let* $\sigma\colon \mathbb{R} \to \mathbb{R}$ *be such that* $\sigma(t)$ *vanishes for all* $t < 0$*, is nonnegative for all* $t \geq 0$*, and such*

[15]Throughout the discussion of Theorem 4.1 and its proof, as well as in the rest of this subsection, the symbol h clearly does not refer to a gauge function.

that its Laplace transform

$$F(s) := \{\mathfrak{L}\sigma\}(s) := \int_0^{+\infty} \mathrm{e}^{-st}\sigma(t)\,\mathrm{d}t \tag{4.1}$$

exists for all $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > 0$. *Furthermore, suppose that for some constants* $A > 0$ *and* $\lambda > 0$, *the function*

$$H(s) := F(s) - \frac{A}{s}, \quad s := x + \mathbb{i}y \quad (x > 0,\, y \in \mathbb{R}), \tag{4.2}$$

converges in $L^1(-\lambda, \lambda)$ *to a boundary function* $H(\mathbb{i}y)$ *as* $x \to 0^+$. *Then, for every real number* $h \geq 2\pi/\lambda$, *we have that*

$$\sigma_h(u) := \frac{1}{h}\int_u^{u+h} \sigma(t)\,\mathrm{d}t \leq CA + o(1) \quad \text{as} \quad u \to +\infty, \tag{4.3}$$

for some positive constant $C < 3$.

Moreover, if the above constant λ *can be taken to be arbitrarily large, then for every fixed* $h > 0$,

$$\sigma_h(u) \to A \quad \text{as} \quad u \to +\infty. \tag{4.4}$$

Let us now state the announced result and then prove it by using the above Tauberian theorem.

Theorem 4.2 (Sufficient condition for Minkowski measurability)**.** *Let* (A, Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *and let* $\overline{D} := \overline{\dim}_B(A, \Omega)$. *Furthermore, suppose that the relative tube zeta function* $\widetilde{\zeta}_{A,\Omega}$ *of* (A, Ω) *can be meromorphically extended to a connected open neighborhood* $U \subseteq \mathbb{C}$ *of the critical line* $\{\operatorname{Re} s = \overline{D}\}$, *with a single pole* $\overline{D}$, *which is assumed to be simple. Then* $D := \dim_B(A, \Omega)$ *exists,* $D = \overline{D}$ *and* (A, Ω) *is Minkowski measurable with Minkowski content given by*

$$\mathcal{M}^D(A, \Omega) = \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, D). \tag{4.5}$$

Moreover, if we assume, in addition, that $\overline{D} < N$, *then the theorem is also valid if we replace the relative tube zeta function* $\widetilde{\zeta}_{A,\Omega}$ *by the relative distance zeta function* $\zeta_{A,\Omega}$ *of* (A, Ω), *and in that case, we have*

$$\mathcal{M}^D(A, \Omega) = \frac{\operatorname{res}(\zeta_{A,\Omega}, D)}{N - D}. \tag{4.6}$$

Proof. Without loss of generality, for the tube zeta function $\widetilde{\zeta}_A(\,\cdot\,, \Omega; \delta)$ we may choose $\delta = 1$ and change the variable of integration by letting $u := 1/t$:

$$\begin{aligned}
\widetilde{\zeta}_{A,\Omega}(s + \overline{D}) &= \int_0^1 t^{s+\overline{D}-1-N}|A_t \cap \Omega|\,\mathrm{d}t \\
&= \int_1^{+\infty} u^{-s-1-\overline{D}+N}|A_{1/u} \cap \Omega|\,\mathrm{d}u \\
&= \int_0^{+\infty} \mathrm{e}^{-sv}\mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}} \cap \Omega|\,\mathrm{d}v = \{\mathfrak{L}\sigma\}(s),
\end{aligned} \tag{4.7}$$

where we have made another change of variable in the second to last equality, namely, $v := \log u$, and we have also let $\sigma(v) := \mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}} \cap \Omega|$. Clearly, the definition of the relative tube zeta function of (A, Ω) implies that the residue of $\widetilde{\zeta}_{A,\Omega}(s)$ at $s = \overline{D}$ must be real and positive. (Note that, a priori, it should be nonnegative, but since by hypothesis, $\overline{D}$ is a pole of the meromorphic continuation of $\widetilde{\zeta}_{A,\Omega}$ to U, the residue at $\overline{D}$ must be different from zero.) Furthermore, since $s=\overline{D}$ is the only pole of $\widetilde{\zeta}_{A,\Omega}$ in U, we conclude that

$$H(s):=\widetilde{\zeta}_{A,\Omega}(s+\overline{D})-\frac{\operatorname{res}(\widetilde{\zeta}_{A,\Omega},\overline{D})}{s} \tag{4.8}$$

is holomorphic in the neighborhood $U_{\overline{D}}:=\{s\in\mathbb{C}\,:\,s+\overline{D}\in U\}$ of the vertical line $\{\operatorname{Re}s=0\}$. In other words, we can apply Theorem 4.1 (for arbitrarily large $\lambda>0$, in the notation of that theorem) and conclude that

$$\sigma_h(u)=\frac{1}{h}\int_u^{u+h}\sigma(v)\,\mathrm{d}v\to\operatorname{res}(\widetilde{\zeta}_{A,\Omega},\overline{D})\quad\text{as}\quad u\to+\infty, \tag{4.9}$$

for every fixed $h>0$. In particular, since $v\mapsto|A_{\mathrm{e}^{-v}}\cap\Omega|$ is nonincreasing, we next consider the following two cases:

Case (a): We assume that $\overline{D}<N$. Hence, we have

$$\begin{aligned}\frac{1}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v&\le\frac{|A_{\mathrm{e}^{-u}}\cap\Omega|}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}\,\mathrm{d}v\\&=\frac{|A_{\mathrm{e}^{-u}}\cap\Omega|}{\mathrm{e}^{-u(N-\overline{D})}}\,\frac{\mathrm{e}^{h(N-\overline{D})}-1}{(N-\overline{D})h}.\end{aligned}$$

By taking the lower limit of both sides as $u\to+\infty$, we obtain that

$$\operatorname{res}(\widetilde{\zeta}_{A,\Omega},\overline{D})\le\mathcal{M}_*^{\overline{D}}(A,\Omega)\frac{\mathrm{e}^{h(N-\overline{D})}-1}{(N-\overline{D})h}. \tag{4.10}$$

Since this is true for every $h>0$, we deduce by letting $h\to0^+$ that

$$\operatorname{res}(\widetilde{\zeta}_{A,\Omega},\overline{D})\le\mathcal{M}_*^{\overline{D}}(A,\Omega). \tag{4.11}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v&\ge\frac{|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|}{h}\int_u^{u+h}\mathrm{e}^{v(N-\overline{D})}\,\mathrm{d}v\\&=\frac{|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|}{\mathrm{e}^{-(u+h)(N-\overline{D})}}\,\frac{1-\mathrm{e}^{-h(N-\overline{D})}}{(N-\overline{D})h}\end{aligned} \tag{4.12}$$

and, similarly as before, by taking the upper limit of both sides as $u\to+\infty$, we obtain that

$$\operatorname{res}(\widetilde{\zeta}_{A,\Omega},\overline{D})\ge\mathcal{M}^{*\overline{D}}(A,\Omega)\frac{1-\mathrm{e}^{-h(N-\overline{D})}}{(N-\overline{D})h}. \tag{4.13}$$

Since this is true for every $h>0$, we can let $h\to0^+$ and conclude that $\operatorname{res}(\widetilde{\zeta}_{A,\Omega},\overline{D})\ge\mathcal{M}^{*\overline{D}}(A,\Omega)$. This latter inequality, combined with (4.11), implies that (A,Ω) is $\overline{D}$-Minkowski measurable which, a fortiori, implies that $D=\dim_B(A,\Omega)=\overline{D}$. Furthermore, we also conclude that $\operatorname{res}(\widetilde{\zeta}_{A,\Omega},D)=\mathcal{M}^D(A,\Omega)$, the Minkowski content of (A,Ω).

Case (b): We will now assume that $\overline{D}=N$. Therefore, in this case we have

$$|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|=\frac{|A_{\mathrm{e}^{-(u+h)}}\cap\Omega|}{\mathrm{e}^{-(u+h)(N-N)}}\le\frac{1}{h}\int_u^{u+h}|A_{\mathrm{e}^{-v}}\cap\Omega|\,\mathrm{d}v\le\frac{|A_{\mathrm{e}^{-u}}\cap\Omega|}{\mathrm{e}^{-u(N-N)}}=|A_{\mathrm{e}^{-u}}\cap\Omega|.$$

Then, by taking, respectively, the lower and upper limits as $u \to +\infty$, we obtain that

$$\mathcal{M}^{*N}(A,\Omega) \le \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, N) \le \mathcal{M}_*^N(A,\Omega). \tag{4.14}$$

Finally, if $D < N$, then the part of the theorem dealing with the distance (instead of the tube) zeta function of (A,Ω) follows at once from case (a) of the proof for $\widetilde{\zeta}_{A,\Omega}$. This is so in light of the functional equation (2.4) connecting $\zeta_{A,\Omega}$ and $\tilde{\zeta}_{A,\Omega}$, or more precisely, of the relation between the residues at the simple pole $s = D$ of the two zeta functions which follows from it (namely, $\operatorname{res}(\zeta_{A,\Omega}, D) = (N-D)\operatorname{res}(\widetilde{\zeta}_{A,\Omega}, D)$). This concludes the proof of the theorem. □

Remark 4.3. In light of Theorem 4.1, the assumptions of Theorem 4.2 can be weakened. More precisely, it suffices to assume that for every fixed $\lambda > 0$, the function $\widetilde{\zeta}_{A,\Omega}(s) - \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, \overline{D})/(s-\overline{D})$ (restricted to the vertical line segment $(-\mathbb{i}\lambda, \mathbb{i}\lambda)$ and viewed as a function of $\tau := \operatorname{Im} s \in (-\lambda,\lambda)$), converges in $L^1(-\lambda,\lambda)$ to a boundary function $H(\mathbb{i}\operatorname{Im} s)$ as $\operatorname{Re} s \to \overline{D}^+$. Hence, $H(\mathbb{i}\tau)$ must then satisfy $\int_{-\lambda}^{\lambda} |H(\mathbb{i}\tau)|\,\mathrm{d}\tau < \infty$, for every $\lambda > 0$.

In the case when, besides $\overline{D}$, there are other singularities on the critical line $\{\operatorname{Re} s = \overline{D}\}$ of the relative fractal drum (A,Ω), we can use Theorem 4.1 to derive an upper bound for the upper $\overline{D}$-dimensional Minkowski content of (A,Ω) expressed in terms of the residue of its relative tube (or distance) zeta function at $s = \overline{D}$, as we now explain in the next result.

Theorem 4.4 (Upper bound for the upper Minkowski content)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and let $\overline{D} := \overline{\dim}_B(A,\Omega)$. Furthermore, assume that the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$ of (A,Ω) can be meromorphically extended to a connected open neighborhood U of the critical line $\{\operatorname{Re} s = \overline{D}\}$ and that $\overline{D}$ is a simple pole of its meromorphic continuation to U. Also assume that the critical line $\{\operatorname{Re} s = \overline{D}\}$ contains another pole of $\widetilde{\zeta}_{A,\Omega}$, different from $\overline{D}$. Furthermore, let*

$$\lambda_{A,\Omega} := \inf\left\{|\overline{D} - \omega| \,:\, \omega \in \dim_{PC}(A,\Omega) \setminus \{\overline{D}\}\right\}. \tag{4.15}$$

Then, if $\overline{D} < N$, we have the following upper bound for the upper $\overline{D}$-dimensional Minkowski content of (A,Ω), expressed in terms of the residue at $s := \overline{D}$ of the relative tube zeta function of (A,Ω):

$$\mathcal{M}^{*\overline{D}}(A,\Omega) \le \frac{C\lambda_{A,\Omega}(N-\overline{D})}{2\pi\left(1 - \mathrm{e}^{-2\pi(N-\overline{D})/\lambda_{A,\Omega}}\right)} \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, \overline{D}). \tag{4.16}$$

Moreover, in the case when $\overline{D} = N$, we have $\mathcal{M}^{*N}(A,\Omega) \le C\operatorname{res}(\widetilde{\zeta}_{A,\Omega}, N)$, where (*both in* (4.16)*, just above and in* (4.17) *just below*) *C is a positive constant such that $C < 3$.*

Finally, if $\overline{D} < N$, we have the following upper bound for the upper $\overline{D}$-dimensional Minkowski content of (A,Ω), expressed in terms of the residue at $s := \overline{D}$ of the relative distance zeta function of (A,Ω):

$$\mathcal{M}^{*\overline{D}}(A,\Omega) \le \frac{C\lambda_{A,\Omega}}{2\pi\left(1 - \mathrm{e}^{-2\pi(N-\overline{D})/\lambda_{A,\Omega}}\right)} \operatorname{res}(\zeta_{A,\Omega}, \overline{D}). \tag{4.17}$$

Proof. We use the same reasoning as in the proof of Theorem 4.2, with the only difference residing in the fact that we may now only use the weaker statement (4.3) of Theorem 4.1 since by hypothesis, there is another pole on the critical line $\{\operatorname{Re} s = \overline{D}\}$, besides $\overline{D}$ itself. More specifically, if $\overline{D} < N$ and $\lambda < \lambda_{A,\Omega}$, then by using (4.12) and (4.3), we show that for every $h \geq 2\pi/\lambda$, we have

$$C \operatorname{res}(\widetilde{\zeta}_{A,\Omega}, \overline{D}) \geq \mathcal{M}^{*\overline{D}}(A,\Omega)\frac{1-\mathrm{e}^{-h(N-\overline{D})}}{(N-\overline{D})h}. \tag{4.18}$$

Since the right-hand side of (4.18) just above is a decreasing function of h, we obtain the best estimate for $h = 2\pi/\lambda$. Furthermore, since this is true for every $\lambda < \lambda_{A,\Omega}$, we obtain (4.16) by letting $\lambda \to \lambda_{A,\Omega}^-$. Moreover, (4.18) is also valid if $\overline{D} = N$, but without the factor that depends on h, by a similar argument as in case (b) of the proof of Theorem 4.2. Finally, if $\overline{D} < N$, the statement about the relative distance zeta function $\zeta_{A,\Omega}$ follows by the same argument as in case (a) of the proof of Theorem 4.2, by also using the functional equation (2.4). ☐

Remark 4.5. Much as in the case of Theorem 4.2 (see Remark 4.3), the hypotheses of Theorem 4.4 can be weakened. However, we have stated Theorem 4.4 in the above form because this is the most common situation which is encountered in our examples of RFDs.

4.2. **Characterization of Minkowski Measurability.** The necessary condition for Minkowski measurability (Theorem 4.11), which will then be combined with Theorem 4.2 in order to yield the desired Minkowski measurability criterion stated in Theorem 4.14, will follow from the distributional tube formula established in [LapRaŽu8]. The version of the distributional tube formula which we will need is stated in terms of the *Mellin zeta function* of a given relative fractal drum. The Mellin zeta function, which is closely related to the tube zeta function, was also introduced in [LapRaŽu8] in order to obtain another distributional tube formula, valid for a larger class of test functions. The latter formula will be needed here in order to prove Theorem 4.11 below.

We now recall the definition of the Mellin zeta function, along with some of its basic properties. For the proof of the following results, we refer the reader to [LapRaŽu8, §5.4].

Definition 4.6. Let (A,Ω) be an RFD in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$. We define the *Mellin zeta function* $\zeta_{A,\Omega}^{\mathfrak{M}}$ of (A,Ω) by

$$\zeta_{A,\Omega}^{\mathfrak{M}}(s) := \int_0^{+\infty} t^{s-N-1}|A_t \cap \Omega|\,\mathrm{d}t, \tag{4.19}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \in (\overline{\dim}_B(A,\Omega), N)$, where the integral is taken in the Lebesgue sense.

Note that the Mellin zeta function of (A,Ω) is actually equal to the Mellin transform of $f(t) := t^{-N}|A_t \cap \Omega|$; i.e., $\zeta_{A,\Omega}^{\mathfrak{M}}(s) = \{\mathfrak{M}f\}(s)$, where we recall that

$$\{\mathfrak{M}f\}(s) := \int_0^{+\infty} t^{s-1} f(t)\,\mathrm{d}t. \tag{4.20}$$

The Mellin zeta function is closely related to the tube and distance zeta functions, as is explained in the next theorem.

Theorem 4.7. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega) < N$. Then, the Mellin zeta function $\zeta^{\mathfrak{M}}_{A,\Omega}$, as given by Equation* (4.19)*, is holomorphic on the open vertical strip $\{\overline{\dim}_B(A,\Omega) < \operatorname{Re} s < N\}$ and*

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta^{\mathfrak{M}}_{A,\Omega}(s) = \int_0^{+\infty} t^{s-N-1}|A_t\cap\Omega|\log t\,\mathrm{d}t, \tag{4.21}$$

for all s in $\{\overline{\dim}_B(A,\Omega) < \operatorname{Re} s < N\}$. Furthermore, $\{\overline{\dim}_B(A,\Omega) < \operatorname{Re} s < N\}$ is the largest vertical strip on which the integral (4.19) *is absolutely convergent* (*i.e., is a convergent Lebesgue integral*)*.*

Moreover, for all $s\in\mathbb{C}$ such that $\overline{\dim}_B(A,\Omega) < \operatorname{Re} s < N$ and for any fixed $\delta > 0$ such that $\Omega\subseteq A_\delta$, $\zeta^{\mathfrak{M}}_{A,\Omega}$ satisfies the following functional equations, connecting it respectively to $\widetilde{\zeta}_{A,\Omega}$ and $\zeta_{A,\Omega}$:

$$\zeta^{\mathfrak{M}}_{A,\Omega}(s) = \widetilde{\zeta}_{A,\Omega}(s;\delta) + \frac{\delta^{s-N}|\Omega|}{N-s} \tag{4.22}$$

and

$$\zeta^{\mathfrak{M}}_{A,\Omega}(s) = \frac{\zeta_{A,\Omega}(s;\delta)}{N-s}. \tag{4.23}$$

For the proof, see [LapRaŽu8, Thm. 5.22]. Note also that as a consequence of the principle of analytic continuation, the functional equations (4.23) and (4.22) continue to hold on any connected open set containing the open vertical strip $\{\overline{\dim}_B(A,\Omega) < \operatorname{Re} s < N\}$ and to which any (and hence, each) of the zeta functions involved has a meromorphic continuation. Consequently, for an RFD (A,Ω) in $\mathbb{R}^N$ and provided that $\overline{\dim}_B(A,\Omega) < N$, we may also define the visible complex dimensions of an RFD as the poles of the meromorphic continuation of its Mellin zeta function in a given domain $U\supseteq\{\operatorname{Re} s = D\}$.

Theorem 4.8. *Assume that (A,Ω) is a nondegenerate RFD in $\mathbb{R}^N$, that is, $0 < \mathcal{M}^D_*(A,\Omega) \le \mathcal{M}^{*D}(A,\Omega) < \infty$ (in particular, $D := \dim_B(A,\Omega)$ exists), and $D < N$. If $\zeta^{\mathfrak{M}}_{A,\Omega}$ can be extended meromorphically to a connected open neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta^{\mathfrak{M}}_{A,\Omega}$ and*

$$\mathcal{M}^D_*(A,\Omega) \le \operatorname{res}(\zeta^{\mathfrak{M}}_{A,\Omega}, D) \le \mathcal{M}^{*D}(A,\Omega). \tag{4.24}$$

Furthermore, if (A,Ω) is Minkowski measurable, then $\operatorname{res}(\zeta^{\mathfrak{M}}_{A,\Omega}, D) = \mathcal{M}^D(A,\Omega)$.

Before stating the distributional tube formula for relative fractal drums in terms of the Mellin zeta function [LapRaŽu8, Thm. 5.28], we have to introduce the notion of distributional asymptotics (see [EsKa, JaffMey, PiStVi] and also, independently, [Lap-vFr2, Def. 5.29] and [Lap-vFr1]).

Let $\mathcal{D}(0,+\infty) := C_c^\infty(0,+\infty)$ be the Schwartz space of infinitely differentiable functions with compact support contained in $(0,+\infty)$. For a test function $\varphi\in\mathcal{D}(0,+\infty)$ and $a>0$, we let $\varphi_a(t) := a^{-1}\varphi\left(a^{-1}t\right)$.

Definition 4.9. Let $\mathcal{R}$ be a distribution in $\mathcal{D}'(0,\delta)$ and let $\alpha\in\mathbb{R}$. We say that $\mathcal{R}$ is of *asymptotic order* at most t^α (resp., less than t^α) as $t\to 0^+$ if applied to an arbitrary test function φ_a in $\mathcal{D}(0,\delta)$, we have that[16]

$$\langle\mathcal{R},\varphi_a\rangle = O(a^\alpha) \quad (\text{resp., } \langle\mathcal{R},\varphi_a\rangle = o(a^\alpha)), \quad \text{as} \quad a\to 0^+. \tag{4.25}$$

We then write that $\mathcal{R}(t) = O(t^\alpha)$ (resp., $\mathcal{R}(t) = o(t^\alpha)$), as $a\to 0^+$.

[16] In this formula, the implicit constant may depend on the test function φ.

Note that it is easy to see that if f is a continuous function satisfying a classic pointwise asymptotics, then f also satisfies the same asymptotics, in the distributional sense of Definition 4.9.

Theorem 4.10 (Distributional fractal tube formula with error term, via $\zeta_{A,\Omega}^{\mathfrak{M}}$; level $k=0$)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega)<N$. Furthermore, assume that $\zeta_{A,\Omega}^{\mathfrak{M}}$ satisfies the languidity conditions* **L1** *and* **L2** *of Definition 2.12, for some $\kappa\in\mathbb{R}$ and $\delta>0$. Then, the regular distribution $\mathcal{V}_{A,\Omega}(t):=|A_t\cap\Omega|$ in $\mathcal{D}'(0,+\infty)$ is given by the following distributional identity in $\mathcal{D}'(0,+\infty)$:*

$$|A_t\cap\Omega|=\sum_{\omega\in\mathcal{P}(\zeta_{A,\Omega}^{\mathfrak{M}},\boldsymbol{W})}\operatorname{res}\left(t^{N-s}\zeta_{A,\Omega}^{\mathfrak{M}}(s),\omega\right)+\mathcal{R}_{A,\Omega}^{\mathfrak{M}}(t). \tag{4.26}$$

That is, the action of $\mathcal{V}_{A,\Omega}$ on an arbitrary test function $\varphi\in\mathcal{D}(0,+\infty)$ is given by

$$\langle\mathcal{V}_{A,\Omega},\varphi\rangle=\sum_{\omega\in\mathcal{P}(\zeta_{A,\Omega}^{\mathfrak{M}},\boldsymbol{W})}\operatorname{res}\Big(\{\mathfrak{M}\varphi\}(N-s+1)\,\zeta_{A,\Omega}^{\mathfrak{M}}(s),\omega\Big)+\langle\mathcal{R}_{A,\Omega}^{\mathfrak{M}},\varphi\rangle. \tag{4.27}$$

Here, the distributional error term $\mathcal{R}_{A,\Omega}^{\mathfrak{M}}$ is the distribution in $\mathcal{D}'(0,+\infty)$ given for all $\varphi\in\mathcal{D}(0,+\infty)$ by

$$\langle\mathcal{R}_{A,\Omega}^{\mathfrak{M}},\varphi\rangle=\frac{1}{2\pi\mathbb{i}}\int_{\boldsymbol{S}}\{\mathfrak{M}\varphi\}(N-s+1)\,\zeta_{A,\Omega}^{\mathfrak{M}}(s)\,\mathrm{d}s. \tag{4.28}$$

Moreover, the distribution $\mathcal{R}_{A,\Omega}^{\mathfrak{M}}(t)$ is of asymptotic order at most $t^{N-\sup S}$ as $t\to 0^+$; i.e., in the sense of Definition 4.9, we have

$$\mathcal{R}_{A,\Omega}^{\mathfrak{M}}(t)=O(t^{N-\sup S})\quad\text{as}\quad t\to 0^+. \tag{4.29}$$

In addition, if $S(\tau)<\sup S$ for all $\tau\in\mathbb{R}$ (that is, if the screen $\boldsymbol{S}$ lies strictly to the left of the vertical line $\{\operatorname{Re}s=\sup S\}$), then $\mathcal{R}_{A,\Omega}^{\mathfrak{M}}(t)$ is of asymptotic order less than $t^{N-\sup S}$; i.e., still in the sense of Definition 4.9,

$$\mathcal{R}_{A,\Omega}^{\mathfrak{M}}(t)=o(t^{N-\sup S})\quad\text{as}\quad t\to 0^+. \tag{4.30}$$

We have now introduced all of the prerequisites needed to state and prove the announced necessary condition for the Minkowski measurability of relative fractal drums. We stress that in the statement of the following theorem, the phrase according to which the Mellin zeta function $\zeta_{A,\Omega}^{\mathfrak{M}}$ is languid means that $\zeta_{A,\Omega}^{\mathfrak{M}}$ satisfies the languidity conditions of Definition 2.12 for some *languidity exponent* $\kappa\in\mathbb{R}$, with the caveat that in condition **L1** we now assume that $c\in(\overline{\dim}_B(A,\Omega),N)$.

Theorem 4.11 (Necessary condition for Minkowski measurability)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D:=\dim_B(A,\Omega)$ exists, $D<N$ and (A,Ω) is Minkowski measurable. Furthermore, assume that its Mellin zeta function $\zeta_{A,\Omega}^{\mathfrak{M}}$ is languid for some screen $\boldsymbol{S}$ passing strictly to the left of the critical line $\{\operatorname{Re}s=D\}$ and strictly to the right of all the complex dimensions of (A,Ω) with real part strictly less than D.*

Then, D is the only pole of $\zeta_{A,\Omega}^{\mathfrak{M}}$ located on the critical line $\{\operatorname{Re}s=D\}$ and it is simple.

Proof. Since (A,Ω) is languid, the hypotheses of Theorem 4.8 are satisfied and, therefore, $s=D$ is a simple pole of $\zeta_{A,\Omega}^{\mathfrak{M}}$. Furthermore, also by Theorem 4.8, we have that $\mathcal{M}:=\mathcal{M}^D(A,\Omega)=\operatorname{res}(\zeta_{A,\Omega}^{\mathfrak{M}},D)$. It remains to show that D is the only pole located on the critical line. First, we deduce at once from the definition of the Mellin zeta function given in Equation (4.19) that $|\zeta^{\mathfrak{M}}_{A,\Omega}(s)| \le \zeta^{\mathfrak{M}}_{A,\Omega}(\operatorname{Re} s)$, for all $s \in \{D < \operatorname{Re} s < N\}$. We conclude from this inequality that if ξ is another pole of $\zeta^{\mathfrak{M}}_{A,\Omega}$ with $\operatorname{Re}\xi = D$, then it must also be simple.

Now, let us denote by $\xi_n := D + \mathbb{i}\gamma_n$, with $\gamma_n \in \mathbb{R}$ and $n \in J$, the potentially infinite sequence of poles of $\zeta^{\mathfrak{M}}_{A,\Omega}$ with real part D (i.e., of principal poles of $\zeta^{\mathfrak{M}}_{A,\Omega}$). (Recall that a meromorphic function has at most countably many poles.) Here, $J \subseteq \mathbb{N}_0$ is a finite or infinite subset of $\mathbb{N}_0$, $0 \in J$, and we use the convention according to which $\gamma_0 := 0$ and hence, $\xi_0 := D$. Since D is simple (i.e., a simple pole of $\zeta^{\mathfrak{M}}_{A,\Omega}$), we must have $\gamma_n \neq 0$ for all $n \in J \setminus \{0\}$.

Observe that in light of the argument given in the first part of the proof, each principal pole ξ_n is then also *simple*, for every $n \in J \setminus \{0\}$; so that we can let $a_n := \operatorname{res}(\zeta^{\mathfrak{M}}_{A,\Omega}, \xi_n)$, for every $n \in J$. Furthermore, as was established at the beginning of the proof, we have $a_0 = \operatorname{res}(\zeta^{\mathfrak{M}}_{A,\Omega}, D) = \mathcal{M}$, the Minkowski content of (A,Ω). (Recall that the RFD (A,Ω) is assumed to be Minkowski measurable.)

Next, we will show that $J \setminus \{0\}$ is empty and therefore, that D is the only principal pole of $\zeta^{\mathfrak{M}}_{A,\Omega}$ (and is simple), as desired. For this purpose, we reason by contradiction and assume that $J \setminus \{0\}$ is nonempty. Then, in light of Theorem 4.10 (the distributional fractal tube formula at level $k = 0$ via $\zeta^{\mathfrak{M}}_{A,\Omega}$, in the terminology of [LapRaŽu8]) applied with the stronger error estimate given by (4.30) and for the same choice of screen $\boldsymbol{S}$ as assumed to exist in the statement of that theorem (and which also exists according to the hypotheses of the present theorem), we have that

$$
\begin{aligned}
|A_t \cap \Omega| &= \sum_{n \in J} a_n t^{N-\xi_n} + o(t^{N-D}) \\
&= \mathcal{M} t^{N-D} + t^{N-D} \sum_{n \in J \setminus \{0\}} a_n t^{-\mathbb{i}\gamma_n} + o(t^{N-D}) \quad \text{as} \quad t \to 0^+,
\end{aligned}
\tag{4.31}
$$

in the distributional sense since, by assumption, the screen $\boldsymbol{S}$ lies strictly to the left of the critical line $\{\operatorname{Re} s = D\}$.

On the other hand, since (A,Ω) is Minkowski measurable, we know that its relative tube function satisfies

$$
|A_t \cap \Omega| = \mathcal{M} t^{N-D} + o(t^{N-D}) \quad \text{as} \quad t \to 0^+, \tag{4.32}
$$

in the usual pointwise sense and hence also, in the distributional sense. Combining (4.31) with (4.32) yields that

$$
\sum_{n \in J \setminus \{0\}} a_n t^{-\mathbb{i}\gamma_n} = o(1) \quad \text{as} \quad t \to 0^+, \tag{4.33}
$$

in the distributional sense. After a (distributional) change of variable $\tau := \log t$,[17] the uniqueness theorem for almost periodic distributions (see [Schw, §VI.9.6, p. 208]) can be applied and now implies that (4.33) can only be true if $a_n = 0$ for all $n \in J \setminus \{0\}$; that is, only if $J \setminus \{0\}$ is empty (which contradicts our assumption) or, equivalently, only if there are no other poles on the critical line, except for $s = D$, as we needed to show. □

Remark 4.12. The above theorem can also be stated in terms of the relative distance zeta function of (A,Ω). This follows from the fact that the functional

[17] Note that $\tau\colon (0,+\infty) \to \mathbb{R}$ is infinitely differentiable.

equation (4.23) which connects the relative distance zeta function and the Mellin zeta function of (A,Ω) imply that if the d-languidity conditions **L1** and **L2** are satisfied by the distance zeta function, then they are also satisfied by the Mellin zeta function. with a possibly different languidity exponent. We may still, however, apply Theorem 4.10 in this case.

Remark 4.13. It clearly follows from the proof of Theorem 4.11 that it would suffice to assume in the statement of that theorem that the RFD (A,Ω) is Minkowski measurable *in the distributional sense* (which specifically means in the present context that Equation (4.32) holds as a distributional identity in $\mathcal{D}'(0,+\infty)$, with $\mathcal{M}\in(0,+\infty)$). We stress, however, that the notion of Minkowski measurability characterized in all of the criteria stated in this paper (namely, Theorem 4.14, Theorem 4.16 and Corollary 4.17) is always that of pointwise (or ordinary) Minkowski measurability.

Finally, we can now state the announced Minkowski measurability criterion, the proof of which follows directly from Theorems 4.2 and 4.11. We will state two versions of this criterion, in Theorem 4.14 and in Theorem 4.16, expressed respectively in terms of the distance and the tube zeta functions.

Theorem 4.14 (Minkowski measurability criterion in terms of $\zeta_{A,\Omega}$)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D:=\dim_B(A,\Omega)$ exists and $D<N$. Furthermore, assume that (A,Ω) is d-languid for a screen passing* strictly *between the critical line $\{\mathrm{Re}\, s=D\}$ and all the complex dimensions of (A,Ω) with real part strictly less than D. Then the following statements are equivalent*:

(a) *The RFD (A,Ω) is Minkowski measurable.*

(b) *D is the only pole of the relative distance zeta function $\zeta_{A,\Omega}$ located on the critical line $\{\mathrm{Re}\, s=D\}$ and it is simple.*

Remark 4.15. The above criterion is also valid if in (b), we replace $\zeta_{A,\Omega}$ with the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$ or the Mellin zeta function $\zeta^{\mathfrak{M}}_{A,\Omega}$. In that case, it suffices to assume that the chosen fractal zeta function satisfies the usual languidity conditions (along with the condition from Theorem 4.14 about the existence of a suitable screen). In fact, if we state the theorem in terms of the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$, we may omit the condition that $\dim_B(A,\Omega)<N$, as we shall see in Theorem 4.16 just below.

Next, we give the counterpart of Theorem 4.14, but now in terms of the tube zeta function $\widetilde{\zeta}_{A,\Omega}$ (instead of the distance zeta function $\zeta_{A,\Omega}$) and with the restriction $\dim_B(A,\Omega)<N$ removed in this case.

Theorem 4.16 (Minkowski measurability criterion in terms of $\widetilde{\zeta}_{A,\Omega}$)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $D:=\dim_B(A,\Omega)$ exists. Furthermore, assume that (A,Ω) is languid for a screen $\boldsymbol{S}$ passing strictly to the left of the critical line $\{\mathrm{Re}\, s=D\}$ and strictly to the right of all the complex dimensions of (A,Ω) with real part strictly less than D. Then the following statements are equivalent*:

(a) *The RFD (A,Ω) is Minkowski measurable.*

(b) *D is the only pole of the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$ located on the critical line $\{\mathrm{Re}\, s=D\}$, and it is simple.*

Proof. First of all, if $D = \dim_B(A, \Omega) < N$, then, again, the conclusion of the theorem follows from Theorems 4.2 and 4.11 together with Remark 4.12.

In the case when $D = N$, one embeds (A, Ω) into $\mathbb{R}^{N+1}$, as was done in [LapRaŽu3, §3.2], and then uses the results of the present section. The fact that (b) implies (a) is a consequence of Theorem 4.2 since there are no restrictions of the type $\dim_B(A, \Omega) < N$ in the hypotheses of that theorem. Actually, it follows directly from the definition of the relative Minkowski content that $\dim_B(A, \Omega) = N$ implies that $\mathcal{M}^N(A, \Omega)$ exists and $\mathcal{M}^N(A, \Omega) = |\overline{A} \cap \Omega|$.

In order to prove that (a) implies (b), we embed (A, Ω) into $\mathbb{R}^{N+1}$ as $(A, \Omega)_1 := (A \times \{0\}, \Omega \times (-\delta, \delta))$, for some suitable $\delta > 0$, and then by [LapRaŽu3, Thm. 3.9], conclude that the relative tube zeta functions of the RFDs (A, Ω) and $(A, \Omega)_1$ are connected by the following approximate functional equation:

$$\tilde{\zeta}_{A\times\{0\},\Omega\times(-\delta,\delta)}(s;\delta) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)}\tilde{\zeta}_{A,\Omega}(s;\delta) + E(s;\delta). \tag{4.34}$$

Here, $\delta > 0$ is chosen such that $\tilde{\zeta}_{A,\Omega}(s; \delta)$ satisfies the languidity hypothesis of the theorem. We will now show that $\tilde{\zeta}_{A\times\{0\},\Omega\times(-\delta,\delta)}(\,\cdot\,; \delta)$ satisfies the needed languidity conditions and use Theorem 4.11 in order to complete the proof. The error function $E(\,\cdot\,; \delta)$ is holomorphic on the open left half-plane $\{\operatorname{Re} s < N + 1\}$ and bounded by $2\delta^{\operatorname{Re} s-N}|A_\delta \cap \Omega|_N \left(\frac{\pi}{2} - 1\right)$ (see the proof of [LapRaŽu3, Thm. 3.9]). In other words, $E(\,\cdot\,; \delta)$ is languid (with a languidity exponent equal to 0). Furthermore, for any $a, b \in \mathbb{C}$ such that $\operatorname{Re}(b - a) > 0$, we have the following pointwise asymptotic expansion:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(a-b+1)}(a)}{n!}\frac{\Gamma(b-a+n)}{\Gamma(b-a)}\frac{1}{z^n} \quad \text{as} \quad |z| \to +\infty, \tag{4.35}$$

in the sector $|\arg z| < \pi$.[18] Substituting $z := \frac{N-s}{2} + 1$, $a := 0$ and $b := 1/2$ into Equation (4.35), we obtain that

$$\frac{\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)} \sim (N-s+2)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(2n)!\sqrt{2}(-1)^n B_n^{(1/2)}(0)}{2^n(n!)^2(N-s+2)^n} \quad \text{as} \quad |s| \to +\infty, \tag{4.36}$$

for all $s \in \mathbb{C} \setminus [N + 2, +\infty)$.[19] In particular, we have that

$$\frac{\Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+1-s}{2}+1\right)} = O(|s|^{-1/2}) \quad \text{as} \quad |s| \to +\infty, \tag{4.37}$$

for all $s \in \mathbb{C} \setminus [N + 2, +\infty)$, from which we conclude that the product of this ratio of gamma functions with the relative tube zeta function $\tilde{\zeta}_{A,\Omega}(\,\cdot\,; \delta)$ is languid with a languidity exponent not greater than $\kappa - 1/2$, where κ is the languidity exponent of $\tilde{\zeta}_{A,\Omega}(\,\cdot\,; \delta)$. This fact, along with the languidity of $E(\,\cdot\,; \delta)$ and Equation (4.34), implies that $\tilde{\zeta}_{A\times\{0\},\Omega\times(-\delta,\delta)}(\,\cdot\,; \delta)$ is languid with the same choice of a double sequence $(T_n)_{n\in\mathbb{Z}\setminus\{0\}}$ and the screen $\boldsymbol{S}$ as for $\tilde{\zeta}_{A,\Omega}(\,\cdot\,; \delta)$ and with a languidity exponent not greater than $\max\{\kappa - 1/2, 0\}$.

[18] Here, $B_n^{(\sigma)}(x)$ is the generalized Bernoulli polynomial (see, e.g., [SriTod] for the exact definition and an explicit expression). See also [Tem, §3.6.2] for this result on the asymptotics of ratios of gamma functions.

[19] We have used here the classic identity $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1/2 + n) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$.

On the other hand, if (A,Ω) is Minkowski measurable, then this is also true for the embedded RFD $(A\times\{0\},\Omega\times(-\delta,\delta))$. This fact follows in a completely analogous way as in the case of bounded subsets of $\mathbb{R}^N$ which was proven in [Res] and extended to RFDs in [LapRaŽu3, §3.2]. We now conclude the proof by invoking Theorem 4.11, or rather, its counterpart expressed in terms of the relative tube zeta function (see Remark 4.12). $\square$

In the next corollary of Theorem 4.14 and 4.16, and in light of [LapRaŽu8, Lem. 5.25][20] and Remark 4.12, we can indifferently interpret the (principal) complex dimensions of the RFD (A,Ω) as being the (principal) poles of either the distance, tube, or Mellin zeta function of (A,Ω). This is the reason why we assume that the hypotheses of both Theorems 4.14 and 4.16 are satisfied (i.e., we assume that $\dim_B(A,\Omega)<N$ in order to avoid the situation when N is a pole of the tube zeta function but is not a pole of the distance zeta function, which may happen).

Corollary 4.17 (Characterization of Minkowski measurability in terms of the complex dimensions)**.** *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$, with $N\geq 1$ arbitrary, such that $D:=\dim_B(A,\Omega)$ exists and $D<N$. Assume also that any of its fractal zeta functions (specifically, $\zeta_{A,\Omega}$ or $\widetilde{\zeta}_{A,\Omega}$, respectively) satisfies the hypotheses of Theorem 4.14 (or of Theorem 4.16, respectively) concerning the languidity and the screen. Then, the following statements are equivalent:*

(*a*) *The RFD (A,Ω) is (strongly) Minkowski measurable.*

(*b*) *D is the only complex dimension of the RFD (A,Ω) with real part equal to D (i.e., located on the critical line $\{\operatorname{Re} s=D\}$), and it is simple.*

5. Complex dimensions and gauge-Minkowski measurability

In this section, we first obtain a very general result (Theorem 5.4) about generating h-Minkowski measurable RFDs, where $h(t):=(\log t^{-1})^{m-1}$ for all $t\in(0,1)$ and m is a positive integer, by using only some information about the principal poles and their multiplicities. Its proof rests on the use of the pointwise fractal tube formula with or without error term; [LapRaŽu8, Thm. 3.2] or [LapRaŽu8, Thm. 3.4], respectively. Especially important is the (pointwise) asymptotic expansion of the tube function stated in Equation (5.7), from which it is possible to deduce the optimal tube function asymptotic expansion for a class of h-Minkowski measurable RFDs, as stated in Theorem 5.6.

First, we recall the definitions of gauge functions and gauge Minkowski content. The latter notion is motivated, geometrically and physically, by the study of non power law scaling behavior which arises in many natural examples. (See [HeLap] and the relevant references therein.)

If (A,Ω) is (Minkowski) degenerate and such that $D:=\dim_B(A,\Omega)$ exists, we assume that

$$|A_t\cap\Omega|=t^{N-D}(F(t)+o(1))\quad\text{as}\quad t\to 0^+,\tag{5.1}$$

where $F:(0,\varepsilon_0)\to(0,+\infty)$, for some sufficiently small $\varepsilon_0>0$.

[20]This lemma states that, provided $\overline{\dim}_B(A,\Omega)<N$, the complex dimensions of a given RFD (A,Ω) in $\mathbb{R}^N$ do not depend on the choice of the associated fractal (i.e., distance or tube) zeta function in terms of which they are defined.

Let (A, Ω) be a degenerate RFD in $\mathbb{R}^N$. Then, (A, Ω) is *weakly degenerate* if $D = \dim_B(A, \Omega)$ exists and either $\mathcal{M}_*^D(A, \Omega) = 0$ (i.e., $\liminf_{t \to 0^+} F(t) = 0$) or $\mathcal{M}^{*D}(A, \Omega) = +\infty$ (i.e., $\limsup_{t \to 0^+} F(t) = +\infty$). See Equation (5.1). (There is a corresponding notion of strongly degenerate RFD but it will not be needed here.)

Weakly degenerate RFDs can be classified by their *gauge functions* h, if they exist (see Definition 5.1). We assume that the function $F(t)$ appearing in Equation (5.1) is of the form

$$F(t) = h(t) \quad \text{or} \quad F(t) = \tfrac{1}{h(t)}, \tag{5.2}$$

where $h : (0, \varepsilon_0) \to (0, +\infty)$, for some small $\varepsilon_0 > 0$, $h(t) \to +\infty$ as $t \to 0^+$ and

$$h(t) = O(t^{0)}) \quad \text{as} \quad t \to 0^+, \quad \text{with} \quad O(t^{0)}) := \bigcap_{\beta < 0} O(t^\beta). \tag{5.3}$$

Note that we need to assume that $h(t) = O(t^{0)})$ as $t \to 0^+$ in order to fix the value $D = \dim_B(A, \Omega)$; see Equation (5.1).

Definition 5.1. If a function $h : (0, \varepsilon_0) \to (0, +\infty)$ is of class $O(t^{0)})$ and converges to infinity as $t \to 0^+$, we then say that h is *of slow growth to infinity* as $t \to 0^+$. Analogously, a function $g : (0, \varepsilon_0) \to (0, +\infty)$ is said to be *of slow decay to zero* as $t \to 0^+$ if it is of the form $g(t) = 1/h(t)$, for some function h which is of slow growth to infinity as $t \to 0^+$. Such functions h and g are called *gauge functions*.

It is easy to see that a function $g : (0, \varepsilon_0) \to (0, +\infty)$ is of slow decay to zero as $t \to 0^+$ if and only if for every $\beta > 0$, $t^\beta = O(g(t))$ as $t \to 0^+$.

Example 5.2. If we define $h_1(t) = \log t^{-1}$, $h_2(t) = \log \log t^{-1}$, and more generally, $h_3(t) = (\log t^{-1})^a$, $h_4(t) = (\log^k t^{-1})^a$, for all $t \in (0, 1)$ (here, $a > 0$, $k \in \mathbb{N}$, and $\log^k$ denotes the k-fold composition of logarithms), then all of these functions are of slow growth to infinity as $t \to 0^+$. Furthermore, their reciprocals are functions of slow decay to 0 as $t \to 0^+$.

Since for a weakly degenerate RFD (A, Ω) we have $\mathcal{M}^{*D}(A, \Omega) = +\infty$ or $\mathcal{M}_*^D(A, \Omega) = 0$, it will be convenient to define (as in [HeLap]) the *upper* and *lower* D-dimensional *Minkowski contents of* (A, Ω) *with respect to a given gauge function* h, as follows:

$$\mathcal{M}^{*D}(A, \Omega, h) = \limsup_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} h(t)}, \quad \mathcal{M}_*^D(A, \Omega, h) \quad = \liminf_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} g(t)}. \tag{5.4}$$

The aim is to find gauge functions h and g so that the upper and lower Minkowski contents of (A, Ω) with respect to h are nondegenerate, that is, belong to $(0, +\infty)$. If $\mathcal{M}_*^D(A, \Omega, h) = \mathcal{M}^{*D}(A, \Omega, h)$, this common value is denoted by $\mathcal{M}^D(A, \Omega, h)$ and called the *h-Minkowski content* of (A, Ω). We then say that (A, Ω) is *h-Minkowski measurable* if, in addition, $\mathcal{M}^D(A, \Omega, h) \in (0, +\infty)$.

Definition 5.3. Assume that (A, Ω) is an RFD in $\mathbb{R}^N$ such that (5.1) holds under one of the conditions stated in (5.2) and that, in addition, (5.3) is satisfied. We then say that $h = h(t)$ or $g = 1/h(t)$ is a *gauge function* of (A, Ω).[21] We also say that the RFD (A, Ω) is *weakly degenerate, of type h or* $1/h$, respectively.

[21] In the case when $F(t) = g(t)$, we also assume that the implied function $o(1)$ appearing in (5.1) satisfies $o(1)/g(t) \to 0$ as $t \to 0^+$.

Note that in the first case of (5.2), we have

$$\mathcal{M}^{*D}(A,\Omega)=+\infty,\quad \mathcal{M}^{*D}(A,\Omega,h)\in(0,+\infty),$$

while in the second case of (5.2), we have

$$\mathcal{M}_*^D(A,\Omega)=0,\quad \mathcal{M}_*^D(A,\Omega,1/h)\in(0,+\infty).$$

Let (A,Ω) be a weakly degenerate RFD in $\mathbb{R}^N$ of type h, in the sense of Definition 5.3. We say that (A,Ω) is *h-Minkowski nondegenerate* if $0<\mathcal{M}_*^D(A,\Omega,h)<\mathcal{M}^{*D}(A,\Omega,h)<\infty$.

We adopt a similar terminology for the gauge function $1/h$ instead of h; see Definition 5.3 above.

Theorem 5.4. (Generating h-Minkowski measurable RFDs). *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ which is languid with languidity exponent $\kappa<-1$ or such that $(\lambda A,\lambda\Omega)$ is strongly languid for some $\lambda>0$ with languidity exponent $\kappa<0$, for a screen $\boldsymbol{S}$ passing strictly between the critical line $\{\operatorname{Re}s=\overline{\dim}_B(A,\Omega)\}$ and all the complex dimensions of (A,Ω) with real part strictly less than $\overline{D}:=\overline{\dim}_B(A,\Omega)$. Furthermore, suppose that $\overline{D}$ is the only pole of its relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$ with real part equal to $\overline{D}$ of order $m\geq 1$ and, additionally, that there exists* (*at most*) *finitely many nonreal poles of $\widetilde{\zeta}_{A,\Omega}$ with real part $\overline{D}$. Moreover, assume that the multiplicity of each of those nonreal poles is of order strictly less than m. Then, $\dim_B(A,\Omega)$ exists and is equal to $D:=\overline{D}$. Further, $\mathcal{M}^D(A,\Omega)$ exists and is equal to $+\infty$; hence, (A,Ω), is Minkowski degenerate.*

Finally, an appropriate gauge function for (A,Ω) is $h(t):=(\log t^{-1})^{m-1}$, for all $t\in(0,1)$, and we have that the RFD (A,Ω) is h-Minkowski measurable with h-Minkowski content given by

$$\mathcal{M}:=\mathcal{M}^D(A,\Omega,h)=\frac{\widetilde{\zeta}_{A,\Omega}[D]_{-m}}{(m-1)!},\tag{5.5}$$

where $\widetilde{\zeta}_{A,\Omega}[D]_{-m}$ denotes the coefficient corresponding to $(s-D)^{-m}$ in the Laurent expansion of $\widetilde{\zeta}_{A,\Omega}$ around $s=D$. Moreover, if there is at least one nonreal complex dimension on the critical line $\{\operatorname{Re}=D\}$, then the tube function $t\mapsto|A_t\cap\Omega|$ satisfies the following pointwise asymptotic estimate,

$$|A_t\cap\Omega|=t^{N-D}h(t)\big(\mathcal{M}+O((\log t^{-1})^{-1})\big)\quad\text{as}\quad t\to 0^+,\tag{5.6}$$

while if D is the unique pole of $\widetilde{\zeta}_{A,\Omega}$ on the critical line (*i.e., the unique principal complex dimension of (A,Ω)*)*, we have the following sharper pointwise estimate*:

$$|A_t\cap\Omega|=t^{N-D}h(t)\big(\mathcal{M}+O(t^{D-\sup S})\big)\quad\text{as}\quad t\to 0^+.\tag{5.7}$$

Proof. Let $\omega_0:=\overline{D}:=\overline{\dim}_B(A,\Omega)$,[22] and let $\omega_j:=\overline{D}+\mathbb{i}\gamma_j$, where $\gamma_j\in\mathbb{R}\setminus\{0\}$ for $j\in J$ and J is a finite subset of $\mathbb{Z}\setminus\{0\}$. That is, $\{\omega_j\}_{j\in J}$ is the (finite) set of all the other poles of $\widetilde{\zeta}_{A,\Omega}$ located on the critical line $\{\operatorname{Re}s=\overline{D}\}$, i.e., with real part $\overline{D}$. We also let $\gamma_0:=0$ and $m_0:=m$, in order to be consistent with the notation introduced just below. Furthermore, for each $j\in J$, let m_j be the multiplicity of ω_j and then, by hypothesis of the theorem, we have that $m_j<m$ for every $j\in J$. By [LapRaŽu8, Thm. 3.2] and since the screen $\boldsymbol{S}$ is strictly to the right of all the

[22]It will follow from the proof that $\dim_B(A,\Omega)$ exists and that $\overline{D}=\dim_B(A,\Omega)$ and hence, is equal to D.

other complex dimensions of (A,Ω) with real part strictly less than $\overline{D}$ and strictly to the left of the critical line $\{\operatorname{Re} s = \overline{D}\}$, we obtain a pointwise tube formula for (A,Ω) with an error term that is of strictly higher asymptotic order as $t \to 0^+$ than the term corresponding to the residue at $s = \overline{D}$; that is, we have the following pointwise tube formula with error term:

$$|A_t \cap \Omega| = \sum_{j \in J\cup\{0\}} \operatorname{res}(t^{N-s}\widetilde{\zeta}_{A,\Omega}(s), \omega_j) + O(t^{N-\sup S}) \quad \text{as } t \to 0^+. \tag{5.8}$$

We next consider the Taylor expansion of t^{N-s} around $s = \omega_j$ (for each $j \in J\cup\{0\}$):

$$t^{N-s} = t^{N-\omega_j}\,\mathrm{e}^{(s-\omega_j)\log t^{-1}} = t^{N-\omega_j}\sum_{n=0}^{\infty}\frac{(\log t^{-1})^n}{n!}(s-\omega_j)^n; \tag{5.9}$$

we then multiply it by the Laurent expansion of $\widetilde{\zeta}_{A,\Omega}(s)$ around $s = \omega_j$ and extract the residue of this product in order to deduce that

$$\operatorname{res}(t^{N-s}\widetilde{\zeta}_{A,\Omega}(s), \omega_j) = t^{N-\omega_j}\sum_{n=0}^{m_j-1}\frac{(\log t^{-1})^n}{n!}\widetilde{\zeta}_{A,\Omega}[\omega_j]_{-n-1}. \tag{5.10}$$

In light of this identity and of (5.8), we conclude that $\dim_B(A,\Omega)$ exists and is equal to $\overline{D}$. Furthermore, since $m_j < m_0 = m$, we conclude that the highest power of $\log t^{-1}$ appearing in the fractal tube formula (5.8) is $m-1$, and that it appears only in the sum (5.10) when $j = 0$. Therefore, if we choose $h(t) := (\log t^{-1})^{m-1}$ for all $t \in (0,1)$ as our gauge function, the statements about the Minkowski content and the gauge Minkowski content (with respect to h) now also follow from the fractal tube formula (5.8).

We also easily deduce Equations (5.6) and (5.7) from (5.8) by rewriting (5.10) as follows:

$$\operatorname{res}(t^{N-s}\widetilde{\zeta}_{A,\Omega}(s), \omega_j) = t^{N-D}h(t)\sum_{n=0}^{m_j-1} t^{D-\omega_j}\frac{(\log t^{-1})^{n-m+1}}{n!}\widetilde{\zeta}_{A,\Omega}[\omega_j]_{-n-1}. \tag{5.11}$$

Indeed, for $j = 0$, we have that $m_0 = m$ and $\omega_0 = D$; so that the term on the right-hand side of (5.11) corresponding to $n = m-1$ is equal to $\frac{\widetilde{\zeta}_{A,\Omega}[\omega_j]_{-m}}{(m-1)!}$ (i.e., to $\mathcal{M}$, the h-Minkowski content of (A,Ω); see Equation (5.5) above), while for any $n \in \{0,\dots,m-2\}$ (if this set is nonempty, i.e., if $m \geq 2$), we are left with a function which is $O((\log t^{-1})^{-1})$ as $t \to 0^+$ (if $m = 1$, the corresponding function is absent; i.e., it is equal to zero). Equations (5.6) and (5.7) then follow because for $j \neq 0$ ($j \in J$), we have that $|t^{D-\omega_j}| = 1$ (since $D - \omega_j$ is a purely imaginary complex number) and $(\log t^{-1})^{n-m+1} = O((\log t^{-1})^{-1})$ as $t \to 0^+$.

This concludes the proof of the theorem. □

Remark 5.5. In light of the proof of Theorem 5.4, the error term $O((\log t^{-1})^{-1})$ in Equation (5.6) can be improved to $O((\log t^{-1})^{n-m+1})$, where n is the largest positive integer such that $n < m-1$ and for which there exists $j \in J$ such that $\widetilde{\zeta}_{A,\Omega}[\omega_j]_{-n-1} \neq 0$.

The further to the left we can meromophically extend the tube zeta function $\widetilde{\zeta}_{A,\Omega}$ of a given RFD (A,Ω) (meaning, the smaller the value of $\sup S$), the sharper the estimate (5.7) in Theorem 5.4. In other words, denoting by $\mathcal{S}(A,\Omega)$ the family of all possible screens $\boldsymbol{S}$ such that the tube zeta function $\widetilde{\zeta}_{A,\Omega}$ admits a meromorphic extension to an open connected neighborhood of the corresponding window $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{S})$, it is natural to define the following (extended) real number:

$$\operatorname{or}(A,\Omega) := \inf_{S\in\mathcal{S}(A,\Omega)} \sup S \in [-\infty, \overline{\dim}_B(A,\Omega)], \tag{5.12}$$

which we call the *order* of the RFD (A,Ω). It is clear that

$$\operatorname{or}(A,\Omega) \leq D_{\rm mer}(\widetilde{\zeta}_{A,\Omega}), \tag{5.13}$$

where $D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$ is the abscissa of meromorphic continuation of the tube zeta function $\widetilde{\zeta}_{A,\Omega}$; i.e., $\{\operatorname{Re} s > D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})\}$ is the largest open right half-plane (of the form $\{\operatorname{Re} s > \alpha\}$, with $\alpha \in \mathbb{R}\cup\{\pm\infty\}$) to which $\widetilde{\zeta}_{A,\Omega}$ can be meromorphically continued. In light of Equation (5.7) in Theorem 5.4, we can then deduce the following significant conclusion.

Theorem 5.6. (Optimal tube function asymptotic expansion). *Let (A,Ω) be a relative fractal drum such that the conditions of Theorem 5.4 are satisfied, with $D := D(\widetilde{\zeta}_{A,\Omega})$ being the unique pole of $\widetilde{\zeta}_{A,\Omega}$ in the open right half-plane $\{\operatorname{Re} s > D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})\}$, of order $m \geq 1$. Then, (A,Ω) is h-Minkowski measurable and for any $\varepsilon > 0$, the tube function $t \mapsto |A_t \cap \Omega|$ has the following pointwise asymptotic expansion, with error term*:

$$|A_t\cap\Omega| = t^{N-D}h(t)\big(\mathcal{M} + O(t^{D-D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})-\varepsilon})\big) \quad \text{as} \quad t\to 0^+, \tag{5.14}$$

where $h(t) := (\log t^{-1})^{m-1}$ for all $t\in(0,1)$ and $\mathcal{M} = \mathcal{M}^D(A,\Omega,h)$ is the h-Minkowski content of (A,Ω).

Furthermore, the asymptotic formula in Equation (5.14) is optimal; that is, the exponent $D - D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$ cannot be replaced by a larger number. Moreover, the order of the RFD (A,Ω) is equal to the abscisssa of meromorphic continuation of the corresponding tube zeta function $\widetilde{\zeta}_{A,\Omega}$; i.e.,

$$\operatorname{or}(A,\Omega) = D_{\rm mer}(\widetilde{\zeta}_{A,\Omega}). \tag{5.15}$$

Proof. First of all, by using Theorem 5.4 and since for any $\varepsilon > 0$ there exists $S\in\mathcal{S}$ such that $\sup S > \operatorname{or}(A,\Omega)+\varepsilon$, we have the following pointwise asymptotic estimate:

$$|A_t\cap\Omega| = t^{N-D}h(t)\big(\mathcal{M} + O(t^{D-\operatorname{or}(A,\Omega)-\varepsilon})\big) \quad \text{as } t\to 0^+. \tag{5.16}$$

Equation (5.14) then follows from (5.13).

In order to establish the *optimality* of the asymptotic formula in Equation (5.14), we reason by contradiction. Assume that we have

$$|A_t\cap\Omega| = t^{N-D}h(t)\big(\mathcal{M} + O(t^{\alpha})\big) \quad \text{as} \quad t\to 0^+, \tag{5.17}$$

for some real number $\alpha > D - D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$. (Here, in this inequality, we can omit ε since without loss of generality, we can always choose a smaller real number α satisfying the same strict inequality.) In light of [LapRaŽu4, Thm. 2.24] (applied with $m-1$ instead of m), we conclude that $\widetilde{\zeta}_{A,\Omega}$ can be meromophically extended (at least) to the open right half-plane $\{\operatorname{Re} s > D-\alpha\}$. It then follows from the definition of the abscissa of meromorphic continuation that $D_{\rm mer}(\widetilde{\zeta}_{A,\Omega}) \leq D-\alpha$. On the other hand, we have assumed $D-\alpha < D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$, which contradicts the previous inequality.

Finally, in order to prove Equation (5.15), it suffices to note that, due to the optimality proved just above, we must have that $D - \operatorname{or}(A,\Omega) \leq D - D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$ (see Equations (5.16) and (5.14)); i.e., $\operatorname{or}(A,\Omega) \geq D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$, which together with (5.13), implies the desired equality (5.15). □

In other words, we have shown that under the hypotheses of Theorem 5.6, the larger the difference $\alpha(A,\Omega) := D(\widetilde{\zeta}_{A,\Omega}) - D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$ between the abscissa of (absolute) convergence and the abscissa of meromorphic continuation of the tube zeta function $\widetilde{\zeta}_{A,\Omega}$ of a given RFD (A,Ω), the better the asymptotic estimate (5.14) of the tube function $t \mapsto |A_t \cap \Omega|$ when $t \to 0^+$.

Our next result, Theorem 5.7, can be viewed as the converse of [LapRaŽu4, Thm. 2.24] (or, in the special case when $m = 1$ and of bounded sets instead of general RFDs, of [LapRaŽu2, Thm. 4.1]) about the existence of a meromorphic continuation of $\tilde{\zeta}_{A,\Omega}$ (or of $\zeta_{A,\Omega}$). It shows that in some precise sense, these results are optimal.

Theorem 5.7. *Let (A,Ω) be an RFD such that the conditions of Theorem 5.4 are satisfied. Furthermore, assume that there exists $\alpha > 0$ such that the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$ can be meromorphically extended to the open right half-plane $\{\operatorname{Re} s > D - \alpha\}$, with $D := D(\widetilde{\zeta}_{A,\Omega})$ being the unique pole of $\widetilde{\zeta}_{A,\Omega}$ in this right half-plane, of order $m \geq 1$. Then, (A,Ω) is h-Minkowski measurable and its tube function $t \mapsto |A_t \cap \Omega|$ has the following asymptotic expansion, with error term of order α:*

$$(5.18) \qquad |A_t \cap \Omega| = t^{N-D} h(t)\big(\mathcal{M} + O(t^\alpha)\big) \quad \text{as} \quad t \to 0^+,$$

where $h(t) := (\log t^{-1})^{m-1}$ for all $t \in (0,1)$ and $\mathcal{M} = \mathcal{M}(A,\Omega,h)$ is the h-Minkowski content of (A,Ω).

Moreover, if we let

$$(5.19) \qquad r(A,\Omega) := \sup\{\operatorname{Re} s : s \in \mathcal{P}(\widetilde{\zeta}_{A,\Omega}) \setminus \{D\}\},$$

then, for any $\varepsilon > 0$, the tube function $t \mapsto |A_t \cap \Omega|$ has the following pointwise asymptotic expansion, with error term:

$$(5.20) \qquad |A_t \cap \Omega| = t^{N-D} h(t)\big(\mathcal{M} + O(t^{D - r(A,\Omega) - \varepsilon})\big) \quad \text{as} \quad t \to 0^+,$$

Proof. Equation (5.18) follows from Equation (5.6) of Theorem 5.4, by choosing the screen $\boldsymbol{S}$ to be the vertical line $\{\operatorname{Re} s = D - \alpha\}$; that is, $S(x) = D - \alpha$ for all $x \in \mathbb{R}$. Indeed, in this case, we have $D - \sup S = D - (D - \alpha) = \alpha$.

Equation (5.20) follows easily from (5.18) by letting $\alpha := D - r(A,\Omega) - \varepsilon$, with $\varepsilon > 0$ small enough. Indeed, for such an ε, $s = D$ is the only pole in the open right half-plane $\{\operatorname{Re} s > r(A,\Omega) + \varepsilon = D - \alpha\}$. □

In light of the functional equation (2.4) connecting the relative distance zeta function with the relative tube zeta function, it is clear that when $\overline{\dim}_B A < N$, the value of $\alpha(A,\Omega)$ can be analogously defined by using the relative distance zeta function $\zeta_{A,\Omega}$ instead of the relative tube zeta function $\widetilde{\zeta}_{A,\Omega}$. Furthermore, $D(\zeta_{A,\Omega}) = D(\widetilde{\zeta}_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$, $D_{\rm mer}(\zeta_{A,\Omega}) = D_{\rm mer}(\widetilde{\zeta}_{A,\Omega})$, and the analog of Theorem 5.6 can be easily stated and proved in the case of the relative distance zeta function of a given RFD (A,Ω), instead of the relative tube zeta function.

Remark 5.8. It may be that the conclusion of Theorem 5.4 (and of Theorem 5.9 below, respectively) is also true in the case when there exists an infinite sequence of nonreal complex dimensions of (A,Ω) with real part $\overline{D}$ such that each of them has multiplicity strictly less than that of $\overline{D}$.[23] The fractal tube formula (5.8) ((5.23), respectively) also holds pointwise in this case, but in order to obtain the conclusion about $\dim_B(A,\Omega)$ and the h-Minkowski measurability of (A,Ω), we have to be able to justify the interchange of the limit as $t\to 0^+$ and the infinite sum which appears in this case in Equation (5.8) (respectively, (5.23)). A priori, we do not have such a justification to our disposal without making additional assumptions on the nature of the convergence of the sum in (5.8) (respectively, (5.23)).

It would be interesting to try to extend the above result and obtain a type of gauge Minkowski measurability criterion, in the likes of Theorem 4.14. (Some of the results obtained in [HeLap] may be useful for this purpose.)[24] See [LapRaŽu4, Thm. 2.24] for a partial converse of the above theorem in the case when the relative tube function satisfies the following pointwise asymptotic expansion, with error term, for some $m\in\mathbb{N}$ and $\alpha>0$:

$$|A_t\cap\Omega| = t^{N-D}(\log t^{-1})^{m-1}(\mathcal{M}+O(t^{\alpha}))\quad\text{as}\quad t\to 0^+. \tag{5.21}$$

As always, we can reformulate the above theorems in terms of the distance (instead of the tube) zeta function. As an example, we state the counterpart for $\zeta_{A,\Omega}$ of Theorem 5.4.

Theorem 5.9. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega)<N$. Also assume that (A,Ω) is d-languid with $\kappa_d<0$ or is such that $(\lambda A,\lambda\Omega)$ is strongly d-languid (in the sense of* [LapRaŽu4]*) for some $\lambda>0$ with $\kappa_d<1$, for a screen $\boldsymbol{S}$ passing strictly between the critical line $\{\operatorname{Re} s=\overline{\dim}_B(A,\Omega)\}$ and all the complex dimensions of (A,Ω) with real part strictly less than $\overline{D}:=\overline{\dim}_B(A,\Omega)$. Furthermore, suppose that $\overline{D}$ is the only pole of the relative distance zeta function $\zeta_{A,\Omega}$ with real part equal to $\overline{D}$ of order $m\geq 1$ and, additionally, that there exists (at most) finitely many nonreal poles of $\zeta_{A,\Omega}$ with real part $\overline{D}$. Moreover, assume that the multiplicity of each of those nonreal poles is of order strictly less than m. Then, $\dim_B(A,\Omega)$ exists and is equal to $D:=\overline{D}$. Further, $\mathcal{M}^D(A,\Omega)$ exists and is equal to $+\infty$; hence, (A,Ω), is Minkowski degenerate.*

In addition, an appropriate gauge function for (A,Ω) is $h(t):=(\log t^{-1})^{m-1}$ for all $t\in(0,1)$ and we have that, the RFD (A,Ω) is h-Minkowski measurable, with h-Minkowski content given by

$$\mathcal{M}^D(A,\Omega,h)=\frac{\zeta_{A,\Omega}[D]_{-m}}{(N-D)(m-1)!}, \tag{5.22}$$

where $\zeta_{A,\Omega}[D]_{-m}$ denotes the coefficient corresponding to $(s-D)^{-m}$ in the Laurent expansion of $\zeta_{A,\Omega}$ around $s=D$.

[23] See Example 3.13 where we are in such a situation and the conclusion of Theorem 5.9 holds.

[24] Recall that in [HeLap], a gauge Minkowski measurability criterion was obtained for fractal strings, extending to the case of non power laws the one obtained (when $h\equiv 1$) in [LapPo2]. This criterion does not involve the notion of complex dimensions and is stated only in terms of the underlying gauge function h and the asymptotic behavior of the lengths of the string.

Finally, the exact same conclusions as in Theorem 5.4 hold concerning the asymptotic expansion of $|A_t \cap \Omega|$ *in either* (5.6) *or* (5.7)*, but with* $\zeta_{A,\Omega}$ *in place of* $\widetilde{\zeta}_{A,\Omega}$ *in the respective hypotheses.*

Proof. We will prove the theorem in the special case when $\overline{D}$ is the only pole with real part equal to $\overline{D}$. The general case then follows analogously as in the proof of Theorem 5.4. Let $\overline{D} := \overline{\dim}_B(A,\Omega)$. By [LapRaŽu8, Thm. 5.11] with $k=0$ in the notation of that theorem, we have the following asymptotic pointwise tube formula, with error term:

$$|A_t \cap \Omega| = \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \overline{D}\right) + O(t^{N-\sup S}) \quad \text{as } t \to 0^+. \tag{5.23}$$

Furthermore, we expand $(N-s)^{-1}$ into a Taylor series around $s = \overline{D}$:

$$\frac{1}{N-s} = \sum_{n=0}^{\infty} \frac{(-1)^n (s-\overline{D})^n}{n!(N-\overline{D})^{n+1}}; \tag{5.24}$$

we then multiply the resulting Taylor series by the expression on the right-hand side of (5.9) in order to obtain the following Taylor expansion of $t^{N-s}/(N-s)$ around $s = \overline{D}$:

$$\frac{t^{N-s}}{N-s} = \sum_{n=0}^{\infty} (s-\overline{D})^n \sum_{k=0}^{n} \frac{(-1)^{n-k}(\log t^{-1})^k}{k!(n-k)!(N-\overline{D})^{n-k+1}}. \tag{5.25}$$

We next multiply the above Taylor series (in (5.25)) by the Laurent expansion of $\zeta_{A,\Omega}(s)$ around $s = \overline{D}$ and extract the residue of the resulting product to deduce that

$$\operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \overline{D}\right) = t^{N-\overline{D}} \sum_{n=0}^{m-1} \sum_{k=0}^{n} \frac{(-1)^{n-k}(\log t^{-1})^k \zeta_{A,\Omega}[\,\overline{D}\,]_{-n-1}}{k!(n-k)!(N-\overline{D})^{n-k+1}}.$$

We then complete the proof of the theorem by reasoning analogously as in the proof of Theorem 5.4. ☐

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