INVARIANT MEASURES OF SCALAR SEMILINEAR PARABOLIC DIFFERENTIAL EQUATIONS

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Abstract. Our goal is to describe ergodic-theoretical properties of scalar semilinear parabolic differential equations on the circle, and on the entire $\mathbb{R}$ without decay to zero at infinity (the extended case), with either autonomous or time-periodic nonlinearity, and implications to dynamics. In all these cases, we show that the union of supports of all the measures on the appropriate space of functions, invariant with respect to evolution of solutions, projects one-to-one to $\mathbb{R}^2$. This holds also in the extended case, if one considers spatially invariant measures with a technical condition of finite density of zeroes. Furthermore, we give general sufficient conditions for uniqueness of the invariant measure, generalizing the results by Sinai for the periodically-forced viscous Burgers equation, and then establish a relatively complete description of asymptotics of dynamics in these cases in both the bounded and the extended case.

The main technique is the zero function lifted to the space of Borel probability measures on the space of functions, which is a Lyapunov function with respect to the evolution of measures induced by two "replicas" of the equation. The approach seems to extend to other monotone scalar dynamical systems either without or with a random force, thus is relevant for questions on existence of physical and SRB measures (the non-random case) and phase transitions (the random case).

1. Introduction

We consider the following equation in various settings:

$$u_t = u_{xx} + g(t, x, u, u_x), \tag{1.1}$$

where g satisfies the usual conditions (A1-3) guaranteeing local existence of solutions, given below. We consider separately the case when g is independent of t (the *DC case*) and the case when g is 1-periodic in t (the *AC case*). Furthermore, we consider (1.1) on the *bounded* domain $\mathbb{S}^1$ parametrized with $[0, 1)$, i.e. with periodic boundary conditions. In that case, the phase space is $\mathcal{X}^\alpha = H^{2\alpha}(\mathbb{S}^1)$, where $\mathcal{X} = L^2(\mathbb{S}^1)$, and $3/4 < \alpha < 1$ is such that $\mathcal{X}^\alpha$ is continuously imbedded in $C^1(\mathbb{S}^1)$. Alternatively, we consider (1.1) in the *extended* case, where the domain is the entire $\mathbb{R}$ without assuming decay to zero at infinity. The phase space is then the fractional uniformly local space $\mathcal{X}^\alpha = H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$, where $\mathcal{X} = L^2_{\mathrm{ul}}(\mathbb{R})$, α is as above (see Appendix for key facts on uniformly local spaces), and then $H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$ is continuously imbedded in $C^1(\mathbb{R})$. The bounded case may be considered as an invariant subset of the extended case, as $H^{2\alpha}(\mathbb{S}^1)$ embeds naturally in $H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$, as the invariant set of spatially periodic solutions.

The asymptotics of (1.1) on the bounded domain with separated boundary conditions has been studied in detail (see [18, 24] and references therein) and is reasonably well-understood. In particular, under assumptions (A1-3), for any global, uniformly bounded orbit, the ω-limit set contains a single orbit (equilibria in the DC or a periodic orbit in the AC case) ([24], Theorem 4.2 and references therein). With periodic boundary conditions, i.e. in our setting in the B/DC case and assuming (A1-3), Fiedler and Mallet-Paret [10] have shown that the ω-limit set of any global, bounded solution projects to a plane, and then has the structure in accordance to the Poincaré-Bendixson theorem.

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That means that it consists of a single periodic orbits, or of equilibria and connecting (homoclinic and heteroclinic) orbits. Tereščák [37] has shown that in the B/AC case, assuming (A1-3), the ω-limit set of any global, bounded orbit also projects injectively and continuously into $\mathbb{R}^2$. The structure of the ω-limit set can then be much more complex, as shown by Fiedler and Sandstede [11, 27].

The structure of the attractor of (1.1) on the bounded domain with separated or periodic boundary conditions in the DC case is as follows: the attractor is then generically Morse-Smale, and can in many cases be classified by the graph structure of the equilibria and their connections ([12, 13, 18, 24] and references therein). Similar questions in the B/AC case, and the extended case seem to be currently beyond reach. When assuming decay to 0 at infinity, the dynamics in some cases (for example for g not depending on x, u_x [9]) is similar to the dynamics on the bounded domain with separated boundary conditions, i.e. uniformly bounded orbits then converge to a single periodic solution. If there is no decay to zero at infinity, the attractor seems to be typically infinite dimensional (assuming sufficiently weak topology so that uniformly locally bounded orbits are relatively compact, see Section 2), and the asymptotics can be very complex even in the "extended gradient case" (see [25] and references therein, also Subsection 10.4).

We propose here a different focus: to describe invariant measures of (1.1), and the union of supports of all the invariant measures, which we propose to call *ergodic attractor*. Prior to stating the results, we want to make three points: that this area of research is fertile, that the results are dynamically relevant, and that they are related to some other important areas of research.

First, we will show that in all four considered cases (B or E / AC or DC), the description of the ergodic attractor seems to be within reach even in the cases when the topological attractor is very complex. In all these cases, the ergodic attractor projects one-to-one to $\mathbb{R}^2$ (subject to a technical restriction of finite average density of zeroes in the extended case, which we believe to be generically true and likely redundant), and that in many cases (generalizing the viscous periodically forced Burgers equation), it is one-dimensional.

Secondly, the dynamical relevance is as follows: in the bounded case, the ergodic attractor contains all *ω-limit sets on average* of all relatively compact orbits (Subsection 3.2). The ω-limit set on average has been proposed in the context of partial differential equations in [15], and contains accumulation points of a relatively compact orbit for non-zero density of times. We argue that physically only these orbits are "observable" (Lemma 3.5), thus the description of the ergodic attractor reasonably completely describes "observable" dynamics. In particular, the ergodic attractor contains any "chaos" if present [34]. In the extended case, the ergodic attractor consists of "space-time observable" orbits (Subsection 3.3), the space-time chaos as constructed in [22, 39] if present [34], and frequently describes asymptotics of μ-a.e. $u \in \mathcal{X}^\alpha$ with respect to any S-invariant Borel probability measure μ on $\mathcal{X}^\alpha$, where $Su(x) = u(x-1)$ is the spatial shift (see results for Burgers like equation below; also Subsections 10.3 and 11.2).

Thirdly, we argue that the techniques developed here also extend to the equations (1.1) with an additional random force term such as for example considered in [7, 30], and also to discrete-space continuous-time, or discrete-space discrete-time 1d monotone systems without and with random force, as further discussed in Section 11. In particular, we hope that the main technique of the paper: the zero-function as a Lyapunov function with respect to evolution of measures induced by the dynamical system, can be useful in characterizing uniqueness of invariant measures, thus questions related to existence of physical and SRB measures in the deterministic case, and phase transitions in the random case of these models.

1.1. **Statements of results: the support of invariant measures.** The standing assumptions on the nonlinearity $g : (t, x, u, \xi) \mapsto g(t, x, u, \xi)$ are as follows:

- (A1) g is continuous in all the variables.
- (A2) g is locally Hölder continuous in t and locally Lipschitz continuous in (u, ξ).
- (A3) g is 1-periodic in x and t.

It is well-known that (A1-3) suffice for local existence of solutions in bounded and extended case (Section 2). In addition, in the first part of the paper, we also assume:

(A4) There exists a set $\mathcal{B}$, closed and bounded in $\mathcal{X}^\alpha$-norm, S-invariant in the extended case, such that if $u(t_0) \in \mathcal{B}$, $t_0 \in \mathbb{R}$, then for all $t \geq t_0$, the solution of (1.1) exists and we have that $u(t) \in \mathcal{B}$.

As recalled in Section 2, conditions (A1-4) suffice for (1.1) to generate a continuous semiflow on $\mathcal{B}$, non-autonomous and denoted by $\hat{T}(t,s)$, $t \geq s$ in the AC case, where $\hat{T}(t,s)u(s) = u(t)$; and autonomous and denoted by $T(t) = \hat{T}(s+t,s)$ in the DC case. Sufficient conditions in various contexts for (A4) to hold are given in [19], Section 7 (see also [24] and references therein). These results also apply in the extended case, in the view of comments in the Appendix.

We write shortly $T = \hat{T}(n, n+1)$ (independent of $n \in \mathbb{Z}$; $T = T(1)$ in the DC case). Note that S is continuous, and that S and $T(t)$, resp. T by (A3) commute. The notion of *invariance* throughout the paper will depend on the considered case: unless otherwise specified, an *invariant set* will be any set invariant with respect to all the actions in the Table 1.2:

$$\begin{array}{lcc} \text{Actions:} & \text{DC} & \text{AC} \\ \text{Bounded (B)} & T(t), t \geq 0; & T \\ \text{Extended (E)} & T(t),\ t \geq 0;\ S & T; S. \end{array} \tag{1.2}$$

We always consider ω-limit sets for the semiflow $T(t)$, $t \geq 0$ in the DC case, and for the sequence of maps T^n, $n \in \mathbb{N}$ in the AC case. In the extended case, we will equip $\mathcal{X}^\alpha$ with a coarser topology, to ensure that all the orbits bounded in $\mathcal{X}^\alpha$ are relatively compact, so that we can consider asymptotics and invariant measures (see Section 2 for the choice of topology and a discussion). We define an *invariant measure* to be a Borel probability measure on $\mathcal{B}$, invariant with respect to all the actions in Table 1.2.

Denote by $\mathcal{E}$ the ergodic attractor, i.e. the union of supports of all the invariant measures. As $\mathcal{E}$ depends on the choice of $\mathcal{B}$ in (A4), we may occasionally write $\mathcal{E}(\mathcal{B})$; the argument $\mathcal{B}$ will be omitted when the chosen $\mathcal{B}$ is clear from the context. The main result in the bounded case is that the set $\mathcal{E}$ is not too large, i.e. that it is at most two dimensional:

Theorem 1.1. Ergodic Poincaré-Bendixson Theorem. *Assume (A1-4) holds in the bounded case. Then $\mathcal{E}$ projects continuously and one-to-one to $\mathbb{R}^2$, with the projection $\pi : \mathcal{E} \to \mathbb{R}^2$ given with*

$$\pi(u) = (u(0), u_x(0)). \tag{1.3}$$

In the bounded, DC case, this already follows from Fiedler and Mallet-Paret Poincaré-Bendixson theorem [10] (see Subsection 10.1 for further comments). In the bounded, AC case, it seems new, and is complementary to the results of Poláčik and Tereščak [37].

To establish an analogous result in the extended case, we require a technical condition of non-degeneracy of $\mathcal{E}$, by which we mean that the average density of zeroes on $\mathcal{E}$ is bounded. It is rigorously given in Definition 6.1; we note here that it suffices that for any two $u, v \in \mathcal{E}$,

$$\liminf_{n\to\infty} \frac{1}{2n} z_{[-n,n)}(u - v) < \infty, \tag{1.4}$$

where $z_{[-a,b)}$ is the number of zeroes of $u(x) - v(x)$ for $x \in [-a, b)$, $a < b$ (we set $z_{[-a,b)}(0) = 0$ by definition).

Theorem 1.2. Extended Ergodic Poincaré-Bendixson Theorem. *Assume (A1-4) holds in the extended case, and assume that $\mathcal{E}$ is non-degenerate. Then $\mathcal{E}$ projects continuously and one-to-one to $\mathbb{R}^2$, with the projection $\pi : \mathcal{E} \to \mathbb{R}^2$ given with (1.3)*

Remark 1.1. Non-degeneracy of $\mathcal{E}$ is expected to hold generically, and possibly always. This follows from the results of Angenent and Chen [1, 4]: as $\mathcal{E}$ consists of entire solutions (Lemma 3.2), we have that for any two $u, v \in \mathcal{E}$, $z_{[-n,n]}(u - v)$ is finite. We characterize non-degeneracy in Subsection 6.2 and give further sufficient conditions for it to hold in Subsection 11.1. For example, we show in

Example 7.1 that non-degeneracy of $\mathcal{E}$ holds for non-linearities $g = -\partial V(x,u)/\partial u$, with $V \in C^2(\mathbb{R}^2)$ and bounded from below.

The main technique in proving Theorems 1.1 and 1.2 is the zero number lifted to the space of measures. The zero-number has been established as a tool to study dynamics of (1.1) mainly due to Matano's work [21] (see [24] and references therein for an overview). We write shortly $z(u-v) := z_{[0,1)}(u-v)$, and say that a zero $u(x) - v(x) = 0$ is multiple, if $u_x(x) - v_x(x) = 0$. (We also say that u and v intersect transversally at x if it is a single, and non-transversally if it is a multiple zero.) In the bounded case, if μ is a Borel probability measure on $\mathcal{X}^\alpha$, we define the zero of μ as

$$Z(\mu) = \int_{\mathcal{X}^\alpha} z(u-v)d\mu(u)d\mu(v). \tag{1.5}$$

We will show that Z on the space of Borel probability measures on $\mathcal{X}^\alpha$ has analogous properties to the zero-function z on $\mathcal{X}^\alpha$ ([10, 24] and references therein): for any $t > 0$, $Z(\mu(t))$ is essentially finite[1] (where $\mu(t)$ is the evolution of $\mu(0) = \mu$ induced by (1.1) on the space of measures); it is non-increasing; and if there is a multiple zero $u - v$ for some u, v in the support of $\mu(t)$, then $Z(\mu(t))$ is strictly decreasing.

Importantly, the same technique applies also in the extended case, if we consider *S-invariant measures.* First, we note that there are many S-invariant measures on $\mathcal{X}^\alpha$ which are not supported only on periodic functions: e.g. consider the Bernoulli measure on the space of bi-infinite sequences of $0, 1$, and associate to each sequence a function u by combining two arbitrary smooth profiles $u^0, u^1 : [0,1] \to \mathbb{R}$, $u^0(0) = u^0(1) = u^1(0) = u^1(1)$, as in Example 7.1.

We again define the zero function as in (1.5), i.e. by considering only zeroes in $[0,1)$ (thus $Z(\mu)$ is typically finite). As the measure is S-invariant, it is the same as considering only zeroes in any $[y, y+1)$, $y \in \mathbb{R}$. The $Z(\mu)$ can be interpreted, and indeed for ergodic[2] μ is the same for μ-a.e. u, v as the *average density of zeroes*

$$\lim_{n\to\infty} \frac{1}{2n} z_{[-n,n)}(u-v)$$

(this follows from the Birkhoff ergodic theorem and measurability of z established in Lemma 4.6). Now, $Z(\mu(t))$ is non-increasing in t, as the flux of zeroes through $x = 0$ and $x = 1$ by the S-invariance of the measure cancels out. Finally, it may be somewhat counter-intuitive that a single multiple zero for some $x \in [0,1)$ causes the entire density of zeroes on the infinite line to decrease. The rationale for this is that by the local structure of zeroes (Lemma 4.1), a multiple zero of $u(t) - v(t)$ persists in an open neighborhod $U \times V \times (t-\delta, t+\delta)$ of (u, v, t). By Poincaré recurrence, if u, v are in the support of a S-invariant measure, one can find a positive measure subset of $W \subset U \times V$ for which a positive density of $S \times S$-translates visit W, thus a single multiple zero implies existence of a set of positive measure with a positive density of multiple zeroes along the real line for times close to t. We make this ad-hoc argument rigorous by using standard ergodic-theoretical tools, combined with the well-established local and global structure of zeroes [1, 4].

Considering S-invariant measures and the ergodic attractor in the extended case is related to analysing asymptotics for μ-a.e. initial condition with respect to any S-invariant measure μ. This approach was already taken by Sinai [28] in his study of the forced viscous Burgers equation, as we discuss below. We establish in Proposition 7.1 an example of a general result in this direction used later: for S-invariant μ and μ-a.e. u, $\omega(u)$ consists of orbits which do not intersect non-transversally a given S, T-invariant solution v (i.e. a spatially and temporally periodic orbit). Note that then by definition $v \in \mathcal{E}$, as the Dirac measure δ_v is S, T-invariant.

[1] We can always adjust the "weights" in the ergodic decomposition to make it finite, see Lemma 5.3.

[2] This holds if $\mu \times \mu$ is $S \times S$-ergodic; see Subsection 6.2.

1.2. **Statements of results: uniqueness of invariant measures.** The second part of the paper is concerned with establishing sufficient conditions for uniqueness of an invariant measure, and with implications of such uniqueness to dynamics. It is motivated by the result by Sinai [28] for the viscous, forced Burgers equation:

$$u_t = u_{xx} - u\,u_x + \hat{g}(x,t), \tag{1.6}$$

where $\hat{g}$ is sufficiently smooth, satisfies (A3), and for all $t \in \mathbb{R}$, $\int_0^1 \hat{g}(x,t)dx = 0$.

Recall the results from [28] (extended to quasi-periodic forcing in [30], higher dimensions on bounded domain and stochastic forcing in [29], and to inviscid limit on bounded domain and stochastic forcing in [7]):

(i) Firstly, it was established that there is a unique, S,T-periodic solution of (1.6) $v^0(t)$, such that for any initial condition $u \in H^{2\alpha}(\mathbb{S}^1)$, $\int_0^1 u(x)dx = 0$, we have that $\lim_{t\to\infty} |u(x,t) - v^0(x,t)| = 0$ (a pointwise convergence) (a special case of [28], Theorem 1).

(ii) Secondly, such asymptotics is shown to hold also on the extended domain *for a.e. initial condition with respect to some probability measure* on the phase space, as long as the probability measure satisfies certain conditions (see Subsection 10.5 for details).

(iii) Thirdly, each probability measure from (ii) converges in weak* topology with respect to the induced semiflow on the space of measures to the Dirac measure concentrated on v^0.

The main technique in [7, 28, 29] is the Cole-Hopf transformation, and the integral representation of the transformed solutions. As already noted in [29], p347, the key property of (1.6) is that $\int_0^1 u(x)dx$ is the invariant. We show here that such invariance (B3) below in essence suffices to establish (i), and modified versions of (ii) and (iii). We assume in addition only certain weak dissipativity conditions (B1-2) ensuring global existence and boundedness of solutions. Generalized versions of (i)-(iii) are established in Corollaries 1.4, 1.5 and 1.6 below. The main tools in the proof are Theorem 1.1 and the zero function on the space of probability measures.

We say that an equation is Burgers-like, if the following holds:

(B1) *Sub-quadratic growth of non-linearity in u_x:* There exists an $\varepsilon > 0$ and a continuous function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,x,u,\xi)| \le c(\rho)\left(1 + |\xi|^{2-\varepsilon}\right)$$
$$(\rho > 0,\ (t,x,u,\xi) \in [0,1]\times[0,1]\times[-\rho,\rho]\times\mathbb{R}).$$

(B2) *Weak dissipation:* There exists an upper semi-continuous function $d : \mathbb{R} \to \mathbb{R}^+$ such that: if $u \in H^{2\alpha}(\mathbb{S}^1)$, $\int_0^1 u(x)dx = y$ and $||u-y||_{L^\infty(\mathbb{S}^1)} \le d(y)$; and if the solution of (1.1), $u(t_0) = u$ exists on $[t_0,t_1)$ for some $t_1 > t_0$, then for every $t \in [t_0,t_1)$ we have $||u(t)-y||_{L^\infty(\mathbb{S}^1)} \le d(y)$. Furthermore, the function d satisfies

$$\lim_{y\to\infty}(y - d(y)) = +\infty, \qquad \lim_{y\to-\infty}(y + d(y)) = -\infty. \tag{1.7}$$

(B3) *Invariance:* For every $u \in H^{2\alpha}(\mathbb{S}^1)$, if the solution of (1.1), $u(t_0) = u$ exists on $[t_0,t_1)$ for some $t_1 > t_0$, then for every $t \in [t_0,t_1)$, we have that $\int_0^1 u(t,x)dx = \int_0^1 u(t_0,x)dx$.

We show in Subsection 10.5 that a generalization of the Burgers equation (1.6) satisfies (B1-3).

Recall the ordering on $\mathcal{X}^\alpha$: we write $u \le v$ if $u(x) \le v(x)$ for all $x \in \mathbb{S}^1$, resp. $x \in \mathbb{R}$; $u \ll v$ if $u(x) < v(x)$ for all $x \in \mathbb{S}^1$, resp. $x \in \mathbb{R}$; and $u < v$ if $u \le v$ but $u \ne v$. A family in $\mathcal{X}^\alpha$ is strongly totally ordered, if for all u,v in the family, we have either $u = v$ or $u \ll v$.

We state results only for the more general AC case; modifications for the DC case are straightforward.

Theorem 1.3. *Assume (A1-3) and (B1-3) in the AC case.*

(i) There exists a set $\mathcal{V} = \{v^y,\ y \in \mathbb{R}\}$, $v^y \in H^{2\alpha}(\mathbb{S}^1)$, satisfying that $y \mapsto v^y$ is continuous in $H^{2\alpha}(\mathbb{S}^1)$, strictly increasing, and that for all $y \in \mathbb{R}$, v^y is T-invariant and $\int_0^1 v^y(x)dx = y$. Furthermore, it is a unique family with these properties.

(ii) In the bounded case, and in the extended case if $\mathcal{E}$ is non-degenerate, we have that $\mathcal{E} = \mathcal{V}$.

(iii) In the bounded case, there is a unique invariant measure on $\mathcal{B}_y := \{u \in \mathcal{X}^\alpha, \int_0^1 u(x)dx = y\}$, concentrated on a single $v^y \in \mathcal{B}_y$.

We can now recover the conclusion (i) by Sinai on asymptotics of the Burgers equation in the bounded case, by applying general techniques of the order-preserving dynamics (in particular the Nonorderedness principle, see e.g. [24], Section 3):

Corollary 1.4. *Assume (A1-3) and (B1-3) in the bounded, AC case. Then for each $u \in \mathcal{X}^\alpha$, $\omega(u) = \{v^{y_0}\}$, where $y_0 = \int_0^1 u(x)dx$ and v^y is as in Theorem 1.3, (i).*

Let $\mathcal{V}$ be as in Theorem 1.3, (i). To establish conclusions in the extended case, we again require a technical condition of finite density of zeroes:

(N1) Assume in the extended case that μ is a S-invariant Borel probability measure on $\mathcal{X}^\alpha$, supported on a set bounded in $\mathcal{X}^\alpha$, such that for every $v \in \mathcal{V}$, and μ-a.e. u, (1.4) holds.

We give examples of many non-trivial measures satisfying (N1) without any a-priori knowledge of $\mathcal{V}$ in Example 7.1.

We denote by $\mathcal{H}$ the family (possibly empty) of all *spatially heteroclinic solutions* associated to $\mathcal{V}$, i.e. such that for $h \in \mathcal{H}$, the solution of (1.1), $h(0) = h$ exists for all $t \in \mathbb{R}$, such that h intersect each $v^y \in \mathcal{V}$ at most once, transversally, and such that for some $y_1 \neq y_2$, and for all $t \in \mathbb{R}$, $\lim_{x\to-\infty} |u(x,t) - v^{y_1}(x,t)| = 0$, $\lim_{x\to\infty} |u(x,t) - v^{y_2}(x,t)| = 0$.

We establish below a weaker form of Sinai's conclusions (ii), (iii) by assuming (N1). To recover full results, we also require an additional control of the average of the quantity conserved in the bounded case:

(N2) Assume in the extended case that μ is a S-invariant Borel-probability measure with $y_0 := \mathbb{E}_\mu[\int_0^1 u(x)dx]$, such that for μ-a.e. u, and for each $w \in \omega(u)$,
$$\lim_{x\to\infty} \frac{1}{x}\int_0^x w(z)dz = \lim_{x\to\infty} \frac{1}{x}\int_{-x}^0 w(z)dz = y_0.$$

Corollary 1.5. *Assume (A1-3), (B1-3), in the extended, AC case, and let μ satisfy (N1).*

(i) For μ-a.e. u, we have that $\omega(u) \subset \mathcal{V} \cup \mathcal{H}$.

(ii) If μ also satisfies (N2), then for μ-a.e. u, we have $\omega(u) = \{v^{y_0}\}$.

Corollary 1.6. *Assume (A1-3), (B1-3), in the extended, AC case, and let μ satisfy (N1).*

(i) ω-limit set of μ in the weak-topology consists of measures supported on $\mathcal{V}$.*

(ii) If μ also satisfies (N2), then the ω-limit set of μ is $\delta_{v^{y_0}}$, i.e. the Dirac measure concentrated on $v^{y_0} \in \mathcal{V}$.

We give an example of a family of measures satisfying (N2) in Example 10.2, and compare our assumptions (N1), (N2) with the ones by Sinai [28].

We actually show in Section 8 that (A1-3), (B1) and existence of a family $\mathcal{V}$ as in Theorem 1.3,(i) suffices for conclusions in Corollaries 1.4, 1.5 and 1.6 to hold without the assumptions (B2) and (B3).

The paper is structured as follows: in Section 2 we give required background on existence of solutions of (1.1), the choice of topologies, and introduce the notation. We introduce the ergodic attractor in Section 3 and summarize how it relates to dynamics. In Section 4 we recall the key properties of the zero number as the key tool, introduce the balance law of zeroes, and show that the zero, zero flux and zero dissipation functions are measurable. We prove Theorem 1.1 in Section 5, and Theorem 1.2 in Section 6. In Sections 7-9 we prove results for Burgers-like equations and extensions, then give a number of examples illustrating general results and discuss open problems. Some required notions on uniformly local spaces are summarized in the Appendix.

Remark 1.2. All the results also hold for the equations $u_t = \varepsilon u_{xx} + g(t,x,u,u_x)$, $\varepsilon > 0$.

Remark 1.3. Theorems 1.1 and 1.2 were already announced in [35], with derived further implications to the entropy of (1.1) in all four cases considered here. All the results in [35] in the extended case hold under an additional assumption of non-degeneracy of $\mathcal{E}$.

2. Preliminaries and notation

It is well-known that the equation (1.1) with assumptions (A1-4) generates a continuous semiflow on $\mathcal{X}^\alpha$, $3/4 < \alpha < 1$ in the bounded case [17, 19, 24], non-autonomous and denoted by $\hat{T}(t,s)$ in the AC case, and by $T(t) = \hat{T}(s+t,s)$ in the DC case. The same holds in the extended case, as discussed in the Appendix. Consider the set $\hat{T}(0,-\delta_0)\mathcal{B}$ for some $0 < \delta_0 \leq 1$ fixed throughout the paper, where $\mathcal{B}$ is as in (A4). In both bounded and extended case, the standard application of the variations of constants formula implies that $\hat{T}(0,-\delta_0)\mathcal{B}$ is bounded in $\mathcal{X}^\gamma$ for any $\alpha < \gamma < 1$ (see [17], Section 3 in the bounded case, Appendix in the extended case).

We always use the graph norm on $\mathcal{X}^\alpha$, $0 < \alpha < 1$:

$$||u||_{\mathcal{X}^\alpha} := ||A_1^\alpha u||_{\mathcal{X}},$$

where $Au = -u_{xx}$ is the linear operator on $\mathcal{X}$ with the domain $D(A) = H^2(\mathbb{S}^1)$, resp. $D(A) = H^2_{\mathrm{ul}}(\mathbb{R})$, with $A_1 = A + I$, and A_1^α is the standard fractional power (see [17], Section 1.4, and the Appendix).

Consider the bounded case first, with $X^\alpha = H^{2\alpha}(\mathbb{S}^1)$, and let $\tilde{\mathcal{B}} = Cl\left(\hat{T}(0,-\delta_0)\mathcal{B}\right)$, where the closure is in X^α. Then by (A4) and the compact imbedding of X^γ in X^α in the bounded case, we have that $\tilde{\mathcal{B}}$ is invariant and compact, thus we can consider dynamics and Borel probability invariant measures on $\tilde{\mathcal{B}}$.

In the extended case, with $X^\alpha = H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$, to establish compactness, we need to consider a coarser topology on $\mathcal{B}$ and $\mathcal{C} := \hat{T}(0,-\delta_0)\mathcal{B} \subset \mathcal{B}$. Typical choices are: the topology of locally uniform convergence, i.e. induced by $L^\infty_{\mathrm{loc}}(\mathbb{R})$; $C^1_{\mathrm{loc}}(\mathbb{R})$; or $H^{2\delta}_{\mathrm{loc}}(\mathbb{R})$ for $1/2 \leq \delta \leq \alpha$ (defined as topology of convergence in $H^{2\gamma}([-n,n])$ for all $n \in \mathbb{N}$). As we have already established that $\mathcal{C}$ is bounded in $\mathcal{X}^\gamma$, one can easily verify that for all these choices of topologies induced on $\mathcal{C}$, $\mathcal{C}$ is relatively compact as a subset of the space whose topology it inherits. The following elementary observation shows that all these choices of topologies are the same:

Lemma 2.1. *Consider a subset $\mathcal{Z} \subset \mathcal{Y}_0 \subset \mathcal{Y}_1$, where $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are metrizable and complete topological spaces with respective topologies τ_0, τ_1, such that $\tau_1|_{\mathcal{Y}_0} \subset \tau_0$. Furthermore, assume that $\mathcal{Z}$ is relatively compact in both $\mathcal{Y}_0$ and $\mathcal{Y}_1$. Then the closure of $\mathcal{Z}$ in $\mathcal{Y}_0$ and $\mathcal{Y}_1$ is the same, and the topologies τ_0 and τ_1 induced on $Cl(\mathcal{Z})$ are the same.*

(We denote by $\tau_1|_{\mathcal{Y}_0}$ the induced topology τ_1 on $\mathcal{Y}_0$).

Proof. We first show that a sequence $u_n \in \mathcal{Z}$ converges in $\mathcal{Y}_0$ if and only if it converges in $\mathcal{Y}_1$. As $\tau_1|_{\mathcal{Y}_0} \subset \tau_0$, the non-trivial direction is that convergence in $\mathcal{Y}_1$ implies convergence in $\mathcal{Y}_0$. Assume the contrary, and let $v \in \mathcal{Y}_1$ be a limit of $u_n \in \mathcal{Z}$ in $\mathcal{Y}_1$. By the assumptions and relative compactness, there exists a subsequence u_{n_k} converging to some $\tilde{v} \in \mathcal{Y}_0$ in $\mathcal{Y}_0$. But then u_{n_k} also converges to $\tilde{v}$ in $\mathcal{Y}_1$, thus $\tilde{v} = v$. We deduce that the closure of $\mathcal{Z}$ in both topologies is the same. As the topology in metric spaces is entirely determined by convergence, it suffices to repeat the argument above for any sequence in $u_n \in Cl(\mathcal{Z})$. □

We apply Lemma 2.1 by setting $\mathcal{Z} = \mathcal{C}$, $\mathcal{Y}_0 = \mathcal{B}$ with the induced $H^{2\alpha}_{\mathrm{loc}}(\mathbb{R})$ topology, and $\mathcal{Y}_1$ be the space of any of the aforementioned "coarser" topologies. We conclude that the closure $\tilde{\mathcal{B}} := Cl(\mathcal{C}) = Cl(\hat{T}(0,-\delta_0)\mathcal{B})$ does not depend of the choice of the coarser topology, and that $\tilde{\mathcal{B}}$ is invariant and compact. We always implicitly assume such a coarser topology, called also a localized topology, on $\tilde{\mathcal{B}}$ in the extended case, and consider asymptotics and Borel probability measures on $\tilde{B}$. Finally we note that that by using the variations of constants formula, that the solutions (1.1) depends continuously on initial conditions in the localized topology, thus $\hat{T}(t,s)$ is continuous, $T(t)$ is a continuous semi-flow in the DC case and $T = T(1)$ a continuous map in the AC case on $\tilde{\mathcal{B}}$.

We frequently use the fact that (1.1) is strongly monotone, i.e. that if $u(t_0) < v(t_0)$, then for all $t \geq t_0$ for which both solutions exist, $u(t) \gg v(t)$.

Assuming (A1-4), from this discussion follows that the equation (1.1) admits an attractor $\mathcal{A}$ on $\tilde{B}$ ([26], Section 2.3), which is unique, characterized as the set of all global solutions on $\tilde{\mathcal{B}}$ (or $\mathcal{B}$), and compact. The attractor $\mathcal{A}$ depends on the choice of $\mathcal{B}$, thus we write $\mathcal{A}(\mathcal{B})$ when the choice of a set $\mathcal{B}$ satisfying (A4) is not clear from the context.

Remark 2.1. Note that we do not assume strong dissipativity conditions on g, such as e.g. (G1-3) in [24], as they would not cover the Burgers-like equations considered in the second part of the paper.

Finally, we note the properties essential for considerations involving the zero number.

Lemma 2.2. *(i) For any $t > t_0$, $x, y \in \mathbb{R}$, $x < y$, the mapping $\tilde{\mathcal{B}} \mapsto C^1([x,y])$ defined with $u(t_0) \mapsto u(.,t)|_{[x,y]}$ is continuous.*

(ii) For any $t > s > t_0$, $x \in \mathbb{R}$, the mapping $\tilde{\mathcal{B}} \mapsto C^1([s,t])$ defined with $u(t_0) \mapsto u(x,.)|_{[s,t]}$ is continuous.

(iii) The claims (i), (ii) hold on $\mathcal{A}$ for all $t \in \mathbb{R}$, resp. $t > s$, $t, s \in \mathbb{R}$.

(In the bounded case, we implicitly assume the natural imbedding of $H^{2\alpha}(\mathbb{S}^1)$ in $H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$ in the statements.)

Proof. The claim (i), as well as continuity of $u(t_0) \mapsto u(x,.)|_{[s,t]}$ as $\tilde{\mathcal{B}} \to C^0([s,t])$, follows from continuous imbedding of $H^{2\alpha}(\mathbb{S}^1)$ in $C^1(\mathbb{S})$, respectively $H^{2\alpha}_{\mathrm{loc}}(\mathbb{R})$ in $C^1_{\mathrm{loc}}(\mathbb{R})$, continuous dependence on initial conditions in $\tilde{\mathcal{B}}$, and continuity of $t \mapsto u(t)$ in $\tilde{B}$ for the latter claim. To complete (ii), it suffices to show continuity of $u(t_0) \mapsto u_t(x,.)$ as $\tilde{\mathcal{B}} \to C^0([s,t])$. This follows from e.g. [17], Theorem 3.5.3, with the choice of the spaces as in the proof of local existence of solutions (in the extended case, we in addition apply continuous dependence on initial conditions in $\tilde{\mathcal{B}}$). We deduce (iii) from the fact that T is a homeomorphism on $\mathcal{A}$, as it suffices to consider first $u(t_0) \mapsto \tilde{u}(t_0) := T^{-n}u(t_0)$ for an integer $n > |s|$, and then apply (i),(ii) to $\tilde{u}(t_0)$. □

Remark 2.2. For an alternative argument enabling applying zero-number techniques for even less smooth g than those satisfying (A1-3), refer to [25], Section 2.

3. Invariant measures and ergodic attractor

In this section we fix $\mathcal{A} \subset \tilde{\mathcal{B}} \subset \mathcal{B}$ as in the previous section. Denote by $\mathcal{M}(\mathcal{B})$ all Borel probability measures on $\mathcal{B}$, invariant with respect to the actions in Table 1.2. We first define and list key properties of an ergodic attractor, and then relate it to asymptotics of dynamics in the bounded, and in the extended case.

3.1. Ergodic attractor.

As in the introduction, define the ergodic attractor with

$$\mathcal{E} = \cup_{\mu \in \mathcal{M}(B)} \operatorname{supp} \mu.$$

As in the introduction, define the ergodic attractor with

$$\mathcal{E} = \cup_{\mu \in \mathcal{M}(B)} \operatorname{supp} \mu.$$

We first show that $\mathcal{E}$ is not empty.

Lemma 3.1. *The set $\mathcal{M}(B)$ is non-empty.*

Proof. Existence of the invariant measure in the bounded case is the classical Krylov - Bogolioubov theorem for continous maps, resp. continuous semiflows on compact metrizable sets [40]. To show it in the E/AC case, i.e. for commuting S, T, it suffices to find a weak*-convergent subsequence of the sequence of Borel probability measures

$$\sum_{m=-L}^{L-1} \sum_{n=1}^{L} \frac{1}{2L^2} (S^m)^* (T^n)^* \delta(u),$$

where $f^*\mu$ is the standard pull of a measure, and $\delta(u)$ is the (Dirac) probability measure concetrated on a single, fixed $u \in \mathcal{A}$. Analogously we prove the claim in the E/DC case by replacing summing T^n from 0 to L with integrating $T(t)$ from 0 to L. □

Lemma 3.2. *(i) $\mathcal{E} \subset \mathcal{A}$, thus it consists of entire orbits,*
(ii) $\mathcal{E}$ is closed and compact,
(iii) $\mathcal{E}$ is invariant,
(iii) In the DC case $T(t)|_{\mathcal{E}}$ is a continuous flow, and $T|_{\mathcal{E}}$, $S|_{\mathcal{E}}$ are homeomorphisms.

Proof. Let $\nu \in \mathcal{M}(\mathcal{B})$. Consider the sequence of sets $\mathcal{B}_n := T^n(\mathcal{B})$. As by (A4), $T(\mathcal{B}) \subset \mathcal{B}$, the sequence $\mathcal{B}_n$ is decreasing. By the characterization of the attractor as the set of entire orbits, $\mathcal{A} = \cap_{k=1}^{\infty} \mathcal{B}_k$. As for $n \geq 1$, $\mathcal{B}_n \subset \tilde{\mathcal{B}}$ is compact, and by T-invariance of μ, all of $\mathcal{B}_n$ are of full measure, thus $\cap_{k=1}^{\infty} \mathcal{B}_k$ is of full measure and closed, so by definition $\operatorname{supp} \nu \subset \cap_{k=1}^{\infty} \mathcal{B}_k = \mathcal{A}$, which completes (i).

Now consider a convergent sequence $u_n \in \mathcal{E} \subset \mathcal{A}$ converging to some $u \in \mathcal{A}$, and the associated invariant measures $\mu_n \in \mathcal{M}(\mathcal{B})$ such that $u_n \in \operatorname{supp} \mu_n$. The measure

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \mu_n$$

is by definition in $\mathcal{M}(\mathcal{B})$, and also by definition the support of μ contains supports of μ_n for all $n \geq 1$, thus $u_n \in \operatorname{supp} \mu$. As $\operatorname{supp} \mu$ is by definition closed, we deduce that $u \in \operatorname{supp} \mu$, thus $\mathcal{E}$ is closed. As it is a subset of a compact set, it is compact. The claim (iii) follows from the invariance of every $\mu \in \mathcal{M}(\mathcal{B})$ and the definition of support, and (iv) follows from (i),(iii) and the properties of $T(t)$, T and S on $\mathcal{B}$. □

3.2. **Asymptotics and the ergodic attractor in the bounded case.** We now show relationship of the ergodic attractor, and the asymtpotics of the dynamics with respect to (1.1), focusing in this subsection only on the bounded case. All the results in this subsection actually hold for any compact metric space $\tilde{\mathcal{B}}$ and a continuous map T or a continuous semiflow $T(t)$ on $\tilde{\mathcal{B}}$.

In [15] we introduced the notion of the *ω-limit set on average*, denoted by $\bar{\omega}(u)$, as the set of $x \in \tilde{\mathcal{B}}$ such that $u(t)$ converges to x for non-zero density of times t. The physical meaning of $\bar{\omega}(u)$ is as follows: if we start from u and wait long enough, we are going to observe only $v \in \bar{\omega}(u)$. Even though there may be $v \in \omega(u) \setminus \bar{\omega}(u)$, any neighborhood of such v is visited only for zero density of times, thus effectively non-observable (see Lemma 3.5). More precisely, in the AC case (i.e. for the map T), we set

$$\bar{\omega}(u) = \left\{ v \,:\, \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\mathcal{U}}(T^k u) > 0 \text{ for all open neigborhoods } \mathcal{U} \text{ of } x \right\},$$

and in the DC case (i.e. for a semiflow $T(t)$), we have (using the notation $T(t)u = u(t)$):

$$\bar{\omega}(u) = \left\{ v \,:\, \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{U}}(T(t)u) dt > 0 \text{ for all open neigborhoods } \mathcal{U} \text{ of } x \right\}.$$

The following lemma shows that the properties of $\bar{\omega}(u)$ reflect those of $\omega(u)$:

Lemma 3.3. *The set $\bar{\omega}(u)$ is non-empty, compact, T- (resp. $T(t)$-) invariant, and $\bar{\omega}(u) \subset \omega(u)$.*

The proof is in [15], Proposition 5.4 (for the semiflow case, the map case is analogous). We also give in [15] an example of (1.1) for which $\omega(u)\backslash\bar{\omega}(u) \neq \emptyset$ (for example, consider u whose $\omega(u)$ consists of exactly two equilibria and their two heteroclinic connections. Then $\bar{\omega}(u)$ is the two equilibria).

Lemma 3.4. *We have that* $\mathcal{E} = Cl\left(\cup_{u\in\tilde{\mathcal{B}}}\bar{\omega}(u)\right)$.

Proof. We first show that $\bar{\omega}(u) \subset \mathcal{E}$. We give the proof only in the AC case, as the DC case is analogous. Let $v \in \bar{\omega}(u)$, and let $\mathcal{U}_m$ be a $1/m$-open ball around v. By definition of $\bar{\omega}(u)$, we can find a subsequence n_j such that

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{k=1}^{n_j} \mathbf{1}_{\mathcal{U}_m}(T^k u) = \kappa > 0. \tag{3.1}$$

Consider Borel probability measures $\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{T^k u}$, where δ_u is the Dirac probability measure concentrated at u. By compactness, we can find a weak*-convergent subsequence n'_j of n_j such that $\mu_{n'_j}$ weak*-converges to μ, which is then T-invariant. By (3.1) we have $\lim_{j\to\infty}\mu_{n'_j}(\mathcal{U}_m) = \kappa$, thus by the well-known property of weak* convergence [40], $\mu(\mathcal{U}_{2m}) \geq \mu(\bar{\mathcal{U}}_m) \geq \kappa > 0$. We can thus find $v_m \in \operatorname{supp}\mu \subset \mathcal{E}$ which is in U_{2m}. We repeat this for all $m \in \mathbb{N}$ and obtain $v_m \in \mathcal{E}$ such that $\lim_{m\to\infty} v_m = v$. However, by Lemma 3.2, (ii), $\mathcal{E}$ is closed, thus $v \in \mathcal{E}$.

To show the other direction, note that by the Birkhoff ergodic theorem applied to $\mathbf{1}_{\mathcal{U}}$ for each open set $\mathcal{U}$ in a chosen countable basis of open sets, we obtain that for any T-ergodic measure, the set of u such that $u \in \bar{\omega}(u)$ has full measure, thus it must be dense in $\mathcal{E}$. □

In particular, $\mathcal{E}$ contains all uniformly recurrent u (see [14]), as for uniformly recurrent u, by definition $u \in \bar{\omega}(u)$.

We conclude the subsection with a statement on "observability" of $\mathcal{E}$.

Lemma 3.5. *Let U be an open neighbourhood of $\mathcal{E}$ in the AC (resp. DC) case. Then for any $u \in \tilde{\mathcal{B}}$, we have* $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{1}_U(T^k u) = 1$ *(resp.* $\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_U(T(t)u)dt = 1$*).*

Proof. In [15], Proposition 5.3, this was shown for any open neighborhood U of $\bar{\omega}(u)$. The claim now follows from Lemma 3.4. □

3.3. **Asymptotics and the ergodic attractor in the extended case.** We now consider the unbounded AC or DC case, or more generally a compact, metric $\tilde{\mathcal{B}}$, and two commuting continuous maps T, S on $\tilde{\mathcal{B}}$ (respectively a commuting continuous semiflow $T(t)$ and a continuous map S), where $\mathcal{E}$ is the union of supports of all T, S-invariant (resp. $T(t), S$-invariant) Borel probability measures. Then $\mathcal{E}$ contains "space-time observable" orbits in the following sense:

Lemma 3.6. *Let U be an open neighbourhood of $\mathcal{E}$ in the AC (resp. DC) case. Then for any $u \in \tilde{\mathcal{B}}$, we have that* $\lim_{n\to\infty}\frac{1}{n^2}\sum_{j,k=0}^{n-1}\mathbf{1}_U(S^jT^k u) = 1$ *(resp.* $\lim_{n\to\infty}\frac{1}{n^2}\sum_{j=0}^{n-1}\int_0^n \mathbf{1}_U(S^jT(t)u)dt = 1$*).*

Proof. See [36], Proposition 4. □

4. Preliminaries on the set of zeroes

We first recall the well-known properties of the set of zeroes of $u(t) - v(t)$, where u, v are two solutions of (1.1) on $\tilde{\mathcal{B}}$. In particular, we define the functions z, f, d on $\tilde{\mathcal{B}}^2$ denoting the number of zeroes in $[0, 1)$, the "flux" of zeroes through $x = 0$, and the "dissipation" of zeroes in $[0, 1)$. The main results are the balance law for the flux of zeroes, the key fact that $d > 0$ persists for small perturbations, and that z, f, d are Borel-measurable.

In this section, assume (A1-4). We fix $u, v \in \tilde{\mathcal{B}}$ for which the solution of (1.1), $u(0) = u$, resp. $v(0) = v$ exists on $(\hat{t}, \infty)$, where $\hat{t} \in [-\infty, 0)$ is fixed throughout the section. Denote by $w = u - v$

and $w(t) = u(t) - v(t)$, $t \in (\hat{t}, \infty)$. Let Z_w be the set of zeroes (or the nodal set), and S_w the set of multiple (or singular) zeroes associated to $w \neq 0$, defined with

$$\begin{aligned} Z_w &:= \{(x,t) \in \mathbb{R} \times (\hat{t}, \infty) \,:\, w(x,t) = 0\}, \\ S_w &:= \{(x,t) \in \mathbb{R} \times (\hat{t}, \infty) \,:\, w(x,t) = w_x(x,t) = 0\}. \end{aligned}$$

For $u = v$, i.e. $w = 0$, we set $S_w = Z_w = \emptyset$.

4.1. **Local, global structure of zeroes and the balance law.** The following local and global structure of zeroes is well-known, proved by Chen [4] (for earlier, less complete description by Angenent and Chen and Poláčik see [1, 3]):

Lemma 4.1. Local structure of zeroes. *If $(x_0, t_0) \in Z_w$, then there is a neighborhood $Q = [x_0 - \varepsilon, x_0 + \varepsilon] \times [t_0 - \delta, t_0 + \delta]$, $\varepsilon, \delta > 0$ of (x_0, t_0) such that the following properties hold:*

(a) *If $(t_0, v_0) \notin S_w$, then $Q \cap Z_w$ equals a single curve $\{(\gamma(t), t) : t \in [t_0 - \delta, t_0 + \delta]\}$, where $\gamma : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ is of class C^1 and $\gamma(t_0) = x_0$.*

(b) *If $(t_0, v_0) \in S_w$, then there is an integer $m \geq 2$ (the degree of the zero) such that the following holds:*

(b1) *For even m, there exist m curves $\gamma_1, ..., \gamma_m : [t_0 - \delta, t_0) \to \mathbb{R}$ of class C^1, such that*

$$\gamma_1(t) < \gamma_2(t) < ... < \gamma_m(t) \qquad \text{for all } t \in [t_0 - \delta, t_0), \tag{4.1}$$

such that $\lim_{t \to t_0^-} \gamma_k(t) = x_0$, $k = 1, ..., m$ and such that $Q \cap Z_w$ equals union of $\{(\gamma_j(t), t) : t \in [t_0 - \delta, t_0)\}$, $j = 1, ..., m$, and $\{(x_0, t_0)\}$.

(b2) *For odd m, there exist m curves $\gamma_1, ..., \gamma_{(m-1)/2}, \gamma_{(m+3)/2}, ..., \gamma_m : [t_0 - \delta, t_0) \to \mathbb{R}$, $\gamma_{(m+1)/2} : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ of class C^1, satisfying (4.1), such that $\lim_{t \to t_0^-} \gamma_j(t) = x_0$, $j = 1, ..., (m-1)/2, (m+3)/2, ..., m$, such that $\gamma_{(m+1)/2}(t_0) = x_0$, and such that $Q \cap Z_w$ equals union of $\{(\gamma_j(t), t) : t \in [t_0 - \delta, t_0)\}$, $j = 1, ..., (m-1)/2, (m+3)/2, ..., m$ and $\{(\gamma_{(m+1)/2)}(t), t) : t \in [t_0 - \delta, t_0 + \delta]\}$.*

In both cases, $\{(t_0, v_0)\}$ is equal to $Q \cap S_w$.

From this we can deduce the following global structure of zeroes.

Lemma 4.2. Global structure of zeroes. *There exist an at most countable family of curves $\gamma_i : (\hat{t}, d_i) \to \mathbb{R}$ of class C^1 associated to w, $d_i \in (\hat{t}, \infty]$, $i \in \mathcal{I}_w$, $\mathcal{I}_w$ a finite set or $\mathbb{N}$, satisfying the following:*

(i) *The sets $\{(\gamma_i(t), t),\ t \in (\hat{t}, d_i)\}$, $i \in \mathcal{I}$ are disjoint,*

(ii) *$S_w = \cup_{i \in \mathcal{I}_w, d_i < \infty} \{(\lim_{t \to d_i^-} \gamma_i(t), d_i)\}$,*

(iii) *$Z_w = \cup_{i \in \mathcal{I}_w} \{(\gamma_i(t), t),\ t \in (\hat{t}, d_i)\} \cup S_w$,*

(iv) *For each compact $Q \subset \mathbb{R}^2$, there exist at most finite $i \in \mathcal{I}$ such that $\{(\gamma_i(t), t), t \in (\hat{t}, d_i)\}$ intersects Q. Specifically, there are at most finitely many multiple zeroes in Q.*

For the proof, see the proof of Lemma 2.3 in [9], taking into account adjustments of the statement fitting our purposes (see Remark 4.1 below).

Note that for simplicity of notation, we drop the dependency on w in the notation of curves of zeroes γ. For $i \in \mathcal{I}_w$ such that $d_i < \infty$, denote by $x_i = \lim_{t \to d_i^-} \gamma_i(t)$, and then $S_w = \{(x_i, d_i), d_i < \infty,\ i \in \mathcal{I}_w\}$. For $d_i < \infty$, let $\overline{\gamma}_i : (-\infty, d_i] \to \mathbb{R}$ be the unique continous extension of γ_i (i.e. such that $\overline{\gamma}_i(d_i) = x_i$), and for $d_i = \infty$ let $\overline{\gamma}_i = \gamma_i$.

We define the number of zeroes Z_w in $[x, y) \times \{t\}$, the flux F_w of zeroes through $\{x\} \times [s, t)$, and the dissipation D_w of zeroes in $[x, y) \times (s, t]$, defined for $s < t$, $x < y$, $s, t, x, y \in \mathbb{R}$, associated to $w \neq 0$, as follows:

$$Z_w(x,y,s) = \sum_i \left(\mathbf{1}(\{\gamma_i(t) \geq y\}) - \mathbf{1}(\{\gamma_i(s) \geq x\})\right),$$
$$F_w(x,s,t) = \sum_{i,d_i>s} \mathbf{1}(\{\overline{\gamma}_i(\min(t,d_i)) \geq x\}) - \sum_{i,d_i>s} \mathbf{1}(\{\gamma_i(s) \geq x\}),$$
$$D_w(x,y,s,t) = \sum_i \mathbf{1}\left(\{(x_i,d_i) \in [x,y) \times (s,t]\}\right).$$

where the sums go only over by Lemma 4.2, (iv) only finitely many $i \in \mathcal{I}_w$ such that the intersection $\{(t,\gamma_i(t)), t \in (\hat{t}, d_i)\}$ and $Q = [x,y] \times [s,t]$ is nonempty, and $\mathbf{1}$ is the characteristic function. For $w = 0$, we set $Z_w = F_w = D_w = 0$ independently of the arguments. Note that the function D_w counts multiple zeroes in $[x,y) \times [s,t)$ with their multiplicity (m times for even, $m-1$ times for odd m).

We frequently use the following abbreviated notation

$$z(w) = Z_w(0,1,1), \qquad f(w) = F_w(0,0,1), \qquad d(w) = D_w(0,1,0,1).$$

Remark 4.1. For technical reasons, our definition of the curves of zeroes γ_i slightly differs from e.g. [9, 10], as the even, multiple zeroes are not in the union of images $(t,\gamma_i(t))$. Also the zero functions Z_w, z, do not "count" even, multiple zeroes. This simplifies definition of the flux and dissipation of zeroes, as the images of γ_i are disjoint. Note that all the multiple zeroes are properly "counted" by the dissipation functions D_w, d.

We now obtain the following balance law:

Lemma 4.3. The balance law for the flux of zeroes. *Let $x,y,s,t \in \mathbb{R}$ such that $0 \leq s < t$. Then*

$$Z_w(x,y,t) - Z_w(x,y,s) = F_w(y,s,t) - F_w(x,s,t) - D_w(x,y,s,t). \tag{4.2}$$

Specifically,

$$z(Tv - Tu) - z(v-u) = f(Sv - Su) - f(v-u) - d(v-u). \tag{4.3}$$

Proof. If $w = 0$, the claim is trivial. For $w \neq 0$, we first note that

$$\sum_{i,d_i>s} \mathbf{1}(\{\overline{\gamma}_i(\min(t,d_i)) \geq x\}) = \sum_{i,d_i>t} \mathbf{1}(\{\gamma_i(t) \geq x\} + \sum_i \mathbf{1}\left(\{(x_i,d_i) \in [x,\infty) \times (s,t]\}\right),$$

where we sum over finitely many $i \in \mathcal{I}$ as in the definition of F_w. It suffices to insert that in the definition of F_w, and to note that in the definition of $Z_w(x,y,t)$ we sum only over i, $d_i > t$, to obtain (4.2). The relation (4.3) is a special case of (4.2), for $(x,y,s,t) = (0,1,0,1)$. $\square$

4.2. **Continuity and measurability of zero, zero flux and zero dissipation.** We first establish continuity of z, f, d under certain "no singular zero on the boundary" assumptions, then persistence of $d \geq 1$ for small perturbations, and finally prove Borel-measurability of z, f, d.

Lemma 4.4. Continuity of zero functions. *Let $x,y,s,t \in \mathbb{R}$ such that $0 \leq s < t$. Then*

(i) *If all zeroes in $[x,y) \times \{t\}$ are regular, and there are no zeroes in $\{(x,t),(y,t)\}$, then there is an open neighborhood $\mathcal{U}$ of (u,v) in $\tilde{\mathcal{B}}^2$ such that for all $\tilde{u},\tilde{v} \in \mathcal{U}$, $\tilde{w} = \tilde{u} - \tilde{v}$, we have $Z_w(x,y,t) = Z_{\tilde{w}}(x,y,t)$.*

(ii) *If all zeroes in $\{x\} \times [s,t)$ are regular, and there are no zeroes in $\{(x,s),(x,t)\}$, then there is an open neighborhood $\mathcal{U}$ of (u,v) in $\tilde{\mathcal{B}}^2$ such that for all $\tilde{u},\tilde{v} \in \mathcal{U}$, $\tilde{w} = \tilde{u} - \tilde{v}$, $F_w(x,s,t) = F_{\tilde{w}}(x,s,t)$.*

(iii) *Assume all the zeroes in ∂Q, where $Q = [x,y] \times [s,t]$, are regular. Then there is an open neighborhood $\mathcal{U}$ of (u,v) in $\tilde{\mathcal{B}}^2$ such that for all $\tilde{u},\tilde{v} \in \mathcal{U}$, $\tilde{w} = \tilde{u} - \tilde{v}$, $D_w(x,y,s,t) = D_{\tilde{w}}(x,y,s,t)$.*

Proof. By embedding of $\mathcal{X}^\alpha$ in $C^1(\mathbb{R})$ and continuous dependence on initial conditions of (1.1), we can find an open neighborhood of $\mathcal{U}$ of (u,v) such that for each $(\tilde{u},\tilde{v}) \in \mathcal{U}$, for $\tilde{w} = \tilde{u}-\tilde{v}$ the assumptions (i) hold. Under the assumptions (i), $Z_w(x,y,t) = |w(.,t)^{-1}(0)\cap(x,y)|$, $|.|$ the cardinal number, and whenever $w(z,t)=0$, we have $w_x(z,t)\neq 0$. By the implicit function theorem and compactness of $[x,y]$, we see that $|\tilde{w}(.,t)^{-1}(0)\cap(x,y)|$ is constant in a $C^1(\mathbb{R})$ neighborhood of $\tilde{w}(.,t)$ on which assumptions (i) hold, thus $Z_w(x,y,t)$ is constant on $\mathcal{U}$.

To prove (ii), consider an open neighborhood $\mathcal{U}_1$ of (u,v), such that for some $\delta_0>0$ small enough, $(\tilde{u},\tilde{v})\in\mathcal{U}_1$, for $\tilde{w}=\tilde{u}-\tilde{v}$ we have $Z_{\tilde{w}}(x-\delta_0,x,s)=0$, $Z_{\tilde{w}}(x-\delta_0,x,t)=0$ and $D_{\tilde{w}}(x-\delta,x,s,t)=0$ (such an $\mathcal{U}_1$ exists by the assumptions (ii)). Now for all $0<\delta\le\delta_0$, by definition of Z_w, D_w, the assumptions (ii) hold on $\{x-\delta\}\times[s,t)$. We claim that we can find $0<\delta_1\le\delta_0$ such that, in addition, for all the zeroes in $\{x-\delta_1\}\times[s,t)$, expressed as $(\tau,\gamma_i(\tau))$, $\gamma_i(\tau)=x-\delta_1$, $s<\tau<t$, we have that $(\gamma_i)_x(\tau)\neq 0$. We apply the Morse-Sard Lemma to every C^1 function γ_i, and establish that the set of critical values x, i.e. x for which $\gamma_i(\tau)=x$ and $(\gamma_i)_x(\tau)=0$ for some $\tau\in[s,t]$, has the Lebesgue measure 0. As there are at most countably many curves of zeroes γ_i, $i\in\mathcal{I}_w$, this completes the proof of existence of such δ_1. It is easy to verify that now

$$F_w(x-\delta_1,s,t) = \sum_{\tau\in(s,t),w(x-\delta_1,\tau)=0} \operatorname{sgn}\,(-w_t(x-\delta_1,\tau)), \tag{4.4}$$

as $\operatorname{sgn}(\gamma_i)_x(\tau) = -w_t(\tilde{x},\tau)$ for regular zeroes $(\tilde{x},\tau)$, where $i\in\mathcal{I}_w$ is such that $\gamma_i(\tau)=\tilde{x}$. We now find an open neighborhood $\mathcal{U}\subset\mathcal{U}_1$ of (u,v) such that for each $(\tilde{u},\tilde{v})\in\mathcal{U}$, $\tilde{w}=\tilde{u}-\tilde{v}$, for all the zeroes of $\tilde{w}$ in $(x-\delta_1,\tau)$ in $\{x-\delta_1\}\times(s,t)$, we have that $\tilde{w}_t(x-\delta_1,\tau)\neq 0$. Applying the implicit function theorem analogously as for z, but now for the function $\tau\mapsto w(x-\delta_1,\tau)$, $\tau\in[s,t]$, we deduce that $F_{\tilde{w}}(x-\delta_1,s,t)$ is constant on $\mathcal{U}$. By the balance law (4.2) and the construction of $\mathcal{U}_1$ we see that $F_{\tilde{w}}(x,s,t)=F_{\tilde{w}}(x-\delta_1,s,t)$ on $\mathcal{U}$, which completes (ii).

To show (iii), note first that for any $\delta_1,\delta_2,\varepsilon_1,\varepsilon_2>0$ small enough, we have

$$D_w(x,y,s,t) = D_w(x+\delta_1,y-\delta_1,s+\varepsilon_1,t-\varepsilon_1) = D_w(x-\delta_2,y+\delta_2,s-\varepsilon_2,t+\varepsilon_2)$$

(this follows from the finiteness of the number of multiple zeroes in any compact Q and the assumptions (iii)). In addition, by the local structure of zeroes we can choose $\delta_1,\delta_2,\varepsilon_1,\varepsilon_2>0$ such that there are no zeroes in the "corners" $\{x+\delta_1,y-\delta_1\}\times\{s+\varepsilon_1,t-\varepsilon_1\}$ and $\{x-\delta_2,y+\delta_2\}\times\{s-\varepsilon_2,t+\varepsilon_2\}$. Now applying twice the balance law (4.2) and (i), (ii) (i.e. on $[x+\delta_1,y-\delta_1]\times[s+\varepsilon_1,t-\varepsilon_1]$ and $[x-\delta_2,y+\delta_2]\times[s-\varepsilon_2,t+\varepsilon_2]$), we can find a neighborhood $\mathcal{U}$ such that

$$D_w(x,y,s,t) = D_{\tilde{w}}(x+\delta_1,y-\delta_1,s+\varepsilon_1,t-\varepsilon_1) = D_{\tilde{w}}(x-\delta_2,y+\delta_2,s-\varepsilon_2,t+\varepsilon_2)$$

for $(\tilde{u},\tilde{v})\in\mathcal{U}$, $\tilde{w}=\tilde{u}-\tilde{v}$. To establish $D_{\tilde{w}}(x,y,s,t)=D_w(x,y,s,t)$ on $\mathcal{U}$, it suffices to note that by the definition of D_w,

$$D_{\tilde{w}}(x+\delta_1,y-\delta_1,s+\varepsilon_1,t-\varepsilon_1) \le D_{\tilde{w}}(x,y,s,t) \le D_{\tilde{w}}(x-\delta_2,y+\delta_2,s-\varepsilon_2,t+\varepsilon_2).$$

□

We will frequently use the notation $\hat{d}(u,v)=d(u-v)$, $\hat{f}(u,v)=f(u-v)$, $\hat{d}(u,v)=d(u-v)$, where $\hat{d},\hat{u},\hat{v}:\tilde{\mathcal{B}}^2\to\mathbb{Z}$, and let $\hat{S},\hat{T}:\tilde{\mathcal{B}}^2\to\tilde{\mathcal{B}}^2$, $\hat{S}(u,v)=(Su,Sv)$, $\hat{T}(u,v)=(Tu,Tv)$. The balance law of zeroes (4.3) can now be written as

$$\hat{z}\circ\hat{T}-\hat{z} = \hat{f}\circ\hat{S}-\hat{f}-\hat{d}. \tag{4.5}$$

Lemma 4.5. *If $u,v\in\tilde{\mathcal{B}}$ are such that $\hat{d}(u,v)>0$, then there exists an open neighbourhood $\mathcal{U}$ of (u,v) in $\tilde{\mathcal{B}}^2$ such that for each $(\tilde{u},\tilde{v})\in\mathcal{U}$, we have*

$$\hat{d}(\tilde{u},\tilde{v})+\hat{d}(\hat{S}^{-1}(\tilde{u},\tilde{v}))+\hat{d}(\hat{T}(\tilde{u},\tilde{v}))+\hat{d}(\hat{S}^{-1}\hat{T}(\tilde{u},\tilde{v}))\ge 1. \tag{4.6}$$

Proof. By finiteness of the number zeroes in a compact set, we can find $0<\delta,\varepsilon<1$ small enough such that for $\tilde{Q}=[-\delta,1-\delta]\times[\varepsilon,1+\varepsilon]$, there are no multiple zeroes in ∂Q, and such that $D_w(-\delta,1-\delta,\varepsilon,1+\varepsilon)=D_w(0,1,0,1)=d(w)$. Now we apply Lemma 4.4, (iii), and find an open neighborhood $\mathcal{U}$

of (u,v) such that for each $(\tilde{u},\tilde{v}) \in \mathcal{U}$, $\tilde{w} = \tilde{u}-\tilde{v}$, we have $D_w(-\delta, 1-\delta, \varepsilon, 1+\varepsilon) = D_{\tilde{w}}(-\delta, 1-\delta, \varepsilon, 1+\varepsilon)$. Finally it suffices to note that

$$\begin{aligned}\hat{d}(\tilde{u},\tilde{v}) + \hat{d}(\hat{S}^{-1}(\tilde{u},\tilde{v})) + \hat{d}(\hat{T}(\tilde{u},\tilde{v})) + \hat{d}(\hat{S}^{-1}\hat{T}(\tilde{u},\tilde{v})) &= D_{\tilde{w}}(-1,1,0,2) \\ &\geq D_{\tilde{w}}(-\delta, 1-\delta, \varepsilon, 1+\varepsilon) \\ &= d(w) \geq 1.\end{aligned}$$

where the second inequality follows from the definition of D_w. □

Lemma 4.6. *The functions $\hat{z}, \hat{d}, \hat{f} : \tilde{\mathcal{B}}^2 \to \mathbb{Z}$ are Borel-measurable. Specifically, $\hat{d}, \hat{z} \geq 0$.*

Proof. The functions $\hat{z}, \hat{d}, \hat{f}$ have values in $\mathbb{Z}$ and $\hat{d}, \hat{z} \geq 0$ by definition and Lemma 4.2, (iv).

Let $w = u - v$. To show measurability of $\hat{z}$, we first show the following: for $n \in \mathbb{N}$, $n \geq n_0(w)$, $n_0(w)$ large enough, we have that $x_n := -1/\sqrt{n}$, $y_n := 1 - 1/\sqrt{n}$, $t_n := 1/n$ satisfy the assumptions in Lemma 4.4, (i), and that $z(w) = Z_w(x_n, y_n, t_n)$. Firstly, by finiteness of the number of multiple zeroes in a compact set, there are no multiple zeroes in $[x_n, y_n) \times \{t_n\}$ for n large enough. Now if $(0,0)$ and $(1,0)$ are not zeroes, or are even, multiple zeroes, the assumptions in Lemma 4.4, (i) hold for n large enough by the local structure of zeroes. Assume $(0,0)$ is a regular or odd multiple zero, thus lying on a C^1 curve of zeroes γ_i, $\gamma_i(0) = 0$. By the local structure of zeroes, (x_n, t_n) can be a zero for n large enough only if $x_n = \gamma_i(t_n)$. However, it is impossible for n large enough, whenever $\sqrt{n} \gg |(\gamma_i)_t(0)|$ (which is well-defined as γ_i is C^1 and defined on an open set). Analogously we show that (y_n, t_n) can not be a zero for n large enough. By an analogous consideration, we see that for all $i \in \mathcal{I}$, $\mathbf{1}(\{\gamma_i(t_n) \geq x_n\}) = \mathbf{1}(\{\gamma_i(0) \geq 0\})$, $\mathbf{1}(\{\gamma_i(t_n) \geq y_n\}) = \mathbf{1}(\{\gamma_i(0) \geq 1\})$ for $n \geq n_0(w)$, $n_0(w)$ large enough, thus by definition and finiteness of the number of relevant $i \in \mathcal{I}$, we have $z(w) = Z_w(x_n, y_n, t_n)$ for $n \geq n_0(w)$ for $n_0(w)$ large enough.

Denote by $\mathcal{U}_n \subset \tilde{\mathcal{B}}^2$ such that x_n, y_n, t_n satisfy assumptions in Lemma 4.4, (i). Clearly, $\mathcal{U}_n$ is measurable (as an intersection of a closed $\tilde{\mathcal{B}}^2$ and an open set for which the assumptions hold). Thus by Lemma 4.4, (i), and separability of $\tilde{\mathcal{B}}^2$, the set $V_{n,k} = \{Z_{u-v}(x_n, y_n, t_n) = k,\ (u,v) \in \mathcal{U}_n\}$ is measurable for any integers $n \geq 1$, $k \geq 0$. By the previous paragraph,

$$\{z(u-v) = k\} = \cap_{n \in \mathbb{N}} V_{n,k}$$

thus $\hat{z}(u,v) = z(u-v)$ is measurable. The proof for $\hat{f}$ is analogous by considering F_w on $(x_n, s_n, t_n) = (-1/n, 0, 1)$. Measurability of $\hat{d}$ follows from (4.5). □

5. The proof of Theorem 1.1 (the bounded case)

In this section we consider only the bounded case, and then show:

Proposition 5.1. *For any $(u,v) \in \mathcal{E}^2$, we have that $d(u-v) = 0$.*

This will trivially imply Theorem 1.1. We prove Proposition 5.1 in the AC case only; the DC case is analogous (by taking the semiflow $T(t)$ instead of the map T). Extending the ideas from the introduction, we define the zero function $\hat{Z}$ of two Borel probability measures on $\tilde{\mathcal{B}}$ (by also using the notation from the previous section) as

$$\hat{Z}(\mu_1, \mu_2) := \int_{\tilde{\mathcal{B}}^2} z(u-v) d\mu_1(u) \mu_2(v) = \int_{\tilde{\mathcal{B}}^2} \hat{z} d\mu_1 d\mu_2.$$

More generally, $\hat{Z}$ is well-defined for any Borel probability measure on $\tilde{\mathcal{B}}^2$. Now $Z(\mu) = \hat{Z}(\mu^2)$. We prove Proposition 5.1 as follows. We first deal with the possibility that $Z(\mu) = \infty$. In the bounded case this is not a difficulty, as we first show in Lemma 5.2 that if μ is T-ergodic, Z is always finite. We actually prove a more general statement for measures on $\tilde{\mathcal{B}}^2$. In the non-ergodic case, we show in Lemma 5.3 that we can always modify "weights" in the ergodic decomposition to make $\hat{Z}$ finite.

The claim will then follow from integrating the balance law of zeroes (4.5), which by S-periodicity in the bounded case reduces to

$$\hat{z} \circ \hat{T} - \hat{z} = -\hat{d}. \tag{5.1}$$

Lemma 5.2. *Let ν be a Borel-probability measure on $\tilde{\mathcal{B}}^2$. If ν is $\hat{T}$-ergodic, then $\hat{Z}(\nu) < \infty$.*

Proof. By ergodicity, any $\hat{T}$-invariant set is of ν-measure 0 or 1. The balance law (5.1) implies that $\hat{z}$ is non-increasing on $\tilde{\mathcal{B}}^2$ with respect to $\hat{T}$, thus the sets $\mathcal{B}_n = \{\hat{z}(u,v) \leq n, (u,v) \in \tilde{\mathcal{B}}^2\}$, $n \geq 0$ an integer, are $\hat{T}$-invariant, and by finiteness of $\hat{z}$, $\tilde{\mathcal{B}}^2 = \cup_{n=0}^{\infty} \mathcal{B}_n$. As $\nu(\tilde{\mathcal{B}}^2) = 1$, by continuity of probability we have that there exists $n_0 \geq 0$ such that $\mu(\mathcal{B}_{n_0}) = 1$, thus $\hat{Z}(\nu) \leq n_0$. □

Lemma 5.3. *Assume ν is $\hat{T}$-invariant, and $(u,v) \in \operatorname{supp} \nu$. Then there exists a $\hat{T}$-invariant $\tilde{\nu}$ such that $\hat{Z}(\tilde{\nu}) < \infty$ and $(u,v) \in \operatorname{supp} \tilde{\nu}$.*

Proof. First we find a sequence of $\hat{T}$-ergodic measures ν_k such that (u,v) is in the closure of $\cup_{k=1}^{\infty} \operatorname{supp} \nu_k$. We do it e.g. by choosing any ergodic measure ν_k such that the ν_k-measure of the $1/k$-ball $\mathcal{B}_k$ around (u,v) is > 0 (it must exist by the ergodic decomposition theorem [40]). Let

$$\tilde{\nu} = \kappa \sum_{k=1}^{\infty} \frac{1}{\max\{\hat{Z}(\nu_k), 2^k\}} \nu_k,$$

where κ is uniquely chosen so that $\tilde{\nu}$ is a probability measure. Indeed, it is possible to choose $\kappa \geq 1$, as by Lemma 5.2, the sum of the factors is

$$0 < \sum_{k=1}^{\infty} 1/\max\{\hat{Z}(\nu_k), 2^k\} \leq 1.$$

Also by definition, $\hat{Z}(\tilde{\nu}) \leq 1/\kappa < \infty$. We see that $(u,v) \in \operatorname{supp} \tilde{\nu}$ by choosing any $w_k \in \mathcal{B}_k \subset \operatorname{supp} \tilde{\nu_k} \subset \operatorname{supp} \tilde{\nu}$. Then w_k converges to (u,v), so (u,v) must be in $\operatorname{supp} \tilde{\nu}$ as the support of a measure is always closed. □

Proof of Proposition 5.1. Assume $(u,v) \in \mathcal{E}^2$, i.e. that $u \in \operatorname{supp} \mu_1$, $v \in \operatorname{supp} \mu_2$ for some T-invariant μ_1, μ_2. Let $\nu = \mu_1 \times \mu_2$, and let $\tilde{\nu}$ be a $\hat{T}$-invariant measure constructed in Lemma 5.3. We can write (5.1) twice to obtain

$$\hat{z} \circ \hat{T}^2 - \hat{z} = -\hat{d} - \hat{d} \circ \hat{T}.$$

Integrating it with respect to $\tilde{\nu}$, and using $\hat{T}$-invariance of $\tilde{\nu}$ and integrability of $\hat{Z}$ (and thus integrability of $\hat{Z} \circ \hat{T}$, $\hat{Z} \circ \hat{T}^2$), we see that

$$\int_{\tilde{\mathcal{B}}^2} \hat{d}\, d\tilde{\nu} + \int_{\tilde{\mathcal{B}}^2} \hat{d} \circ \hat{T} d\tilde{\nu} = 0. \tag{5.2}$$

Now, assume that $d(u-v) = \hat{d}(u,v) > 0$. We now find an open neighborhood $\mathcal{U}$ of (u,v) such that (4.6) holds, i.e. by S-invariance of all u in the bounded case, such that $\hat{d}(\tilde{u},\tilde{v}) + \hat{d} \circ \hat{T}(\tilde{u},\tilde{v})$. As (u,v) is in the support of $\tilde{\nu}$, we have that $\tilde{\nu}(\mathcal{U}) \geq \varepsilon$ for some $\varepsilon > 0$, thus as always $\hat{d} \geq 0$,

$$\int_{\tilde{\mathcal{B}}^2} \hat{d}\, d\tilde{\nu} + \int_{\tilde{\mathcal{B}}^2} \hat{d} \circ \hat{T} d\tilde{\nu} \geq \varepsilon,$$

which contradicts (5.2). □

Proof of Theorem 1.1. By definition, if $(u(0), u_x(0)) = (v(0), v_x(0))$, then by definition of d, we have $d(T^{-1}u - T^{-1}v) \geq 1$, and by T-invariance of $\mathcal{E}$, $(T^{-1}u, T^{-1}v) \in \mathcal{E}$. This is by Proposition 5.1 impossible. □

6. The proof of Theorem 1.2 (the extended case)

In this section, we consider only the extended, AC case (the DC case is analogous). The main difficulty in the proof of Theorem 1.2 is possible non-integrability of the zero and flux functions $\hat{z}, \hat{f}$. For integrable $\hat{z}, \hat{f}$, by integrating (4.5) with respect to a product of any two S, T-invariant measures μ_1, μ_2, we see that $d = 0$, $\mu_1 \times \mu_2$-a.e.. The claim then follows analogously as in the bounded case. To address it in the extended case, we apply ergodic-theoretical tools for two commuting transformations $\hat{S}, \hat{T}$, on the product $\tilde{\mathcal{B}}^2$ with the Borel σ-algebra. In other words, we consider ergodic properties of two "replicas" of elements in $\tilde{\mathcal{B}}$ with associated actions induced by (1.1) and the spatial shift S.

Given a $\hat{S}$-invariant measure ν on $\tilde{\mathcal{B}}^2$, the zero function is defined with

$$\hat{Z}(\nu) = \int_{\tilde{\mathcal{B}}^2} \hat{z}\, d\nu$$

(ν is now an arbitrary $\hat{S}$-invariant measure, not necessarily a product of two measures on $\tilde{\mathcal{B}}$). In the first sub-section we prove a balance law of zeroes on average, i.e. that the flux in (4.5) cancels out when (4.5) is integrated with respect to a $\hat{S}$-invariant measure. In the second subsection we consider properties of the average density of zeroes defined as

$$\hat{\zeta}(u, v) = \liminf_{n\to\infty} \frac{1}{2n} z_{[-n,n)}(u - v).$$

By the Birkhoff ergodic theorem, for any $\hat{S}$-invariant measure ν on $\tilde{\mathcal{B}}^2$, for ν-a.e. (u, v), the $\liminf$ in the definition of $\hat{\zeta}$ can be replaced with lim, though we can not exclude the possibility that the value of $\hat{\zeta}$ is $+\infty$. We then characterize the case of $\hat{\zeta}$ being ν-a.e. finite. We use these tools to complete the proof of Theorem 1.2 analogously as in the bounded case.

6.1. The balance law of zeroes on average.

We prove the following:

Proposition 6.1. The balance law of zeroes on average. *Assume $\nu = \nu(0)$ is a $\hat{S}$-invariant measure on $\tilde{\mathcal{B}}^2$, such that $\hat{Z}(\mu) < \infty$. Then*

$$\hat{Z}(\nu(0)) = \hat{Z}(\nu(1)) + \int_{\tilde{\mathcal{B}}^2} \hat{d}\, d\nu(0). \tag{6.1}$$

Here $\nu(t)$ denotes the induced evolution of $\nu(0)$ on $\tilde{\mathcal{B}}^2$ with respect to two "replicas" of (1.1). The proposition will follow from a general ergodic theoretical argument, which is needed to show that the flux f in (4.5) cancels out when integrated with respect to a $\hat{S}$-invariant measure, even in the case when f is not integrable:

Proposition 6.2. *Assume $(\Omega, \mathcal{F}, \nu)$ is a probability space, and that $\hat{\sigma} : \Omega \to \Omega$ is a measurable, ν-invariant map. Assume that $\varphi, \zeta : \Omega \to \mathbb{R}$ are measurable, and that ζ is ν-integrable. Furthermore, assume that ν-a.e.,*

$$\varphi \circ \hat{\sigma} - \varphi \geq -\zeta. \tag{6.2}$$

Then $(\varphi \circ \hat{\sigma} - \varphi)$ is integrable and $\int_\Omega (\varphi \circ \hat{\sigma} - \varphi) d\nu = 0$.

Proof. Let $\mathcal{U}_m$ be the set of all $u \in \Omega$ such that $\varphi(\hat{\sigma}^n(u)) \leq m$ for infinitely many $n \in \mathbb{N}$. Then it is easy to see that $\mathcal{U}_m$ is $\hat{\sigma}$-invariant, and by the Poincaré recurrence theorem applied to sets $\{u : \varphi(u) \leq m\}$, that

$$\nu\{\bigcup_{m=1}^{\infty} \mathcal{U}_m\} = 1. \tag{6.3}$$

Consider functions

$$\begin{aligned} u \mapsto & h(u) := \varphi(\hat{\sigma}(u)) - \varphi(u) + \zeta(u), \\ u \mapsto & h_m(u) := \mathbf{1}_{\mathcal{U}_m}(u)\{\varphi(\hat{\sigma}(u)) - \varphi(u) + \zeta(u)\} \wedge m, \end{aligned}$$

where $\mathbf{1}_{\mathcal{U}_m}$ is the characteristic function and $\wedge$ the minimum. By the assumptions, $h \geq 0$, thus $h_m \geq 0$, and by construction and (6.3), h_m is an increasing sequence of functions converging ν-a.e. to h.

We will first show that $\mathbb{E}[h_m] \leq \mathbb{E}[\zeta]$, where $\mathbb{E}[.]$ denotes the expectation, i.e. the Lebesgue integral with respect to ν. Let $\mathcal{S}$ be the σ-algebra of $\hat{\sigma}$-invariant sets. It suffices to show that for all $m \geq 0$,

$$\mathbb{E}[h_m|\mathcal{S}] \leq \mathbb{E}[\zeta|\mathcal{S}], \quad \nu - a.e., \tag{6.4}$$

where $\mathbb{E}[.|\mathcal{S}]$ denotes the conditional expectation [6]. As $0 \leq h_m \leq m$, h_m is integrable, thus by the Birkhoff ergodic theorem, we have that ν-a.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_m \circ \hat{\sigma}^k = \mathbb{E}[h_m|\mathcal{S}]. \tag{6.5}$$

Without loss of generality $\nu(\mathcal{U}_m) > 0$ (otherwise $h_m = 0$ ν-a.e.). Choose $u \in \mathcal{U}_m$, and one of infinitely n_j such that $\varphi(\hat{\sigma}^{n_j}(u)) \leq m$. Then it is easy to see that

$$\begin{aligned} \frac{1}{n_j} \sum_{k=0}^{n_j-1} h_m(\hat{\sigma}^k(u)) \leq \frac{1}{n_j} \sum_{k=0}^{n_j-1} h(\hat{\sigma}(u)) &= \frac{1}{n_j}(\varphi(\hat{\sigma}^{n_j}(u)) - \varphi(u)) + \frac{1}{n_j} \sum_{k=0}^{n_j-1} \zeta(\hat{\sigma}^k(u)) \\ &\leq \frac{1}{n_j}(m - \varphi(u)) + \frac{1}{n_j} \sum_{k=0}^{n_j-1} \zeta(\hat{\sigma}^k(u)). \end{aligned} \tag{6.6}$$

Now by the Birkhoff ergodic theorem applied to ζ, we see that the right-hand side of (6.6) converges to $\mathbb{E}[\zeta|\mathcal{S}]$ as $n_j \to \infty$. Combined with (6.5), we deduce that for ν-a.e. $u \in \mathcal{U}_m$, we have that $\mathbb{E}[h_m|\mathcal{S}] \leq \mathbb{E}[\zeta|\mathcal{S}]$. As for $u \in \mathcal{U}_m^c$, $h_m(u) = 0$ and $\mathcal{U}_m^c$ is $\hat{\sigma}$-invariant, we conclude that (6.4) holds also for ν-a.e. $u \in \mathcal{U}_m^c$.

Now, by the definition of the conditional expectation, (6.4) implies that for all $m \in \mathbb{N}$, $\mathbb{E}[h_m] \leq \mathbb{E}[\zeta]$, thus by the Lebesgue monotone convergence theorem, h is integrable and $\mathbb{E}[h] \leq \mathbb{E}[\zeta]$. As we can now apply the Birkhoff ergodic theorem also to h, we repeat the argument as in (6.5) and (6.6) applied to h instead of h_m to conclude that $\mathbb{E}[h] = \mathbb{E}[\zeta]$. As now $h - \zeta$ is integrable and $\mathbb{E}[h - \zeta] = 0$, the proof is complete. □

Proof of Proposition 6.1. We insert in Proposition 6.2 the following: $\Omega = \tilde{\mathcal{B}}^2$ with the Borel σ-algebra, $\hat{\sigma} = \hat{S}$, $\varphi = \hat{f}$ and $\zeta = \hat{z}$. By (4.5), the assumptions of Proposition 6.2 hold, thus $(\hat{f} \circ \hat{S} - \hat{f})$ is ν-integrable and

$$\int_{\tilde{\mathcal{B}}^2} \left(\hat{f} \circ \hat{S} - \hat{f} \right) d\nu = 0.$$

Inserting it into (4.5) integrated with respect to ν, we obtain (6.1), where by non-negativity of $\hat{z}, \hat{d}$, we have that $\hat{z} \circ \hat{T}$ and $\hat{d}$ are also ν-integrable. □

6.2. **Density of zeroes and non-degeneracy of invariant measures.** First we establish that density of zeroes is a.e. non-decreasing, and then define and characterize non-degeneracy of invariant measures.

Lemma 6.3. *Assume ν is a $\hat{S}$-invariant measure. Then for ν-a.e. (u, v),*

$$\hat{\zeta}(Tu, Tv) \leq \hat{\zeta}(u, v). \tag{6.7}$$

Proof. It suffices to prove that (6.7) holds a.e. with respect to every $\hat{S}$-ergodic measure ν_0, as the claim then follows by the ergodic decomposition theorem. This follows from the Birkhoff ergodic theorem and (6.1) if $\hat{Z}(\nu_0) < \infty$, and trivially if $\hat{Z}(\nu_0) = \infty$, as then $\hat{\zeta}(u, v) = \infty$ ν_0-a.e. □

Definition 6.1. We say that a $\hat{S}$-invariant measure on $\tilde{\mathcal{B}}^2$ is non-degenerate, if for ν-a.e. (u,v), $\hat{\zeta}(u,v)<\infty$. We say that a pair (μ_1,μ_2) of S-invariant measures on $\tilde{\mathcal{B}}$ is non-degenerate, if $\mu_1\times\mu_2$ is non-degenerate. An S-invariant measure μ on $\tilde{\mathcal{B}}$ is non-degenerate, if the pair (μ,μ) is non-degenerate. A family of S-invariant measures $\mathcal{N}$ on $\tilde{\mathcal{B}}$ is non-degenerate, if every $\mu\in\mathcal{N}$ is non-degenerate. The ergodic attractor $\mathcal{E}$ is non-degenerate, if $\mathcal{M}(\mathcal{B})$ is non-degenerate.

We note that we do not know of any examples of degenerate measures on $\tilde{\mathcal{B}}^2$. We discuss it further in Section 11.

In the following lemma, we use the ergodic decomposition of a measure with respect to two commuting transformations. We say that a measure is ergodic with respect to two commuting transformations $\hat{S},\hat{T}$, if any $\hat{S},\hat{T}$-invariant measurable set has measure 0 or 1. We can decompose a $\hat{S},\hat{T}$-invariant measure on $\tilde{\mathcal{B}}^2$ into $\hat{S},\hat{T}$-ergodic measures, with the standard decomposition formula [40], Section 6.2, as the Choquet theorem applies. We will require the following generalization of Lemma 5.2 to the extended case.

Lemma 6.4. *Let ν be a $\hat{S}$-invariant measure on $\tilde{\mathcal{B}}^2$.*

(i) ν is non-degenerate, if and only if for a.e. measure ν_0 in its ergodic decomposition into $\hat{S}$-ergodic measures, $\hat{Z}(\nu_0)<\infty$.

(ii) Assume ν is $\hat{S},\hat{T}$-invariant. Then ν is non-degenerate, if and only if for a.e. measure ν_0 in its ergodic decomposition into $\hat{S},\hat{T}$-ergodic measures, $\hat{Z}(\nu_0)<\infty$.

Proof. Assume ν is non-degenerate, and take any measure ν_0 from its $\hat{S}$-ergodic decomposition such that $\hat{\zeta}<\infty$ ν_0-a.e. (it holds for a.e. measure in the ergodic decomposition.) As for each n, the set $\{(u,v),\ \hat{\zeta}(u,v)\le n\}$ is $\hat{S}$-invariant, it has ν_0-measure 0 or 1, thus we can find n_0 large enough such that $\nu_0(\hat{\zeta}(u,v)\le n_0)=1$, so $\hat{Z}(\nu_0)\le n_0$. The other implication in (i) follows from the ergodic decomposition theorem.

To show (ii), it suffices to note that by Lemma 6.3, for every $\hat{S},\hat{T}$-ergodic measure ν_0, the sets $\{(u,v),\ \hat{\zeta}(u,v)\le n\}$ are ν_0-a.e. $\hat{S},\hat{T}$-invariant. The rest of the proof is analogous to the case (i). □

6.3. **Proof of Theorem 1.2.** We prove the following slightly generalized version of Theorem 1.2:

Proposition 6.5. *Assume $\mathcal{M}_0(\mathcal{B})$ is a non-degenerate family of measures, closed for finite or countable convex combinations, and let $\mathcal{E}_0=\cup_{\mu\in\mathcal{M}_0(\mathcal{B})}\operatorname{supp}\mu$. Then for any $(u,v)\in\mathcal{E}_0$, we have that $d(u-v)=0$.*

Proof. Let $u\in\operatorname{supp}\mu_1$, $v\in\operatorname{supp}\mu_2$, and let $\nu=\frac{1}{4}(\mu_1+\mu_2)^2$ be a $\hat{S},\hat{T}$-invariant measure on $\tilde{\mathcal{B}}^2$, by assumptions non-degenerate. Analogously as in Lemma 5.3, by applying Lemma 6.4, we can construct a $\hat{S},\hat{T}$-invariant $\tilde{\nu}$ such that $\hat{Z}(\nu)<\infty$, and such that $(u,v)\in\operatorname{supp}\tilde{\nu}$. As $\tilde{\nu}$ is $\hat{S},\hat{T}$-invariant, (6.1) implies that $\hat{d}=0$, $\tilde{\nu}$-a.e. The rest of the argument is analogous to the proof of Proposition 5.1. □

Example 6.1. Assume $\mathcal{M}_0(\tilde{B})$ consists of measures μ, such that for μ-a.e. $(u,v)\in\tilde{B}^2$ there exists an integer k such that $(S^k u,S^k v)=(u,v)$. Then $\mathcal{M}_0(\tilde{B})$ is non-degenerate and closed for finite or countable convex combinations.

Proof of Theorem 1.2. It is analogous to the proof of Theorem 1.1, by inserting $\mathcal{M}(\mathcal{B})=\mathcal{M}_0(\mathcal{B})$ in Proposition 6.5. □

7. Non-transversal intersections of an equilibrium in the extended case

Prior to discussing uniqueness of an invariant measure, we demonstrate here a universal property of non-transversality of intersections of ω-limit set and a S,T-equilibrium (i.e. a spatially and temporally periodic solution). We consider only extended, AC case in this section, assume (A1-4), where $\mathcal{B}$ is as in (A4), and $\tilde{\mathcal{B}}=\hat{T}(0,-\delta_0)\mathcal{B}$ for some $\delta_0>0$ as in Section 2. Let $v\in\mathcal{B}$ such that

$v = S(v) = T(v)$, thus $v \in \tilde{\mathcal{B}}$. Recall that the pair (μ, δ_v) is non-degenerate, if for μ-a.e. u, (6.1) holds. We first give examples of μ such that non-degeneracy holds.

Example 7.1. (i) Let $u_0, u_1 \in \tilde{\mathcal{B}}$, S-invariant (i.e. spatially periodic), such that they intersect for some $x_0 \in [0,1)$ (and then they intersect for all $x_0 + k$, $k \in \mathbb{Z}$); without loss of generality let $x_0 = 0$. We can obtain them by taking any S-periodic $\tilde{u}_0$, $\tilde{u}_1$ with a non-transversal intersection and choosing δ_0 small enough, $u_j = \hat{T}(0, -\delta_0)\tilde{u}_j$, $j = 0,1$. Let $\Omega = \{(\omega_k)_{k\in\mathbb{Z}}, \omega_k \in \{0,1\}\}$, be the Bernoulli space with the standard σ-algebra and the Bernoulli measure μ_0. We embed μ_0 into $\mathcal{B}$ with $\iota : \omega \mapsto u_\omega(x) := u_{\omega_{\lfloor x \rfloor}}(x)$, where $\lfloor x \rfloor$ is the largest integer $\leq x$, and defining $\mu = \iota^*\mu_0$, ι^* the standard pull of measures. Then as $\hat{\zeta}(u_1, v) < \infty$, $\hat{\zeta}(u_2, v) < \infty$, it is easy to check that $\hat{\zeta}(u, v)$ is finite, uniformly bounded for $u \in \operatorname{supp} \mu$.

(ii) Further examples can easily be constructed by using the sufficient conditions for non-degeneracy in Lemma 11.1 and Remark 11.1.

Proposition 7.1. *Assume that μ is an S-invariant measure on $\tilde{\mathcal{B}}$ such that (μ, δ_v) is non-degenerate. Then for μ-a.e. u, $\omega(u)$ consists of z such that $z(t) - v(t)$ can not have a multiple zero for any $(x,t) \in \mathbb{R}^2$.*

Proof. First note that it suffices to prove the claim for S-ergodic μ, as by Lemma 6.4, every measure ν_0 in the $\hat{S}$-ergodic decomposition of $\mu \times \delta_v$ is non-degenerate, and a.e. measures in the $\hat{S}$-ergodic decomposition of $\mu \times \delta_v$ are of the form $\mu_0 \times \delta_v$, μ_0 S-ergodic. Thus assume μ is S-ergodic, so by the non-degeneracy assumption $\hat{Z}(\mu \times \delta_v) < \infty$.

We will first show that there exists an open set $\mathcal{U} \subset \tilde{\mathcal{B}}$ satisfying

$$\{\tilde{u},\ \hat{d}(\tilde{u}, v) \geq 1\} \subset \mathcal{U} \subset \{\tilde{u},\ \hat{d}(\tilde{u}, v) + \hat{d}(S^{-1}\tilde{u}, v) + \hat{d}(T\tilde{u}, v) + \hat{d}(S^{-1}T\tilde{u}, v) \geq 1\}. \tag{7.1}$$

Then we show that

$$\sum_{k=1}^{\infty} \mu(T^k(u) \in \mathcal{U}) < \infty, \tag{7.2}$$

and finally we complete the proof by an application of the first Borel-Cantelli lemma.

To prove the first claim, for any z such that $\hat{d}(z, v) \geq 1$ we can by an application of Lemma 4.5 find an open neighborhood $\tilde{\mathcal{U}}(z) \subset \tilde{\mathcal{B}}$ such that for each $\tilde{u} \in \tilde{\mathcal{U}}(z)$, and for $\tilde{v} = v$, (4.6) holds, thus as $v = Tv = S^{-1}v$,

$$\hat{d}(\tilde{u}, v) + \hat{d}(S^{-1}\tilde{u}, v) + \hat{d}(T\tilde{u}, v) + \hat{d}(S^{-1}T\tilde{u}, v) \geq 1. \tag{7.3}$$

The set $\mathcal{U} = \cup_{z \in \tilde{\mathcal{B}}, \hat{d}(z,v) \geq 1} \tilde{\mathcal{U}}(z)$ now satisfies (7.1).

Applying (6.1) for all $\nu(k)$, $k \geq 0$ an integer, where $\nu = \mu \times \delta_v$, $\nu(0) = \nu$, thus $\nu(k) = \mu(k) \times \delta_v$, we obtain that the left-hand sum below is convergent and that

$$\sum_{k=0}^{\infty} \int_{\tilde{\mathcal{B}}} \hat{d}(T^k(u), v) d\mu(u) \leq \hat{Z}(\mu \times \delta_v) < \infty. \tag{7.4}$$

However, by definition of $\hat{d}$ and the right-hand side of (7.1), we have that

$$\begin{aligned} \nu(T^k(u) \in \mathcal{U}) &\leq \int_{\tilde{\mathcal{B}}} \left(\hat{d}(T^k\tilde{u}, v) + \hat{d}(S^{-1}T^k\tilde{u}, v) + \hat{d}(T^{k+1}\tilde{u}, v) + \hat{d}(S^{-1}T^{k+1}\tilde{u}, v) \right) d\mu \\ &= 2 \int_{\tilde{\mathcal{B}}} \left(\hat{d}(T^k\tilde{u}, v) + \hat{d}(T^{k+1}\tilde{u}, v) \right) d\mu, \end{aligned} \tag{7.5}$$

where in the second row we applied the S-invariance of μ. Inserting (7.5) into (7.4) we obtain (7.2). By the Borel-Cantelli lemma [6], (6.1), the set of u such that $T^k(u) \in \mathcal{U}$ for infinitely many $k \in \mathbb{N}$

has μ-measure 0, thus by openness of $\mathcal{U}$, $\mu(\{u, \omega(u) \cap \mathcal{U} = \emptyset\}) = 1$. Now by S-invariance of μ we have

$$\mu\left(\bigcap_{k\in\mathbb{Z}} \{u,\, \omega(S^k(u)) \cap \mathcal{U} = \emptyset\}\right) = 1.$$

By T-invariance of ω-limit set and (7.1), this completes the proof. □

8. A 1D family of equilibria as an ergodic attractor and asymptotics

Here we give a more general condition (C1), sufficient for the conclusions of Theorem 1.3 and Corollaries 1.4, 1.5 and 1.6 to hold. Specifically, we assume existence of a 1D family of periodic solutions as follows:

(C1) (i) There exists a set $\mathcal{V} = \{v^y,\, y \in \mathbb{R},\, v^y \in H^{2\alpha}(\mathbb{S}^1)$, satisfying that $y \mapsto v^y$ is continuous in $H^{2\alpha}(\mathbb{S}^1)$, strictly increasing, and that for all $y \in \mathbb{R}$, v^y is T-invariant, $\int_0^1 v^y(x)dx = y$.

(ii) The functions

$$y \mapsto \underline{v}(y) := \min\{v^y(t,x),\ (x,t) \in \mathbb{S}^1 \times [0,1]\},$$
$$y \mapsto \overline{v}(y) := \max\{v^y(t,x),\ (x,t) \in \mathbb{S}^1 \times [0,1]\}$$

are onto $\mathbb{R}$.

(Here $v^y(t,x)$ denotes the solution of (1.1), $u(0) = v^y$.)

The standing assumptions in this section are (A1-3), (B1) and (C1), with $\alpha > 1 - \varepsilon/2$, where ε is as in (B1). We prove results in the AC case only (both bounded and extended), the DC case is analogous. Under these assumptions we establish all the claims of Subsection 1.2 in a series of Lemmas. We first establish that $\mathcal{X}^\alpha$ can be decomposed into an increasing union of sets on which (A4) holds.

Lemma 8.1. *Assume $u \in \mathcal{X}^\alpha$ in either bounded or extended case, such that $||u||_{\mathcal{X}^\alpha} \le c_0$. Then there exists a constant $c_1 > 0$ (depending on c_0, non-linearity g and family $\mathcal{V}$), such that for any $t_0 \in \mathbb{R}$, the solution of (1.1), $u(t_0) = u$, exists for all $t \ge t_0$ and $||u(t)||_{\mathcal{X}^\alpha} \le c_1$.*

Proof. By (C1),(ii), we can find $y_1 < y_2$ such that $\overline{v}(y_1) \le u \le \underline{v}(y_2)$. By the maximum principle, if the solution of (1.1) exists on the interval $[t_0, t_1)$, then for each $t \in [t_0, t_1)$, we have that $\underline{v}(y_1) \le u(t) \le \overline{v}(y_2)$, thus $u(t)$ is uniformly bounded in the $L^\infty(\mathbb{S}^1)$, resp. $L^\infty(\mathbb{R})$ norm. This and (C1) imply the claim by the standard argument, e.g. [19], Proposition 7.2.2. (alternatively, see [24], Section 2), which is in the view of the comments in the Appendix also valid in the extended case. □

Let $\tilde{\mathcal{B}}_k$ be the set of all $u \in \hat{T}(0, -\delta_0)\mathcal{X}^\alpha$ such that for all $t \ge 0$, $||u(t)||_{\mathcal{X}^\alpha} \le k$. Then by Lemma 8.1, $\hat{T}(0, -\delta_0)\mathcal{X}^\alpha = \cup_{k=1}^\infty \tilde{\mathcal{B}}_k$, and by the discussion in Section 2, $\tilde{\mathcal{B}}_k$ is compact and invariant. In this section we write $\mathcal{E} = \cup_{k=1}^\infty \mathcal{E}(\tilde{\mathcal{B}}_k)$.

Lemma 8.2. *In the bounded case, and in the extended case if $\mathcal{E}$ is non-degenerate, we have that $\mathcal{E} = \mathcal{V}$.*

Proof. Consider first the bounded case. Fix $k \in \mathbb{N}$, and consider

$$\tilde{\mathcal{B}} := \tilde{\mathcal{B}}_k \cup \{v^y,\, y \in [y^-, y^+]\},$$

$\tilde{\mathcal{B}} \subset H^{2\alpha}(\mathbb{S}^1)$, where $y^- < y^+$ were chosen so that for all $u \in \tilde{\mathcal{B}}_k$, $v^{y^-} < u < v^{y^+}$ (this is possible, as by definition, $\tilde{\mathcal{B}}_k$ is uniformly bounded in $L^\infty(\mathbb{R})$, and by (C1),(ii)). Clearly $\{v^y, y \in [y^-, y^+]\} \subset \mathcal{E}(\tilde{\mathcal{B}}_k)$, as the Dirac measure δ_{v^y} is T-invariant. Assume μ is any T-invariant measure on $\tilde{\mathcal{B}}$, and let $u \in \operatorname{supp}\mu$. Let $y_1 < y_2$ be chosen so that $y_1 = \max\{y, v^y \le u\}$, and $y_2 = \min\{y, u \le v^y\}$ (such minimum and maximum exist by the compactness of the domain $\mathbb{S}^1$). If $y_1 \ne y_2$, we easily see that both $u - v^{y_1}$ and $u - v^{y_2}$ have a multiple zero, which is impossible by Proposition 5.1 applied to $(T^{-1}u, T^{-1}v^{y_1})$ or $(T^{-1}u, T^{-1}v^{y_2})$. The only possibility is $u = v^{y_1} = v^{y_2}$, thus $u \in \mathcal{V}$.

Consider now the extended case with the non-degeneracy assumption, with $\tilde{\mathcal{B}}$ as above, but now $\tilde{\mathcal{B}} \subset H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$. Again we see that $\{v^y, y \in [y^-, y^+]\} \subset \mathcal{E}(\tilde{\mathcal{B}}_k)$, as the Dirac measure δ_{v^y} is S,T-invariant. Let μ be any S,T-invariant measure on $\tilde{\mathcal{B}}$, and let $u \in \operatorname{supp} \mu$. Now suppose that u intersects some v^{y_0} twice at $x_1 < x_2$. Find $y_1 < y_2$ so that $y_1 = \max\{y, v^y(x) \leq u(x), x \in [x_1, x_2]\}$, and $y_2 = \min\{y, u(x) \leq v^y(x), x \in [x_1, x_2]\}$ (again such minimum and maximum exist by compactness of $[x_1, x_2]$). Thus by Proposition 6.5, we deduce analogously as in the bounded case that the only possibility is $u|_{[x_1,x_2]} = v^{y_1}|_{[x_1,x_2]} = v^{y_2}|_{[x_1,x_2]}$, thus by the local structure of zeroes, $u = v^{y_1} = u^{y_2}$, i.e. $u \in \mathcal{E}$. We conclude that u can intersect every $v \in \mathcal{V}$ at most once, transversally, so it is easy to see that the only alternative to $u \in \mathcal{V}$ is $u \in \mathcal{H}$, $\mathcal{H}$ the set of spatially heteroclinic solutions defined in the Introduction. By definition, no $h \in \mathcal{H}$ is S-recurrent, thus by the Poincaré recurrence theorem, $\mu(\mathcal{H}) = 0$, thus $\mu(\mathcal{V}) = 1$. As $\mathcal{V}$ is a closed set, μ must be supported on $\mathcal{V}$, which eliminates the possibility $u \in \mathcal{H}$ and concludes the proof also in the extended case. □

Lemma 8.3. *For each $u \in \mathcal{X}^\alpha$ in the bounded case, there exists $y_0 \in \mathbb{R}$ such that $\omega(u) = \{v^{y_0}\}$.*

Proof. As $\bar{\omega}(u)$ is by Lemma 3.3 non-empty, by Lemma 3.4 and Lemma 8.2, there exists some $y_0 \in \mathbb{R}$ such that $v^{y_0} \in \mathcal{E} \cap \bar{\omega}(u)$, thus $v^{y_0} \in \omega(u)$. Now by (C1), for each $\delta > 0$ there exists a sufficiently large $k_0 \in \mathbb{N}$ such that $v^{y_0-\delta} \leq T^{k_0}(u) \leq v^{y_0+\delta}$. As all v^y are T-invariant, by the maximum principle we have that for all $k \geq k_0$, $v^{y_0-\delta} \leq T^k(u) \leq v^{y_0+\delta}$, thus $\omega(u)$ contains only v^{y_0}. □

Lemma 8.4. *Assume in the extended case that μ satisfies (N1). Then there exists a set $\mathcal{U}$ of full measure such that for $u \in \mathcal{U}$ and for any $z \in \omega(u)$, $z(t) - v^y(t)$ can not have a multiple zero for any $y, x, t \in \mathbb{R}$.*

Proof. By Proposition 7.1, for a given $y \in \mathbb{R}$, there exists a set of full measure $\mathcal{U}_y$ such that if $u \in \mathcal{U}_y$ and $z \in \omega(u)$, $z(t) - v^y(t)$ can not have a multiple zero for any $x, t \in \mathbb{R}$. Now the set $\mathcal{U} = \cap_{y \in \mathbb{Q}} \mathcal{U}_y$ also satisfies $\mu(\mathcal{U}) = 1$. Assume there is $u \in \mathcal{U}$ such that for some $z \in \omega(u)$ and some $y_0 \in \mathbb{R}$, $z(t) - v^{y_0}(t)$ have a multiple zero for some $t, x \in \mathbb{R}$. However, by an analogous argument as in Lemma 4.5, we can find δ_0 such that for each $y \in (y_0 - \delta, y_0 + \delta)$, there exists $\tilde{t}$ such that $z(\tilde{t}) - v^y(\tilde{t})$ has a multiple zero, which is impossible for rational y, thus a contradiction. □

Lemma 8.5. *Assume in the extended case that μ satisfies (N1). For μ-a.e. u, we have that $\omega(u) \subset \mathcal{V} \cup \mathcal{H}$.*

Proof. To show (i), we show analogously as in the proof of Lemma 8.2 in the extended case, by applying Lemma 8.4, that there exists a set of full measure $\mathcal{U}$ so that for any $u \in \mathcal{U}$ and any $z \in \omega(u)$, $z - v^y$ can not have a multiple zero for any $y \in \mathbb{R}$. Analogously as in the same proof, we obtain $\omega(u) \subset \mathcal{V} \cup \mathcal{H}$ (the possibility that $u \in \mathcal{H}$ can not be eliminated). □

Lemma 8.6. *Assume in the extended case that μ satisfies (N1). Then ω-limit set of μ in the weak*-topology consists of measures supported on $\mathcal{V}$.*

Proof. It is a standard ergodic theoretical fact that if $\nu \in \omega(\mu)$ (ω-limit set with respect to the weak* topology of iterations of μ induced by T), then $\operatorname{supp} \nu \subset \cup_{u \in \operatorname{supp} \mu} \omega(u)$ [40], thus ν is by Lemma 8.5 supported on $\mathcal{V} \cup \mathcal{H}$. It is easy to see that ν must be S-invariant, thus as no $h \in \mathcal{H}$ is S-recurrent, by Poincaré recurrence theorem, $\nu(\mathcal{H}) = 0$. Now $\nu(\mathcal{V}) = 1$, and as $\mathcal{V}$ is closed, $\operatorname{supp} \nu \subset \mathcal{V}$. □

9. Burgers-like equations

We now complete the proofs of all the theorems stated in Subsection 1.2, by establishing that (C1) holds under the invariance assumption (B3). This will follow from an application of the Schauder fixed point theorem, assisted by standard interpolation estimates to establish equicontinuity, thus compactness by the Arzelà-Ascoli theorem. Without loss of generality, we set the fixed $\alpha > 1 - \varepsilon/2$, where ε is as in (B1).

Lemma 9.1. *If (A1-3) and (B1-3) hold, then (C1) holds.*

Proof. Throughout the proof, we consider the dynamics of (1.1) in the bounded case $\mathcal{X} = L^2(\mathbb{S}^1)$ only, and assume (A1-3), (B1-3). Fix $n \in \mathbb{N}$ and a function $c : [\alpha, 1) \to (n, \infty)$, and consider the family $\mathcal{V}_{n,c}$ of continuous functions $w : [-n, n] \to \mathcal{X}^\alpha$, $y \mapsto w^y$, satisfying the following properties for all $y, z \in [-n, n]$:

$$\int_0^1 w^y(x)dx = y, \tag{9.1a}$$

$$||w^y - y||_{L^\infty(\mathbb{R})} \le d(y), \tag{9.1b}$$

$$y \le z \Rightarrow w^y \le w^z, \tag{9.1c}$$

$$||w^y||_{\mathcal{X}^\gamma} \le c(\gamma) \quad \text{for all } \gamma \in [\alpha, 1). \tag{9.1d}$$

Clearly $\mathcal{V}_{n,c}$ is convex. We first show it is closed in $C([-n, n], \mathcal{X}^\alpha)$. The only non-trivial claim that it is closed with respect to (9.1d), for $\alpha < \gamma < 1$. It suffices to show that if for some $y \in [-n, n]$, a sequence w_n^y satisfying (9.1d) converges in $\mathcal{X}^\alpha$ to $z^y \in \mathcal{X}^\alpha$, that then $z^y \in \mathcal{X}^\gamma$ and $||z^y||_{\mathcal{X}^\gamma} \le c(\gamma)$. By taking some $\gamma' > \gamma$, by compact imbedding of $\mathcal{X}^{\gamma'}$ in $\mathcal{X}^\gamma$ we deduce that the family w_n^y, $n \in \mathbb{N}$ is relatively compact in $\mathcal{X}^\gamma$. As its every convergent subsequence in $\mathcal{X}^\gamma$ converges also in $\mathcal{X}^\alpha$, it must converge to z^y, thus $z^y \in \mathcal{X}^\gamma$, w_n^y converges to z^y in $\mathcal{X}^\gamma$ and (9.1d) holds in the limit.

We now show that for each $n \in \mathbb{N}$, there exists a function c as above such that $\mathcal{V}_{n,c}$ is non-empty, and such that the function $\tau : \mathcal{V}_{n,c} \to \mathcal{V}_{n,c}$ given with $(\tau(w))^y = T(w^y)$ is well defined, i.e. that $\tau(w) \in \mathcal{V}_{n,c}$. The properties (9.1a) and (9.1b) are preserved by (B2), (B3) respectively; and (9.1c) by the order-preserving property of (1.1). To show τ-invariance of (9.1d), consider $c_\infty := \max_{y \in [-n,n]}(|y| + d(y))$ (which exists by the upper semi-continuity of d). Then by (B1), (B2) and [19], Proposition 7.2.2, the solution of (1.1), $u(0) = w^y$ exists for all $t \ge 0$ as long as w^y satisfies (9.1b), and for all $t \ge 0$, $||u(t)||_{L^\infty(\mathbb{R})} \le c_\infty$. Furthermore, by [19], Lemma 7.0.3 and Proposition 7.2.2, we can find $c(\alpha) > n$ large enough, such that if $||w^y||_{\mathcal{X}^\alpha} \le c(\alpha)$, then $||T(w^y)||_{\mathcal{X}^\alpha} \le c(\alpha)$. Finally, we obtain the required $c(\gamma) > n$ for each $\alpha < \gamma < 1$ by integrating the variation of constants formula over $t \in [0, 1]$ while applying (B1) and a-priori bounds on the solution in $\mathcal{X}^\alpha$ for $t \in [0, 1]$ obtained in [19], Proposition 7.2.2. Clearly now for $w^y \equiv y$ we have that $w \in \mathcal{V}_{n,c}$, thus $\mathcal{V}_{n,c}$ is non-empty. We fix the constructed c.

Finally, we show that $\mathcal{V}_{n,c}$ is compact in $C([-n, n], \mathcal{X}^\alpha)$. First note that by (9.1a) and (9.1c), for $y < z$ we have that $||w^z - w^y||_{L^1(\mathbb{S}^1)} = z - y$, thus

$$||w^z - w^y||_{L^2(\mathbb{S}^1)} \le c_\infty^{1/2}(z - y)^{1/2}. \tag{9.2}$$

Fix a γ, $\alpha < \gamma < 1$. By the interpolation formula [17], p27, we have $||u||_{\mathcal{X}^\alpha} \le c_1 ||u||_{\mathcal{X}^\gamma}^{\alpha/\gamma} ||u||_{\mathcal{X}}^{1-\alpha/\gamma}$ for some fixed constant $c_1 > 0$, thus by (9.1d) and (9.2),

$$||w^z - w^y||_{\mathcal{X}^\alpha} \le 2^{\alpha/\gamma} c_1 c_\infty^{1/2 - \alpha/(2\gamma)} c(\gamma)^{\alpha/\gamma} |z - y|^{1/2 - \alpha/(2\gamma)}.$$

We see that $\mathcal{V}_{n,c}$ is equicontinuous, thus by the Arzelà-Ascoli theorem, it is compact.

By the continuous dependence on initial conditions, $\tau : \mathcal{V}_{n,c} \to \mathcal{V}_{n,c}$ is continuous. Now we can apply the Schauder fixed point theorem to find a fixed point of τ, which was required. We can extend w^y to the entire $y \in \mathbb{R}$ by choosing an increasing sequence of $n_k \in \mathbb{N}$ such that $n_k > \max_{y \in [-n_{k-1}, n_{k-1}]}(|y| + d(y))$, and proving that $\{w^y, y \in [-n_j, n_j]\}$ is then independent of n_k, $k > j$, analogously as in the proof of Lemma 8.2. This completes (C1),(i). We obtain (C2),(ii) from (1.7) and the construction. ☐

Proofs of claims in Subsection 1.2. Theorem 1.3, (i) is a restated Lemma 9.1; (ii) is Lemma 8.2, and (iii) can easily be deduced from (ii), as then $\mathcal{B}_y \cap \mathcal{E} = \{v^y\}$. Corollary 1.4 follows directly from Lemma 8.3, where we obtain the choice of y_0 directly from (B3). Corollaries 1.5 and 1.6 follow from Lemmas 8.5 and 8.6, where we obtain the part (ii) of the claims from an easy application of (N2) to (i). ☐

10. Some examples

10.1. **The B/DC case.** In the B/DC case, $\mathcal{E}$ is equal to the closure of the set of equilibria and periodic orbits in $\mathcal{B}$. This follows from the Poincaré-Bendixson theorem [10] and Lemma 3.4, as by [10], Theorem 1, the only recurrent orbits in the B/DC case are equilibria and periodic orbits.

Theorem 1.1 in the B/DC case can be deduced from results in [10], Theorems 1, 2 and Lemma 3.3.

10.2. **Embedded vector fields in the B/AC case.** Consider planar vector fields constructed by Fiedler and Sandstede [11], embedded in the bounded, AC case of (1.1). Then the union of supports of invariant measures of these vector fields are mapped into a subset of $\mathcal{E}$. This complements well Theorem 1.1, in the sense that $\mathcal{E}$ can have arbitrary complexity of a 2d vector field. In particular, one can embed in $\mathcal{E}$ invariant measures with positive metric entropy with respect to T.

10.3. **Extended gradient systems.** Consider $g = -\partial V(x,u)/\partial u$, with a C^2 V, 1-periodic in x, bounded from below, in the extended case. Then g satisfies (A1-3), and is an example of an *extended gradient system*, introduced in [15]. Under an additional assumption that for any $u(0) \in H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$, $3/4 < \alpha < 1$, the solution exists for all $t \geq 0$ and is uniformly bounded in $\mathcal{X}^\alpha := H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$ (see [15],[16] for further details), we can establish the following:

Theorem 10.1. *(i) The ergodic attractor consists of equilibria, i.e. it is given with* $\mathcal{E} = \{u \in H^2_{ul}(\mathbb{R}), u_{xx} = \partial V(x,u)/\partial u\}$*, and* $\pi : \mathcal{E} \to \mathbb{R}^2$ *given with (1.3) is one-to-one.*

(ii) For all $u \in H^{2\alpha}_{ul}(\mathbb{R})$*,* $3/4 < \alpha \leq 1$*, we have that* $\bar{\omega}(u) \subset \mathcal{E}$*.*

(iii) Given any S*-invariant measure* μ *on* $H^{2\alpha}_{ul}(\mathbb{R})$*, for* μ*-a.e.* $u \in H^{2\alpha}_{ul}(\mathbb{R})$*, we have that* $\omega(u) \subset \mathcal{E}$*.*

(iv) Given any S*-invariant measure* μ *on* $H^{2\alpha}_{ul}(\mathbb{R})$*, its* ω*-limit set in the weak* topology of the induced semiflow on the space of measures consists of measures supported on* $\mathcal{E}$*.*

The claims (i),(iii),(iv) were proved in [31, 32] (the fact that π is one-to-one follows from uniqueness of the solutions of the ordinary differential equation in the description of $\mathcal{E}$), and (ii) was shown in [15], [16].

Theorem 10.1, (i) is an example of a family for which Theorem 1.2 holds without a non-degeneracy restriction; (ii) strengthens in this particular case the properties of the ergodic attractor in the extended case from Subsection 3.3; and (iii), (iv) give an example of another family of nonlinearities g for which the claims in Corollaries 1.5 and 1.6 hold. The main tool in the proof of (i),(iii),(iv) is the following Lyapunov function on the space of S-invariant measures on $\mathcal{X}^\alpha$:

$$L(\mu) = \int_{\mathcal{X}^\alpha} \int_0^1 \left(\frac{u_x^2(x)}{2} + \frac{\partial V(x,u)}{\partial u} \right) dx \, d\mu(u),$$

which plays an analogous role as the zero function in this paper.

10.4. **The Allen-Cahn equation.** We give an example why (N2) is required to obtain sharper conclusions (ii) in the claims of Corollaries 1.5 and 1.6. Even though it does not strictly belong to the class of Burgers-like equations, we believe it is illustrative.

Example 10.1. Consider the nonlinearity as in Subsection 10.3, with $V = \frac{1}{4}u^4 - \frac{1}{2}u^2$, thus $g = u - u^3$. As done in [25], the phase-plane analysis of the family of equilibria and Theorem 10.1 show that $\mathcal{E}$ consists of the following equilibria: $u^- \equiv -1$, $u^+ \equiv 1$, a two families of spatially heteroclinic functions h_y^+, h_y^-, such that $\lim_{x\to-\infty} h_y^-(x) = \lim_{x\to\infty} h_y^+(x) = 1$, $\lim_{x\to-\infty} h_y^+(x) = \lim_{x\to\infty} h_y^-(x) = -1$, characterized by $h_y^+(y) = h_y^-(y) = 0$, and further spatially periodic functions with various periods and values in $(-1,1)$.

Similarly as in [25], consider a smooth profile $v^0 : [-n,n] \to \mathbb{R}$, $-1 < v^0 \leq 0$, $v^0(-n) = v^0(n) = 0$, such that $\frac{1}{2n}\int_{-n}^n v^0(x)dx \leq -1 + \delta$ for $\delta > 0$ small enough, let $v^1 = -v^0$, and embed the Bernoulli measure as in Example 7.1 such that to each sequence $(\omega_k)_{k\in\mathbb{Z}}$ we associate a function u by combining profiles $v^{\omega(k)}$ to obtain a S^{2n}-invariant measure. We easily obtain a S-invariant measure μ by taking

$2n$ copies of its translates. Poláčik [25] has shown that there exists $u \in \operatorname{supp} \mu$ such that $\omega(u)$ contains orbits not in $\mathcal{E}$. However, one can show by applying Theorem 10.1, (iii) and techniques from [25], that for n large enough and $\delta > 0$ small enough, for μ-a.e. u, $\omega(u) = \{u^+, u^-, h_y^+, y \in \mathbb{R}, h_y^-, y \in \mathbb{R}\}$, thus spatially heteroclinic functions in ω-limit sets in the sense of Corollary 1.5,(i) can not be avoided in general. We also obtain that for μ-a.e. u, $\bar{\omega}(u) = \{u^+, u^-\}$, and that the ω-limit set of μ in the weak*-topology consists of a single measure $\frac{1}{2}\delta_{u^-} + \frac{1}{2}\delta_{u^+}$. This shows that the ω-limit measure in the sense of Corollary 1.6 is not necessarily supported on a single function.

10.5. **The Burgers and Burger-like equations.** Consider the family of nonlinearities generalizing the nonlinearity in (1.6):

$$g(t, x, u, u_x) = -h(u)u_x + \hat{g}(t, x),$$

where $h, \hat{g}$ are continuous, $\hat{g}$ is locally Hölder continuous in t and 1-periodic in x, t, such that for all $t \in \mathbb{R}$, $\int_0^1 \hat{g}(t, x)dx = 0$; and h is locally Lipschitz continuous. Then g satisfies (A1-3). We now show that this is a Burgers-like nonlinearity:

Lemma 10.2. *The equation (1.1) with the nonlinearity g as above satisfies (B1-3).*

Proof. To show (B1-3), it suffices to consider (1.1) in the bounded case, for $u \in \mathcal{X}^\alpha = H^{2\alpha}(\mathbb{S}^1)$. First note that for any continuous $\hat{h} : \mathbb{R} \to \mathbb{R}$, by considering $H(y) = \int_0^1 \hat{h}(z)dz$, thus $dH(u)/dx = \hat{h}(u)u_x$, we get

$$\int_0^1 \hat{h}(u)u_x dx = 0. \tag{10.1}$$

Now (B1) is self-evident, and (B3) follows easily by differentiating $\int_0^1 u(x)dx$ with respect to t and using (10.1) with $\hat{h} = h$ and partial integration. To show (B2), let $c_0 = \max_{x,t\in[0,1]} |\hat{g}(x, t)|$, fix $y \in \mathbb{R}$ and choose $u \in \mathcal{X}^\alpha$ such that $\int_0^1 u(x)dx = y$. Let $t_0 \in \mathbb{R}$, and assume the solution of (1.1), $u(t_0) = u$ exists on $[t_0, t_1)$. We differentiate for an integer $p \geq 1$:

$$\begin{aligned} \frac{d}{dt}\frac{1}{2p}\int_0^1 (u(x) - y)^{2p}dx &= \int_0^1 (u(x) - y)^{2p-1}u_{xx} - \int_0^1 (u(x) - y)^{2p-1}h(u)u_x + \int_0^1 (u(x) - y)^{2p-1}\hat{g}(t, x) \\ &= -(2p - 1)\int_0^1 (u(x) - y)^{2p-2}u_x^2(x)dx + \int_0^1 (u(x) - y)^{2p-1}\hat{g}(t, x), \end{aligned} \tag{10.2}$$

where in the second row we partially integrated the first term and used that $u(x)$ is 1-periodic, and also (10.1) with $\hat{h}(u) = (u - y)^{2p-1}h(u)$ applied to the second term.

As $w(x) := (u(x) - y)^p$, $w \in C^1(\mathbb{S}^1)$ has a zero for some $x \in \mathbb{S}^1$, we can apply the L^2-Poincaré inequality to w to obtain

$$\int_0^1 (u(x) - y)^{2p}dx \leq \frac{p^2}{\pi^2}\int_0^1 (u(x) - y)^{2p-2}u_x^2(x)dx. \tag{10.3}$$

By the weighted Young's inequality applied to the integrand in the last term in (10.2), we get

$$(u(x) - y)^{2p-1}\hat{g}(t, x) \leq \frac{(2p - 1)\pi}{2p^2}(u(x) - y)^{2p} + \frac{1}{2\pi}c_0^{2p}. \tag{10.4}$$

Inserting (10.3) and (10.4) into (10.2), we now have

$$\frac{d}{dt}\frac{1}{2p}\int_0^1 (u(x) - y)^{2p}dx \leq -\frac{(2p - 1)\pi}{2p^2}\int_0^1 (u(x) - y)^{2p}dx + \frac{1}{2\pi}c_0^{2p},$$

thus by the Gronwall inequality,

$$||u(t_0) - y||_{L^{2p}(\mathbb{S}^1)} \leq c_{2p} \Rightarrow ||u(t) - y||_{L^{2p}(\mathbb{S}^1)} \leq c_{2p},\ t \in [t_0, t_1), \tag{10.5}$$

where

$$c_{2p} = \left(\frac{p^2}{(2p-1)\pi^2} \right)^{\frac{1}{2p}} c_0.$$

Now if $||u(t_0) - y||_{L^\infty(\mathbb{S}^1)} \leq c_0$, we have that for all integer $p \geq 2\pi^2$ and all $t \in [t_0, t_1)$, $||u(t) - y||_{L^{2p}(\mathbb{S}^1)} \leq c_{2p}$. As $\lim_{p\to\infty} c_{2p} = c_0$, we conclude that for all $t \in [t_0, t_1)$, $||u(t) - y||_{L^\infty(\mathbb{S}^1)} \leq c_0$, thus (C1) holds with $d(y) := c_0$, where (C1),(ii) follows from (1.7). □

Example 10.2. Consider the Burgers equation (1.6). If we use the Cole-Hopf substitution $u = -2\varphi_x/\varphi$, $\varphi(x) = \exp\left(-\frac{1}{2}\int_0^x u(y)dy\right)$, as in [28, 29, 30], and get for φ the linear equation

$$\varphi_t = \varphi_{xx} - \frac{1}{2}\hat{g}(x,t)\varphi. \tag{10.6}$$

Let φ^y be the transformed family v^y, $y \in \mathbb{R}$, as in (C1), which exists by Lemmas 9.1 and 10.2. By definition, φ^0 is non-negative, continuous and 1-periodic in t, x, thus $\varphi^0(t)$ is uniformly bounded in $L^\infty(\mathbb{R})$. Assume now ν is a S-invariant measure supported on sufficiently smooth φ, such that for some $0 < c_1 < c_2$, for ν-a.e. φ, $c_1 \leq \varphi \leq c_2$. Then we can find $0 < c_3 < 1$ such that $c_3\varphi^0 \leq \varphi \leq \frac{1}{c_3}\varphi^0$, thus by the maximum principle and linearity of (10.6), $\varphi(t)$ is bounded uniformly in t in $L^\infty(\mathbb{R})$. It is easy to check that then the measure μ which is the pull of ν with respect to the Cole-Hopf substitution satisfies (N2) with $y_0 = 0$.

The assumptions of Sinai in [28] on the probability measure can be understood as analogous to ours, as his Assumption 2 (the spatial invariance of expectation) is somewhat weaker form of S-invariance, his Assumption 1 when combined with the maximum principle as above implies (N2), and the Assumption 3 seems to be related to the finiteness of density of zeroes in (N1), yet to be understood.

11. Open problems

11.1. **Non-degeneracy of measures.** To further characterize and possibly remove the non-degeneracy restrictions to the results in the extended case, we propose two approaches. First, the following general ergodic-theoretical conjecture (a generalization of Proposition 6.2) would imply Theorem 1.2 without a non-degeneracy restriction:

Conjecture 11.1. Assume $(\Omega, \mathcal{F}, \nu)$ is a probability space, and that $\hat{\sigma}, \hat{\tau} : \Omega \to \Omega$ are commuting, measurable, ν-invariant maps. Assume that $\varphi, \zeta, \delta : \Omega \to \mathbb{R}$ are measurable, that $\zeta, \delta \geq 0$ and that ν-a.e.,

$$\varphi \circ \hat{\sigma} - \varphi + \zeta \circ \hat{\tau} - \zeta \geq \delta. \tag{11.1}$$

Then $\delta = 0$, ν-a.e..

Problem (1). Prove, or disprove Conjecture 11.1.

An alternative approach is to characterize non-linearities g and invariant sets for which all the S-invariant measures are non-degenerate. Let $\pi_1 : C(\mathbb{R}) \to C([0,1])$, $\pi_1(u) = u|_{[0,1]}$, let $3/4 < \alpha < \gamma < 1$ and let $\mathcal{Y} := H^{2\gamma}_{\mathrm{ul}}(\mathbb{R}) \cap \hat{T}(0, -\delta_0) H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$ for some $\delta_0 > 0$. For example, we have the following:

Lemma 11.1. *Assume μ_1, μ_2 are S-invariant measures supported on a subset of $\mathcal{Y}$ bounded in $H^{2\gamma}_{ul}(\mathbb{R})$, such that*

$$\sup \frac{||\pi_1(u) - \pi_1(v)||_{H^{2\gamma}([0,1])}}{||\pi_1(u) - \pi_1(v)||_{H^{2\alpha}([0,1])}} < \infty, \tag{11.2}$$

where supremum goes over $u \in \operatorname{supp}\mu_1, v \in \operatorname{supp}\mu_2, u \neq v$. Then we have that for any such u, v, $\hat{\zeta}(u,v) < \infty$. Furthermore, $\mu_1 \times \mu_2$ is non-degenerate.

Remark 11.1. For example, this holds if μ_1, μ_2 are supported on disjoint sets in $\mathcal{Y}$, bounded in $H^{2\gamma}_{\mathrm{ul}}(\mathbb{R})$; or alternatively if they are supported on finite sets in $\mathcal{Y}$, such as in the Example 7.1, (i).

Proof. By assumptions, the set

$$\mathcal{C} := \left\{ \frac{\pi_1(u) - \pi_1(v)}{||\pi_1(u) - \pi_1(v)||_{H^{2\alpha}([0,1])}},\ u \in \operatorname{supp} \mu_1, v \in \operatorname{supp} \mu_2, u \neq v \right\}.$$

is compact in $H^{2\alpha}([0,1])$. By the local structure of zeroes and the fact that for all $w = u - v$, the solution exists backward in time on the interval $(-\delta_0, 0]$, we can find an open, and by compactness finite cover $\mathcal{U}_j$ of $\mathcal{C}$, $j = 1, ..., m$, such that $z(w)$ is uniformly bounded for $w \in \mathcal{U}_j$. This and S-invariance of μ implies a finite uniform bound on $z_{[n,n+1]}(u - v)$, for $u \in \operatorname{supp} \mu_1$, $v \in \operatorname{supp} \mu_2$, $n \in \mathbb{Z}$. □

Example 11.1. The ergodic attractor for nonlinearities from subsection (10.3) is non-degenerate. Indeed, consider a $S, T(t)$-invariant measure μ supported on a set $\tilde{\mathcal{B}}$ bounded in $H^{2\gamma}_{\mathrm{ul}}(\mathbb{R})$, thus bounded in $L^\infty(\mathbb{R})$ by a constant $c_1 > 0$, and let $u, v \in \operatorname{supp} \mu$. By Theorem 10.1, (i), $u_{xx} = \partial V(x,u)/\partial u$, $v_{xx} = \partial V(x,v)/\partial v$, thus by the Mean Value Theorem,

$$||(\pi_1(u) - \pi_1(v))_{xx}||_{L^1([0,1])} \leq \max_{x \in [0,1], |\xi| \leq c_1} \left| \frac{\partial^2 V(x,\xi)}{\partial \xi^2} \right| ||\pi_1(u) - \pi_1(v)||_{L^1([0,1])}.$$

We can now deduce (11.2) by applying the standard interpolation and embedding estimates.

Now it would suffice to answer the following:

Problem (2). Characterize nonlinearities g such that for any $z = z(0) \in H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$, there exists $t > 0$ and an invariant set $\tilde{\mathcal{B}}$, bounded in $H^{2\gamma}_{\mathrm{ul}}(\mathbb{R})$, such that any $u, v \in \tilde{\mathcal{B}}$, $u \neq v$, satisfy (11.2), and such that $z(t) \in \tilde{\mathcal{B}}$.

Problem (3). Characterize nonlinearities g such that the attractor $\mathcal{A}$ (i.e. the set of the entire solutions) in the extended case consists of u, v, $u \neq v$ satisfying (11.2).

11.2. **Further extended gradient systems.** As noted by Zelenyak [41], and extended by Matano, Fiedler, Poláčik, Rocha and others ([12], [24] and references therein), there is a number of examples of nonlinearities g with a Lyapunov function on the bounded domain (with periodic or other boundary conditions). The discussion in Subsection 10.3 thus naturally leads to the following:

Problem (4). Prove (or disprove) that for all nonlinearities g for which there exists a Lyapunov function in the bounded case (i.e. with periodic boundary conditions), the conclusions (i)-(iv) of Theorem 10.1 hold.

For example, one can show that it holds for the cases considered in in [12].

11.3. **Related problems.** We believe the application of the zero function on the space of measures could be applied to other classes of dynamical systems, and systems with a random force:

Problem (5). Extend results for the Burgers like equations to the quasi-periodic force case considered in [30].

Problem (6). *Work in progress.* Extend results for the Burgers like equations to the random force case considered in [7, 29], by using the fact that for the difference of two weak solutions $u(t)$, $v(t)$ with the same random force, the random force cancels out and the difference is smooth enough to apply the zero function method.

Problem (7). Investigate whether the results for the equations

$$u_t = \varepsilon u_{xx} + g(t, x, u, u_x),$$

g a Burgers like nonlinearity, extend to the entropy solutions in the inviscid limit $\varepsilon \to 0$, as considered in [7], by e.g. using in addition the zero function techniques for perturbations of parabolic differential equations developed by Poláčik and Tereščak [37], or another method.

Problem (8). Consider all the problems in this paper and apply zero-function techniques on the space of measures for analogous 1d, order-preserving discrete-space, continuous-time problems without and with a random force (the Frenkel-Kontorova models, [2, 34] and references therein), or order-preserving discrete-space, discrete-time models (monotone coupled map lattices and probabilistic cellular automata, [5, 38] and references therein).

This program has already been initiated in the case of the Frenkel-Kontorova models [35, 36].

12. Appendix - Fractional uniformly local spaces

We recall the key facts on uniformly local spaces used throughout the paper in the extended case. Let $\varphi^y(u)(x) = u(x+y)$ be the translation, $y \in \mathbb{R}$. The uniformly local spaces are given with:

$$\begin{aligned}
||u||_{L^2_{\mathrm{ul}}(\mathbb{R})} &= \sup_{y\in\mathbb{R}} \left(\int_{\mathbb{R}} e^{-|x+y|} u(x)^2 dx \right)^{1/2}, \\
L^2_{\mathrm{ul}}(\mathbb{R}) &= \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}),\ ||u||_{L^2_{\mathrm{ul}}(\mathbb{R})} < \infty,\ \lim_{y\to 0} ||\varphi^y u - u||_{L^2_{\mathrm{ul}}(\mathbb{R})} = 0 \right\}, \\
H^k_{\mathrm{ul}}(\mathbb{R}) &= \left\{ u \in L^2_{\mathrm{ul}}(\mathbb{R}) \mid \partial_t^j u \in L^2_{\mathrm{ul}}(\mathbb{R}) \text{ for all } j \le k \right\}.
\end{aligned}$$

It is straightforward to show that the unbounded linear operator $Aq = -u_{tt}$ on $\mathcal{X} := L^2_{\mathrm{ul}}(\mathbb{R})$ has the domain $D(A) = H^2_{\mathrm{ul}}(\mathbb{R})$, and that by using an explicit expression of the heat kernel, A generates an analytic semigroup $\exp(-tA)$ on $\mathcal{X}$ with the usual a-priori bounds, thus it is sectorial ([17], Sec. 3). We can thus set $A_1 = A + I$, and then $\sigma(A_1) \ge 1 > 0$, and define the fractional powers A_1^α, $0 < \alpha < 1$, and the space $\mathcal{X}^\alpha := D(A_1^\alpha)$ as in [17], Section 1.4. We occasionally write $H^{2\alpha}_{\mathrm{ul}}(\mathbb{R})$ instead of $\mathcal{X}^\alpha$ to distinguish it from the bounded case. We always use the graph norm on $\mathcal{X}^\alpha$

$$||u||_{\mathcal{X}^\alpha} := ||A_1^\alpha u||_{L^2_{\mathrm{ul}}(\mathbb{R})}.$$

Now local existence, regularity and continuity with respect to initial conditions of (1.1) holds on $\mathcal{X}^\alpha$, with the usual definitions of the mild solution; and the variations of constants formula holds (see [15], Section 7.2 for details).

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