THE TRANSPORT SPEED AND OPTIMAL WORK IN PULSATING FRENKEL-KONTOROVA MODELS

BRASLAV RABAR AND SINIŠA SLIJEPČEVIĆ

Abstract. We consider a generalized one-dimensional chain in a periodic potential (the Frenkel-Kontorova model), with dissipative, pulsating (or ratchet) dynamics as a model of transport when the average force on the system is zero. We find lower bounds on the transport speed under mild assumptions on the asymmetricity and steepness of the site potential. Physically relevant applications include explicit estimates of the pulse frequencies and mean spacings for which the transport is non-zero, and more specifically the pulse frequencies which maximize work. The bounds explicitly depend on the pulse period and subtle number-theoretical properties of the mean spacing. The used method is a study of the time evolution of spatially invariant measures in the space of measures equipped with the L^1-Wasserstein metric. Even though it is applied here in a deterministic setting, thus could perhaps also be of use for ratchet models in random fields.

1. Introduction.

Our main motivation is to analyze the transport in spatially periodic systems far from equilibria, in the cases when there is no a-priori driving bias in any direction. Relevant physical examples range from molecular motors and molecular pumps, photovoltaic and photorefractive effects in materials, Josephson-Johnson arrays and many other examples (see [19] for overview and references).

We consider perhaps the simplest model exhibiting collective ratchet behavior and enabling rigorous results. The generalized Frenkel-Kontorova model [2, 4, 7] is a one-dimensional chain of particles with neighboring sites coupled with a convex interaction potential W, in a periodical potential V. It is given by the formal Hamiltonian

$$H(u) = \sum_{k=-\infty}^{\infty} \left(W(u_{k+1} - u_k)) - V(u_k)\right),$$

where $u \in \mathbb{R}^{\mathbb{Z}}$ is a *configuration* of the chain. The classical Frenkel-Kontorova model is defined with $W(x) = x^2$, $V(x) = k\cos(2\pi x)$, $k > 0$ a parameter. Our standing assumption is that W, V are C^2, that W is strictly convex (i.e. that W'' is positive and bounded away from zero), and that $V(x+1) = V(x)$ for all x.

We consider its dissipative (over-damped), *pulsating* (or *ratchet*) dynamics, with the pulsating potential. The dynamics is given with

$$\frac{d}{dt}u_j(t) = W'(u_{k+1} - u_k) - W'(u_k - u_{k-1}) + K(t)V'(u_k), \tag{1}$$

where $K(t)$ is always assumed to be a measurable, bounded, periodic *pulse* with period 2τ. We note here that all the results and calculations with straightforward modifications also hold in the case of the pulsating interaction, with the equations of motion

$$\frac{d}{dt}u_j(t) = K(t)\left[W'(u_{k+1} - u_k) - W'(u_k - u_{k-1})\right] + V'(u_k). \tag{2}$$

Following initial results holding in more general cases, we omit the detailed estimates for (2) and discuss only the case (1).

Key words and phrases. Frenkel-Kontovora model, Transport processes, Ratchet dynamics, Pumping, Attractor, Invariant measures.

Our focus is study of the existence of transport. The main result is an explicit lower bound on the transport speed $v(\rho)$. Here ρ is the mean spacing of a configuration u given with

$$\rho(u) = \lim_{n-m\to\infty} \frac{u_n - u_m}{n-m}$$

(whenever the limit exists) and $v(\rho)$ is defined in Section 4.

It has already been shown analytically in [9] and numerically in [8] that transport can exist, i.e. that $v(\rho) > 0$ is possible for models (1), (2). The authors showed in [9] that for a large class of potentials V satisfying certain asymmetricity condition, the transport exists in the limit $\tau \to \infty$. The used method, however, does not provide much information on the transport speed $v(\rho)$, as it considers large τ and assumes that the model completely relaxes between pulse "switches". (The approach is based on comparison of ground states of the Frenkel-Kontorova model with a given potential V, and with $V \equiv 0$, using the tools from Aubry-Mather theory [2, 5, 11, 13].)

Our approach is dynamical, and gives lower bounds on the transport speed as long as $K \cdot V$ is sufficiently asymmetric and steep, also in the cases when the period of the pulse τ is small and we always remain relatively far away from equilibria. We are thus able to estimate minimal τ so that the transport speed is > 0, and also heuristically discuss which τ maximizes $v(\rho)$. The physical meaning of this is determination of the optimal pulse frequency for a given model and mean spacing which optimizes work.

We confirm and refine findings [8, 9] that $v(\rho)$ depends non-trivially on number-theoretical properties of the mean spacing ρ, and show how it explicitly depends on the sequence of convergents of ρ.

The paper is structured as follows. In Section 2 we state the key results on lower bounds on the transport speed v, and discuss their implications. We give lower bounds for any τ, ρ, but also evaluate simpler formulas for large τ (as compared to the second derivative of W) and generic ρ. Generic mean spacing is here understood in the number-theoretical sense: these (irrational) ρ for which a version of the central limit theorem for its sequence of continued fraction approximations hold (the Khinchin-Lévy Theorem).

We then develop the required theory in several steps. First we develop a general theory of existence of synchronized solutions of the equations (1), (2). These are the solutions which are periodic or quasi-periodic with respect to both spatial and temporal translations, and can be understood as *dynamical ground states* of the dynamics, as we explain in more detail in Section 3. We give only an abbreviated version of the related proofs, as the ideas are essentially these of Wen Xin-Qin [17] (on existence of synchronized solutions), and already presented in [22, 23] (on stability of synchronized solutions).

We then define the transport speed v, and show in Section 4 that it depends only on the mean spacing $\rho \in \mathbb{R}$. Furthermore, we show that v depends continuously on both the mean spacing ρ and on the vector field in (1), (2). Here we introduce the ergodic-theoretical approach, and consider evolution of (spatially) invariant measures rather than of individual configurations.

While the results of Sections 3, 4 hold in the general case, we focus in Sections 5, 6 and 7 to the special case of $K(t)$ being a step function, and prove lower bounds on $v(\rho)$ in the pulsating potential case (1).

In addition to the standard tools of the study of over-damped Frenkel-Kontorova dynamics mainly relying on the monotonicity of the dynamics [3, 4], we rely on two novel ideas. First, we study evolution of an ensemble of configurations, or more precisely evolution of shift-invariant probability measures on the space of configurations. This enables us to develop estimates of the average energy of the configuration and the variance of the distance of a configuration from the straight line $k\rho + a$ in the $K = 0$ "off"-mode. To control the dynamics in the $K = \kappa$, "on"-mode, we then estimate the Wasserstein distance between the distribution of the positions of the configuration sites $\mod 1$ and the Lebesgue measure on $\mathbb{S}^1$. The number-theoretical properties of the mean spacing emerge as important in this particular step of the calculation.

2. Statement of the results and optimal work

The main result on the transport speed in the pulsating potential case (1) relies on the following assumptions (standing in this Subsection and Sections 5). Let $\kappa \geq 0$, and let K be the simple step-function pulse given with

$$K(t) = \begin{cases} 0, & 2n\tau \le t < (2n+1)\tau,\ n \in \mathbb{Z}, \\ \kappa & (2n+1)\tau \le t < (2n+2)\tau,\ n \in \mathbb{Z}. \end{cases} \tag{3}$$

Furthermore, assume that there exist $\beta > 0$, $1/2 > \alpha > 0$ so that on some interval $[a, b]$, $1/2 - \alpha < b - a < 1$ we have

$$\kappa \cdot V'(x) \ge \delta^+ + \beta, \tag{4}$$

Here δ^-, δ^+ will always denote the minimum, respectively maximum of $W''(x)$ for $x \in [\rho - 1, \rho + 1]$. Also, $\beta > 0$ is a fixed constant. (The condition (4) is the required "steepness and asymmetricity" assumption.) Finally, let $\gamma_{\rho,\tau}$ be the function depending on τ and number-theoretical properties of ρ in the following way:

$$\gamma_{\rho,\tau} = \sqrt{3} \min \left\{ \left(C_\rho \frac{q_n}{\sqrt{\tau}} + \frac{1}{q_n} \right)^{1/2},\ p_n/q_n \text{ a convergent of } \rho \right\}, \tag{5}$$
$$C_\rho = \frac{2\sqrt{6}\,\delta^+}{3(\delta^-)^{3/2}}.$$

We prove in Section 7 the following:

Theorem 2.1. *Assuming (3), (4), we have*

$$v(\rho) \ge \frac{1}{\tau} \left(\alpha - \gamma_{\rho,\tau} + \frac{1}{2}\gamma_{\rho,\tau}^2 - \frac{1}{2} \left[(1/2 - \beta\tau - \alpha - \gamma_{\rho,\tau}) \vee 0 \right]^2 \right). \tag{6}$$

(The symbols $\vee$, $\wedge$ always denote the maximum, respectively minimum of two values.) First note that the last two terms in (6) are negligible as compared to the first two terms as $\tau \to \infty$, and that for irrational ρ, $\gamma_{\rho,\tau} \to 0$ as $\tau \to \infty$. Thus we see that (6) gives an effective > 0 bound for irrational ρ and sufficiently large τ.

We first give several examples, then simplify (6) in the case of a generic ρ and large τ, and finally discuss the optimal work of (1).

Examples 2.1. *(i) Assume ρ is an integer. It is well-known that then the ratchet speed is then always zero* [8, 9]*. Consider $\gamma_{\rho,\tau}$ and (6). The only convergent denominator $q_n = 1$, so $\gamma_{\rho,\tau} > \sqrt{3}$ (and $\gamma_{\rho,\tau} \to \sqrt{3}$ as $\tau \to \infty$). It is easy to check the right-hand side of (6) is always < 0, which is consistent with explicitly known $v = 0$.*

(ii) Let $\rho = p/q$, p, q relatively prime integers, $q > 0$. Now $\gamma_{\rho,\tau} \to \sqrt{3}/q^{1/2}$ as $\tau \to \infty$. Thus (6) gives an effective bound implying $v(\rho) > 0$ for τ sufficiently large and $q \ge 3/\alpha^2$. Furthermore, we can from (6) easily calculate a τ_ρ, so that for $\tau > \tau_\rho$ we have $v(\rho) > 0$.

(iii) Let ρ be the golden mean, $\rho = (1 + \sqrt{5})/2$. Then the sequence of denominators of the sequence of convergents of ρ is the Fibonacci sequence $q_n = q_{n-1} + q_{n-2}$, $q_0 = q_1 = 1$. As then $q_n \sim ((1 + \sqrt{5})/2)^n$, we can for any $\tau > 0$ (for which the right-hand side below is ≥ 1) find a q_n in the interval

$$\frac{\tau^{1/4}}{2C_\rho^{1/2}} \le q_n \le \frac{\tau^{1/4}}{C_\rho^{1/2}}.$$

Inserting that in (5), (6) we get for all $\tau > (1/2 - \alpha)/\beta \vee C_\rho^2$,

$$v(\rho) \ge \frac{1}{\tau} \left(\alpha - 3C_\rho^{1/2} \tau^{-1/8} \right).$$

(iv) Assume ρ is irrational. Then analogously as above we see that $\gamma_{\rho,\tau} \to 0$ as $\tau \to \infty$. Thus asymptotically

$$v(\rho)\tau \ge \alpha + o(1)$$

as $\tau \to \infty$, which is analogous to the result from [9] *in the $\tau \to \infty$ limit for irrational ρ. The behavior of the error term $o(1)$ depends on the continuous fraction expansion of ρ.*

We now discuss the case of generic ρ. We use the Khinchin-Lévy Theorem [12], stating that for the full Lebesgue measure of $\rho \in \mathbb{R}$, the sequence of denominators q_n of convergents of ρ satisfies

$$\lim_{n\to\infty} q_n^{1/n} = \gamma_L,$$

where $\gamma_L = \exp(\pi^2/(12\ln 2))$ is the Lévy's constant. Thus for Lebesgue-a.e. $\rho \in \mathbb{R}$ and any $\varepsilon > 0$, for sufficiently large $\tau^{1/2}/C_\rho$ we can find q_n in the interval

$$\frac{\tau^{1/4}}{(\gamma_L+\varepsilon)C_\rho^{1/2}} \le q_n \le \frac{\tau^{1/4}}{C_\rho^{1/2}}.$$

Inserting that in (5), (6) and applying the Khinchin-Lévy Theorem, we deduce the following:

Corollary 1. *For Lebesgue-a.e. $\rho \in \mathbb{R}$ and any $\varepsilon > 0$,*

$$v(\rho) \ge \frac{1}{\tau}\left[\alpha - (3C_\rho(\gamma_L+\varepsilon+1))^{1/2}\,\tau^{-1/8} + o_{\rho,\varepsilon}(\tau^{-1/8})\right]. \tag{7}$$

We can explicitly estimate the error term in (7) for arbitrarily large sets $R \subset \mathbb{R}$. (Large in terms of the Lebesgue measure, i.e. these for which $\lambda(R \cap [0,1])$ is positive but arbitrarily small. Note that the error term depends only on $\rho \mod 1$. Then the error term will be a function of $\lambda(R\cap[0,1])$.) This can be done by using estimates on the variance of

$$Z = \frac{1}{n}\log q_n - \log\gamma_L$$

by applying of a variant of the Central Limit Theorem, as e.g. in [15].

Thus we heuristically predict by differentiating (7) that for Lebesgue-a.e. $\rho \in \mathbb{R}$, and for C_ρ small enough, we get the optimal work (i.e. the maximal speed $v(\rho)$) for

$$\tau \sim C_\rho^4 \alpha^{-8}. \tag{8}$$

3. Preliminaries and synchronized orbits

In this section we focus on existence of the synchronized orbits, which play the key role in the asymptotic dynamics of (1), (2). The main result is that synchronized solutions with any mean spacing ρ exist. The existence is proved by an application of the Schauder fixed point theorem, following the ideas developed by Wen-Xin Qin [17].

Prior to precisely stating the main result, we introduce the notation. We will always consider the dynamics on the phase space of configurations $\tilde{\mathcal{X}}$ of the bounded slope, that is the configurations $u \in \mathbb{R}^{\mathbb{Z}}$ such that

$$\sup_{k\in\mathbb{Z}} |u_k - u_{k+1}| < \infty.$$

We will denote by $\mathcal{X}$ the quotient space of $\tilde{\mathcal{X}}$, where we identify u and $u+n$ for all integers n. We always assume the product topology on $\tilde{\mathcal{X}}$ (i.e. the topology of pointwise convergence), and the induced product topology on $\mathcal{X}$. The semiflow generated by (1) or (2), established in Lemma 3.1 below, generates a continuous semiflow φ on $\tilde{\mathcal{X}}$ and $\mathcal{X}$, $\varphi^t(u(t')) = u(t'+t)$ for $t \ge 0$.

Let $T = \varphi^{2\tau}$ be the Poincaré map with respect to φ, where 2τ is the period of the pulse K. Let S be the shift-map, $S(u)_k = u_{k-1}$. Clearly S, T commute and are well defined on $\mathcal{X}$.

We say that $u, v \in \tilde{\mathcal{X}}$ do not intersect, if either $u = v$, $u > v$ or $u < v$, where the partial order on $\tilde{\mathcal{X}}$ is defined in the natural way (we write $u \ge v$ if $u_n \ge v_n$ for all $n \in \mathbb{Z}$, and $u > v$ if $u_n > v_n$ for all $n \in \mathbb{Z}$.) We say that $u, v \in \mathcal{X}$ do not intersect, if their representations in $\tilde{\mathcal{X}}$ never intersect.

We recall the definition of rotationally ordered solutions, standard in the study of Frenkel-Kontorova models [4, 3, 8, 9, 6, 7, 10, 16], and of synchronized solutions [17, 22, 23].

Definition 3.1. *We say that $u \in \mathcal{X}$ is rotationally ordered, if for any $n \in \mathbb{Z}$, u and $S^n u$ do not intersect.*

A solution $u(t) \in \mathcal{X}$, $t \in \mathbb{R}$ is synchronized, if for any $t \in \mathbb{R}$ and any integers $n, m \in \mathbb{Z}$, $u(t)$ and $S^n T^m u(t)$ do not intersect.

It is easy to show [22, 23] that each synchronized solution has a well-defined mean spacing ρ independent of t, and that it is of bounded width 1, namely for each $t \in \mathbb{R}$, and integers $n < m$,

$$|u_n(t) - u_m(t) - (n-m)\rho| \le 1 \tag{9}$$

This holds for all rotationally ordered configurations (see e.g. [5]). The fact that ρ is independent of t follows from Lemma 3.1, (ii) below.

The main result is now:

Theorem 3.2. *Given any mean spacing $\rho \in \mathbb{R}$, there exist synchronized solutions of (1) and of (2) with the mean spacing ρ.*

This will be a corollary of an analogous result for a more general system of equations

$$\frac{d}{dt}u_j(t) = -V_2(u_{j-1}(t), u_j(t), t) - V_1(u_j(t), u_{j+1}(t), t), \qquad j \in \mathbb{Z}, \tag{10}$$

such that $V\colon \mathbb{R}^3 \to \mathbb{R}$, $(t,u,v) \mapsto V(t,u,v)$ is C^2 in variables u, v, and bounded and measurable in t for fixed u, v. We assume the following in the general setting (standing assumptions in this and the next Section).

(i) Periodicity: $V(u,v,t) = V(u+1, v+1, t)$, $V(u,v,t) = V(u,v,t+2\tau)$ for some $\tau > 0$,

(ii) The twist condition: for some fixed $\delta > 0$,

$$\frac{V(u,v,t)}{\partial u \partial v} \le -\delta < 0.$$

(iii) Boundedness: There exists $C > 0$ such that $\|D^2_{u,v}V(u,v,t)\| \le C$.

Clearly (1) and (2) are now special cases of (10), thus it suffices to prove:

Theorem 3.3. *Given any mean spacing $\rho \in \mathbb{R}$, there exists a synchronized solution of (10) satisfying (i),(ii),(iii), with the mean spacing ρ.*

Following Qin [17], we prove it in several steps. After establishing the existence of the semiflow, the existence of synchronized solutions for rational mean spacing is proved by an application of the Schauder Fixed Point theorem. Irrational mean spacings are covered by a limiting procedure.

It is easy to check [21, 17] that $\tilde{\mathcal{X}}$, $\mathcal{X}$ are metrizable, and the sets $\mathcal{K}_n \subset \mathcal{X}$, $n \in \mathbb{N}$ of all $u \in \mathcal{X}$ satisfying

$$\sup_{k \in \mathbb{Z}} |u_k - u_{k+1}| \le n$$

are compact.

Lemma 3.1. *(i) The system (10) generates a continuous semiflow φ on $\mathcal{X}$,*

(ii) The solution $t \to u(t) - u(0)$ is continuous in $l^\infty(\mathbb{Z})$,

(iii) The sets $\mathcal{K}_n$ are positively invariant for φ,

(iv) The semiflow is strongly order-preserving, i.e. if $u(0) \ge v(0)$ but not equal, then for all $t > 0$, $u(t) > v(t)$.

Here positive invariance means that if $u \in \mathcal{K}_n$, then for all $t \ge 0$, if $u \in \mathcal{K}_n$, then $\varphi^t(u) \in \mathcal{K}_n$.

Proof. Local existence, uniqueness and continuity of solutions follows from the standard results on existence of solutions of ODE in Banach spaces [18, 21]. The order-preserving property follows as in [4], as the off-diagonal elements of the linearization of the right-hand side of (10) are strictly positive and bounded from below. The claim (iii) follows from the order-preserving property [21]; global existence and existence of semiflow is then straightforward. □

Let us first show the existence of synchronized solutions for rational mean spacings p/q, where p, q are relatively prime integers, $q > 0$. We fix p, q, and following Qin, we denote by Ω the set of functions $h : \mathbb{R} \to \tilde{\mathcal{X}}$ satisfying:

(i) h is continuous;

(ii) h is increasing, that is if $s > t$, then $h(s) > h(t)$;

(iii) The image of h consists of periodic configurations of the type p, q. That means that if $u = h(t)$, then $S^q u = u + p$;

(iv) The image of h consists of configurations of the width 1. More precisely, if $u = h(t)$, then for all integers $m < n$, $|u_m - u_n - (m-n)p/q| \leq 1$;

(v) h is centered in the sense that for all $t \in \mathbb{R}$,

$$t = \frac{1}{q}\sum_{k=0}^{q-1} u_k(t).$$

(vi) The image of h is invariant for the transformations $u \to S^m u + n$ for any integers m, n.

Lemma 3.2. *The set Ω with the induced C^0 topology is non-empty, convex and compact.*

Proof. Convexity follows from [17], Lemma 3.2 (properties (iv) and (vi) are the only non-trivial ones to be shown), and compactness is analogous to [17], Lemma 3.3. □

By the Schauder Fixed Point theorem, any continuous map on Ω thus has a fixed point. We can naturally extend and adapt the Poincaré map $T = \varphi^{2\tau}$ to a continuous map $h : \Omega \to \Omega$, by appropriately reparametrizing the time so that the property (v) is preserved (the other properties hold by the order-preserving property of φ). Thus h has a fixed point $h_{p,q}$ (see [17], Section 3.1 for details in a completely analogous argument).

Note that the image of $h_{p,q}$ is not necessarily an orbit of φ. By the order-preserving property and the definition of $h_{p,q}$, we see that the orbit of $h_{p,q}(t)$ for any $t \in \mathbb{R}$ is synchronized with mean spacing p/q.

Synchronized orbits with irrational mean spacings are obtained analogously as in [17], Section 3.2, as a limit of a sequence of synchronized orbits with rational mean spacings.

4. The transport speed

In this section we define the transport speed v, and show that it depends only on the mean spacing ρ. Furthermore, we show that v is continuous, and give a strong characterization of the existence of transport: $v(\rho) = 0$ if and only if the motion is in a certain sense asymptotically periodic. To achieve that, we rely on the notion of the dynamical ground state, and more generally, consider invariant measures with respect to (10).

Note that the Poincaré map T and the shift S commute. As in [22, 23], we will find S, T-invariant, Borel probability measures on $\mathcal{X}$ important and useful. More specifically, we are interested in synchronized measures, that is the S, T-invariant measures supported on synchronized configurations. We first establish their existence:

Lemma 4.1. *For each $\rho \in \mathbb{R}$, there exists a synchronized measure supported on synchronized orbits with the mean spacing ρ.*

Proof. The set of synchronized orbits with the mean spacing ρ is a compact invariant subset of $\mathcal{X}$, thus by a well-known extension of the Bogolybov-Krylov argument, as S and T commute, there exists a S, T-invariant measure supported on that set. □

Denote by $\mathcal{S}_\rho$ the set of all configurations in the support of a synchronized measure, with the mean spacing ρ. As discussed in detail in [22, 23], the configurations in $\mathcal{S}_\rho$ can be considered as dynamical ground states. In the pinned phase (see [18, 23] for details), configurations in $\mathcal{S}_\rho$ are in a certain sense globally attracting, and are always locally attracting.

Now if μ is a synchronized measure with the mean spacing ρ, we define the transport speed $v(\rho)$ as

$$v(\rho) = \frac{1}{2\tau}\int_{\mathcal{X}} ((Tu)_0) - u_0) d\mu. \tag{11}$$

Clearly by the S-invariance of μ and by the fact that T, S commute, the site index 0 in the definition of $v(\rho)$ can be replaced by any $k \in \mathbb{Z}$.

We now establish several elementary properties of the transport speed $v(\rho)$. We show that it is well-defined (i.e. independent of the choice of a synchronized measure), and that it is the asymptotic speed for any configuration with the same mean spacing (and finite bounded width - see below for definitions). Furthermore, we show that it is continuous in ρ. It also continuously changes with a parameter. Finally, we show that the transport speed is zero if and only if T restricted to $\mathcal{S}_\rho$ is constant.

Lemma 4.2. *Choose $\rho \in \mathbb{R}$. For any $u \in \mathcal{S}_\rho$, the limit*

$$\lim_{n\to\infty} \frac{1}{n}\left(T^n(u)_k - u_k\right) \tag{12}$$

exists, and is independent of the choice of $k \in \mathbb{Z}$ and of $u \in \mathcal{S}_\rho$. Furthermore, the right-hand side of (11) is independent of the choice of a measure μ supported on $\mathcal{S}_\rho$.

Proof. We first prove the claim for $k = 0$. Choose any S, T-invariant measure μ supported on $\mathcal{S}_\rho$, and denote by $v(\mu)$ the transport speed defined by the right-hand side of (11). By the Birkhoff Ergodic Theorem, there exists u in the support of μ, thus in $\mathcal{S}_\rho$, so that the limit (12) exists, denote it $v(u)$. Now if u, w are any two configurations in $\mathcal{S}_\rho$, because of (9) we can find their representations in $\tilde{\mathcal{X}}$, denoted for simplicity again by u, w, so that

$$u \le w \le u + 2.$$

Now by the order-preserving property of the dynamics (Lemma 3.1, (iv)), we deduce that $v(w)$ is well defined and that $v(w) = v(u)$. By the Birkhoff Ergodic Theorem, we easily see now that $v(\mu) = v(u) = v(w)$. As $w \in \mathcal{S}_\rho$ was arbitrary, the claim is proved for $k = 0$.

This holds for all indices $k \in \mathbb{Z}$ by the S-invariance of μ. □

As a direct consequence of Lemma (4.2), we see that $v(\rho)$ is well-defined, i.e. independent of the choice of a S, T-invariant measure.

Now, let us choose any $u \in \mathcal{X}$ of the mean spacing ρ and of *bounded width.* That is, assume that there exists an integer constant $k > 0$ so that for any integers $n > m$,

$$|u_n - u_m - \rho(n - m)| \le k.$$

Lemma 4.3. *If u has the mean spacing ρ and is of bounded width, the limit (12) exists and is equal to $v(\rho)$.*

Proof. By definition, for any $v \in \mathcal{X}$ we can find representatives of u, v (denoted again by u, v) so that $v \le u \le v + k + 1$. The proof now follows analogously as for Lemma 4.2. □

Now we show that the transport speed is continuous.

Proposition 4.1. *(i) The function $v(\rho)$ is continuous in ρ,*

(ii) If the right-hand side of (1) or (2) depends continuously on a parameter λ, then the function $\lambda \mapsto v(\lambda, \rho)$ is continuous for all $\rho \in \mathbb{R}$.

We will need the following Lemma:

Lemma 4.4. *The set of all synchronized measures is closed in the weak*-topology. The set of synchronized measures supported on configurations with the mean spacing in a closed, bounded set of real numbers is weak*-compact.*

Proof. See [23]. □

Proof of Proposition 4.1. We show (i) by first choosing a convergent sequence $\rho_k \to \rho$ as $k \to \infty$. If the claim does not hold, we can find a subsequence, denoted again by ρ_k, so that $v(\rho_k)$ is bounded away from $v(\rho)$ for all $k \in \mathbb{N}$. Choose synchronized measures μ_k with the mean spacing ρ_k. By Lemma 4.4, μ_k has a convergent subsequence μ_{k_n} in the weak*-topology, converging to a synchronized measure with the mean spacing ρ. As the function $u \mapsto (Tu)_0 - u_0$ is continuous in $\mathcal{X}$, by the definition of the weak*-convergence, $v(\rho_{k_n}) \to v(\rho)$ as $n \to \infty$, which is a contradiction.

The claim (ii) is shown analogously. □

We close the section with a nice characterisation of the existence of transport.

Proposition 4.2. *The transport speed $v(\rho) = 0$ if and only if T restricted to $\mathcal{S}_\rho$ is constant.*

Proof. The non-trivial implication is that $v(\rho) = 0$ implies $Tu = u$ for all $u \in \mathcal{S}_\rho$. We prove it by using ergodic-theoretical tools for two commuting continuous maps on metric spaces (applied to maps T, S on $\mathcal{X}$), analogous to the one-map case.

Assume $v(\rho) = 0$, and choose without loss of generality a S, T-ergodic measure μ supported on $\mathcal{S}_\rho$. By a well-known result, we can find $u \in \mathcal{S}_\rho$ so that its S, T-orbit (i.e. the set of all $S^n T^m u$, n, m integers) is dense in the support of μ (see [11], Proposition 4.1.18, (2)). As u is synchronized, $Tu \geq u$ or $Tu \leq u$. By the order-preserving property and continuity, we have that for all w in the support of $\mathcal{S}_\rho$, $Tw - w \leq 0$ or $Tw - w \geq 0$ and the sign is always the same. Without loss of generality, assume $Tw \geq w$ on the support of μ, thus $(Tw)_0 - w_0 \geq 0$. We however assumed that

$$\int ((Tw)_0 - w_0)\, d\mu = 0,$$

thus $(Tw)_0 = w_0$ μ-a.e.. Now by continuity and S-invariance of μ, we have $Tw = w$ on the support of μ. As the union of supports of all S, T-ergodic measures is dense in $\mathcal{S}_\rho$, the claim must hold on $\mathcal{S}_\rho$. □

5. Dynamics in the off-phase

In the rest of the paper we prove Theorem 2.1. We fix a mean spacing $\rho \in \mathbb{R}$. Recall that $v(\rho)$ is independent of the choice of a configuration in $\mathcal{S}_\rho$, and more suitably for our purposes, independent of the choice of an S, T-invariant measure supported on $\mathcal{S}_\rho$. Thus we fix such a S, T-invariant measure μ. We consider the evolution $\mu(t)$ of $\mu(0) = \mu$ for $t \geq 0$, where $\mu(t)$ is the pull of the measure $\mu(0)$ with respect to the semiflow ϕ^t generated by (1).

In this Section we focus on evolution of such a measure μ in the no-pulse, "off"-phase, i.e. for $t \in [0, \tau]$. The main result is an explicit bound on the *average width* function defined with

$$V(\mu) = \int_{\mathcal{X}} (u_1 - u_0 - \rho)^2 d\mu, \tag{13}$$

for $\mu = \mu(\tau)$, i.e. at the end of the off-phase. The bound itself is derived in Lemma 5.4, following several interim results which rely on the fact that the dynamics is dissipative, and explicitly estimate the decay of the "energy"

$$\int_{\mathcal{X}} W(u_1 - u_0) d\mu.$$

This is combined with a discrete-space analogue of a Poincaré inequality (Lemma 5.3), inequalities following from convexity of W (Lemma 5.3), as well as the fact that the transport speed in the off-phase is 0 (Lemma 5.1).

We first comment on the properties of evolution of a S, T invariant measure μ, and on the meaning of $V(\mu)$. It is easy to see that for each t, $\mu(t)$ is S-invariant. Note that $\mu(t)$ is typically not constant, as μ is invariant with respect to the Poincaré map (thus $\mu(T) = \mu(0)$), but not with respect to the semiflow ϕ^t. As $\mu = \mu(t)$ is for all t supported on configurations with the mean spacing ρ and bounded width 1, the expectation $\mathrm{E}_\mu[u_1 - u_0] = \rho$ and the variance $\mathrm{Var}_\mu[u_1 - u_0] = V(\mu)$, which justifies the notation.

Whenever our integrals are over the space $\mathcal{X}$ (which will typically be the case), we omit $\mathcal{X}$ in the following. We first discuss some facts on defining integrals with respect to $\mu(t)$, such as (13). Recall that $\mathcal{X}$ is the quotient space of configurations. Thus one must carefully check that the integrands are well-defined functions on $\mathcal{X}$, i.e. independent of the choice of the integer n in the representation $u + n$, $u \in \tilde{\mathcal{X}}$. It is easy to check that the following functions are well-defined on $\mathcal{X}$ for all $k \in \mathbb{Z}$

$$\begin{gathered} u_k \mod 1, \\ u_k - u_{k-1}, \\ u_{k+1} - 2u_k + u_{k-1}, \\ V'(u_k), \\ du_k(t)/dt. \end{gathered}$$

In addition,

$$u_k(t_2) - u_k(t_1)$$

is also well defined for any $k \in \mathbb{Z}$, $t_1, t_2 \geq 0$, where we choose consistent representations of $u(t)$ (i.e. make sure that the representation of $u(t_2)$ is the time-evolution of the representation of $u(t_1)$).

We will also need the *width* function, defined on $\mathcal{S} = \cup_{\rho\in\mathbb{R}} S_\rho$. Given $u \in \mathcal{X}$, let

$$w_j(u) = u_j - \rho j - a_0,$$

where u is now a representative in $\tilde{\mathcal{X}}$, and a_0 is the supremum of all $a \in \mathbb{R}$ such that for all $k \in \mathbb{Z}$, $u_k \geq k\rho + a$. It is easy to check that $w_j(u)$ is well defined on $\mathcal{X}$ (i.e. independent of a representative in $\tilde{\mathcal{X}}$), that for $u \in \mathcal{S}$, by (9) we have $|w_j(u)| < 1$, and finally

$$w_j(u) - w_{j-1}(u) = w_j(u) - w_j(Su) = u_j - u_{j-1} - \rho. \tag{14}$$

Lemma 5.1. *For any $k \in \mathbb{Z}$, and for $\mu = \mu(0)$ as above,*

$$\int (u_k(\tau) - u_k(0))\, d\mu = 0.$$

(This lemma is equivalent to the fact used in [9], that the average position of the site mod 1 for a chosen rotationally ordered configuration does not change in the off-phase.)

Proof. We calculate using the Fubini theorem:

$$\begin{aligned} \int (u_k(\tau) - u_k(0))\, d\mu &= \int \int_0^\tau (du_k(t)/dt) dt\, d\mu = \int_0^\tau \left(\int (du_k(t)/dt) d\mu \right) dt \\ &= \int_0^\tau \left(\int W'(u_{k+1}(t) - u_k(t)) d\mu - \int W'(u_k(t) - u_{k-1}(t)) d\mu \right) dt \\ &= \int_0^\tau \left(\int W'(u_{k+1} - u_k) d\mu(t) - \int W'(u_k - u_{k-1}) d\mu(t) \right) dt, \end{aligned}$$

which is by the S-invariance of $\mu(t)$ equal to 0. □

We now need to establish a discrete-space version of the Poincaré inequality, bounding a L^2-norm of $u_1 - u_0 - \rho$ by a L^2-norm of its discrete-space derivative $u_1 - 2u_0 + u_{-1}$.

Lemma 5.2. *Assume η is a S-invariant measure supported on the set of rotationally ordered configurations in $\mathcal{X}$ with the mean spacing ρ. Then*

$$\int (u_1 - u_0 - \rho)^2 d\eta \leq \left(\int (u_1 - 2u_0 + u_{-1})^2 d\eta \right)^{1/2}. \tag{15}$$

Proof. First note the following identity holding for any sequences $v_j, w_j \in \mathbb{R}$, $j \in \mathbb{Z}$:

$$v_j w_j - v_{j-1} w_{j-1} = v_j (w_j - w_{j-1}) + (v_j - v_{j-1}) w_{j-1}.$$

Now for some $u \in \mathcal{S}$, insert $v_j = (u_j - u_{j-1} - \rho)$, $w_j = w_j(u)$. Applying (14) we easily get

$$(u_j - u_{j-1} - \rho) w_j(u) - (u_{j-1} - u_{j-2} - \rho) w_{j-1}(u) = (u_j - u_{j-1} - \rho)^2 + (u_j - 2u_{j-1} + u_{j-2}) w_{j-1}(u).$$

As all the terms are well defined functions on $\mathcal{X}$, we can integrate with respect to η, for $j = 1$. Note that because of the S-invariance of η, the term on the left-hand side vanishes, so we have

$$\int (u_1 - u_0 - \rho)^2 d\eta = - \int (u_1 - 2u_0 + u_{-1}) w_0(u) d\eta.$$

It now suffices to apply the Cauchy-Schwartz inequality to the right-hand side, and the fact that $|w_0(u)| < 1$. □

Lemma 5.3. *For a given $\rho \in \mathbb{R}$, let δ^-, δ^+ be the minimum, respectively the maximum of $W''(x)$ for $x \in [\rho-1, \rho+1]$. Then for any S-invairant measure η supported on $\mathcal{S}_\rho$,*

$$(\delta^-)^2 \int (u_1 - 2u_0 + u_{-1})^2 d\eta \leq \int \left(W'(u_1-u_0) - W'(u_0-u_{-1})\right)^2 d\eta, \tag{16}$$

$$\int W(u_1-u_0)d\eta - W(\rho) \leq \delta^+ \int (u_1-u_0-\rho)^2 d\eta, \tag{17}$$

$$\delta^- \int (u_1-u_0-\rho)^2 d\eta \leq \int W(u_1-u_0)d\eta - W(\rho). \tag{18}$$

Proof. The relation (16) follows from the Mean Value Theorem applied to the function W'.

It suffices to prove (18), as the proof of (17) is analogous. Let $x_1, ..., x_n$ be any numbers in $[\rho-1, \rho+1]$, and let $\bar{x} = (x_1 + ... + x_n)/n$, for some integer $n > 0$. By the second order Taylor theorem, we have

$$\begin{aligned} W(x_k) - W(\bar{x}) &\geq W(\bar{x})(x_k - \bar{x}) + \delta^-(x_k-\bar{x})^2, \\ \frac{1}{n}\sum_{k=1}^n W(x_k) - W(\bar{x}) &\geq \frac{\delta^-}{n}\sum_{k=1}^n (x_k-\bar{x})^2. \end{aligned} \tag{19}$$

Now we insert $x_k = u_k - u_{k-1}$, note that then $\bar{x} = (u_n - u_0)/n$, integrate (19) with respect to $d\eta$, and obtain

$$\frac{1}{n}\sum_{k=1}^n \int W(u_k - u_{k-1})d\eta - \int W\left(\frac{u_n-u_0}{n}\right)d\eta \geq \frac{\delta^-}{n}\sum_{k=1}^n \int \left(u_k - u_{k-1} - \frac{u_n-u_0}{n}\right)^2 d\eta.$$

By S-invariance of η, we get

$$\int W(u_1-u_0)d\eta - \int W\left(\frac{u_n-u_0}{n}\right)d\eta \geq \frac{\delta^-}{n}\sum_{k=1}^n \int \left(u_1 - u_0 - \frac{u_{n-k+1}-u_{1-k}}{n}\right)^2 d\eta. \tag{20}$$

Now it suffices to consider $n \to \infty$: as η is supported on $\mathcal{S}_\rho$, the relation (14) and the Lebesgue Dominated Convergence Theorem imply (18). ☐

In the following Lemma, we use the fact noted in [20] that the following function defined on S-invariant measures

$$W(\mu) = \int W(u_1-u_0)d\mu$$

is the Lyapunov function in the off-phase for the induced semiflow on the space of S-invariant measures.

Lemma 5.4. *We have*

$$V(\mu(\tau)) \leq \frac{(\delta^+)^2}{2(\delta^-)^3\tau + \delta^+\delta^-}. \tag{21}$$

Note that in the standard case $W(x) = \kappa x^2$, (21) becomes $V(\mu(\tau)) \leq 1/(2\kappa\tau+1)$.

Proof. We first express $dW(\mu(t))/dt$, then use Lemmas 5.2, 5.3 to obtain a differential inequality, and then we solve it.

Denoting by $\dot{u}_k(t) = du_k(t)/dt$, we see that

$$\begin{aligned} \frac{dW(u_1(t)-u_0(t))}{dt} &= 2W'(u_1(t)-u_0(t))(\dot{u}_1(t)-\dot{u}_0(t)) \\ &= 2W'(u_1(t)-u_0(t))\dot{u}_1(t) - 2W'(u_0(t)-u_{-1}(t))\dot{u}_0(t) \\ &\quad -2\left(W'(u_1(t)-u_0(t)) - W'(u_0(t)-u_{-1}(t))\right)\dot{u}_0(t). \end{aligned}$$

Now integrating it with respect to a S-invariant measure μ supported on $\mathcal{S}_\rho$, the first two summands on the right-hand side cancel out because of S-invariance of μ. As $\dot{u}_0 = W'(u_1(t)-u_0(t)) - W'(u_0(t)-u_{-1}(t)$, we see that

$$\frac{dW(\mu(t))}{dt} = -2\int (W'(u_1(t)-u_0(t)) - W'(u_0(t)-u_{-1}(t))^2 d\mu. \tag{22}$$

We now apply 16, 15 and finally 17 to the right-hand side, and get the differential inequality

$$\frac{dW(\mu(t))}{dt} \leq -2\left(\frac{\delta^-}{\delta^+}\right)^2 [W(\mu(t)) - W(\rho)]^2 . \tag{23}$$

Note that by Jensen's inequality and as $\mu(t)$ is supported on $\mathcal{S}_\rho$, we have $W(\mu(t)) \geq W(\rho)$. Unless $\mu(t)$ is for all t supported on quasiperiodic configurations of the type $u_k = \rho \cdot k + a$ (in which case the dynamics is trivial and $V(\mu(\tau)) = 0$), we also deduce that $W(\mu(t)) > W(\rho)$. Thus we can solve the differential inequality (23), and obtain

$$W(\mu(\tau)) - W(\rho) \leq \frac{1}{2\left(\frac{\delta^-}{\delta^+}\right)^2 \tau + \frac{1}{W(\mu(0))-W(\rho)}}.$$

Combining (9) and (17) we get $W(\mu(0)) - W(\rho) \leq \delta^+$, thus

$$W(\mu(\tau)) - W(\rho) \leq \frac{(\delta^+)^2}{2(\delta^-)^2\tau + \delta^+}.$$

Now it suffices to apply (18) and the definition of $V(\mu)$ to complete the proof. □

6. Dynamics in the on-phase

We focus now on the dynamics of the synchronized measure μ on the on-phase $t \in [\tau, 2\tau]$. The main tool is the projection of the u_0 to the unit circle $\mathbb{S}^1$ defined with $\pi(u_0) = u_0 \mod 1$, and more generally the associated pull of a measure μ on $\mathcal{X}$ to a measure on $\mathbb{S}^1$. We always use the notation $\mu^* = \pi^*\mu$ for the pulled measure on $\mathbb{S}^1$ (it is well-defined by the definition of $\mathcal{X}$ as a quotient space). Note also that, as μ is S-invariant, the definition of the measure μ^* is independent of the choice of the coordinate at which we project to $\mathbb{S}^1$.

The second tool we need is the L^1-Wasserstein distance on the space of Borel probability measures on $\mathbb{S}^1$. Recall the definition [1]: if μ^*, ν^* are two Borel-probability measures on $\mathbb{S}^1$, then the L^1-Wasserstein distance is defined with

$$d_1(\mu^*, \nu^*) = \inf_{\gamma \in \Gamma(\mu^*, \nu^*)} \int_{\mathbb{S}^1 \times \mathbb{S}^1} d(x, y) d\gamma(x, y),$$

where γ is a Borel probability measure on $\mathbb{S}^1 \times \mathbb{S}^1$ whose marginals are μ^*, ν^* respectively. Furthermore, d_1 is a metric on the space of Borel-probability measures on $\mathbb{S}^1$.

We will develop estimates for irrational ρ, and will then extend them by continuity also to rational ρ.

The estimate of the dynamics in the on-phase is as follows: in the Lemma 6.1 we develop an upper bound on $d_1(\mu(\tau), \lambda)$, where λ will always denote the Lebesgue measure on $\mathbb{S}^1$. Note that for irrational ρ, λ is an invariant and ergodic measure for the irrational translation $x \mapsto x + \rho \mod 1$ on $\mathbb{S}^1$.

Then we estimate $v(\tau)$ as a function of ρ and $d_1(\mu(\tau), \lambda)$, first for a generic distribution $\mu^*(\tau)$, then in the case $\mu^* = \lambda$, and finally using the bound on $d_1(\mu(\tau), \lambda)$.

Lemma 6.1. *Assume $|\rho - p/q| \leq 1/q^2$, p/q rational, $q > 0$, and assume $\int (u_1 - u_0 - \rho)^2 d\mu \leq \varepsilon^2$, where μ is S-invariant supported on $\mathcal{S}_\rho$. Then*

$$d_1(\mu^*, \lambda) \leq \frac{q}{\sqrt{3}}\varepsilon + \frac{3}{4q}. \tag{24}$$

Proof. The strategy of the proof is as follows: if u is in the support of μ, we consider the measure $\mu_q^*(u)$ on $\mathbb{S}^1$ equi-supported on $u_0, u_1, ..., u_{q-1} \mod q$. We then bound the distance $d_1(\mu_q^*(u), \lambda)$ in terms of the expression

$$V_q(u) = \left(\frac{1}{q}\sum_{k=0}^{q-1}(u_{k+1} - u_k - \rho)^2\right)^{1/2}.$$

We obtain the final bound by estimating the integral of $V_q(u)$ with respect to $d\mu$.

Let $\nu_{p/q}$, ν_ρ be the probability measures on $\mathbb{S}^1$, equi-supported on

$$u_0, u_0 + p/q, ..., u_0 + p/q \cdot (q-1) \qquad \text{mod } 1,$$
$$u_0, u_0 + \rho, ..., u_0 + \rho \cdot (q-1) \qquad \text{mod } 1$$

respectively. By applying the Cauchy-Schwartz inequality in the second line, it is easy to see that

$$\begin{aligned} d_1(\mu_q^*(u), \nu_\rho) &\leq \frac{1}{q}\sum_{k=0}^{q-1}(q-1-k)|u_{k+1} - u_k - \rho| \\ &\leq \frac{1}{q}\left(\sum_{k=0}^{q-1} k^2\right)^{1/2} \cdot q^{1/2} V_q(u) \leq \frac{q}{\sqrt{3}} V_q(u), \\ d_1(\nu_\rho, \nu_{p/q}) &\leq \frac{1}{q}\sum_{k=0}^{q-1}(q-1-k)|p/q - \rho| \leq \frac{1}{2q}. \end{aligned}$$

Finally, from the definition of $\nu_{p/q}$ we easily get

$$d_1(\nu_{p/q}, \lambda) \leq \frac{1}{4q}.$$

Summing all of the above and applying the triangle inequality, we obtain

$$d_1(\mu_q^*(u), \lambda) \leq \frac{q}{\sqrt{3}} V_q(u) + \frac{3}{4q}. \tag{25}$$

By definition of the L^1-Wasserstein distance, it suffices to estimate the following:

$$\begin{aligned} \int V_q(u) d\mu &= \int \left(\frac{1}{q}\sum_{k=0}^{q-1}(u_{k+1} - u_k - \rho)^2\right)^{1/2} d\mu \\ &\leq \left(\int \left(\frac{1}{q}\sum_{k=0}^{q-1}(u_{k+1} - u_k - \rho)^2\right) d\mu\right)^{1/2} \\ &= \left(\int (u_1 - u_0 - \rho)^2 d\mu\right)^{1/2} \leq \varepsilon, \end{aligned}$$

where we applied the Jensen's inequality in the second line, and the S-invariance of μ in the third line. Combining this, (25) and the fact that L^1-Wasserstein distance is a metric, we complete the proof. □

Assume from now on in this section without loss of generality that (4) holds on the interval $[1/2 - \alpha, 1]$. We will need the function $\varphi : \mathbb{S}^1 \to \mathbb{R}$ defined with

$$\varphi(x) = \begin{cases} \beta\tau \wedge (1-x) & x > 1/2 - \alpha, \\ -x & x \leq 1/2 - \alpha, \end{cases}$$

where $\mathbb{S}^1$ is parametrized with $x \in [0, 1)$.

Lemma 6.2. *Assuming all as above, we have*

$$v(\rho) \geq \int_{\mathbb{S}^1} \varphi(x) d\mu^*(\tau).$$

Proof. First note that if $\pi(u) \in (1/2 - \alpha, 1]$, then

$$\begin{aligned} \frac{d}{dt} u_0(t) &= -W'(u_1 - u_0) + W'(u_0 - u_1) + KV'(u_k) \\ &\geq -\delta^+|u_1 - 2u_0 + u_{-1}| + (\delta^+ + \beta) \\ &\geq \beta, \end{aligned}$$

where we applied (9) in the last line. Let $x = \pi(u(\tau))$. We easily deduce that, if $x \in [1/2 - \alpha, 1)$, then

$$u_0(2\tau) - u_0(\tau) \geq \beta\tau \wedge (1 - x).$$

Similarly, if $x \in [0, 1/2 - \alpha)$,

$$u_0(2\tau) - u_0(\tau) \geq -x.$$

We thus see that

$$\int (u_0(2\tau) - u_0(\tau))d\mu \geq \int \varphi(\pi(u))d\mu(u) = \int_{\mathbb{S}^1} \varphi(x)d\mu^*(\tau).$$

It suffices to combine this with Lemma 5.1 and the definition of the transport speed $v(\rho)$. ☐

Lemma 6.3. *Assume ρ is irrational, and $d_1(\mu^*(\tau), \lambda) \leq \varepsilon$. Then*

$$v(\rho) \geq \alpha - 2\varepsilon^{1/2} + 2\varepsilon - \frac{1}{2}\left[\left(\frac{1}{2} + \alpha - \beta\tau - 2(\varepsilon^*)^{1/2}\right) \vee 0\right]^2. \tag{26}$$

Proof. As φ is piece-wise linear, as it attains its minimum at $x = 1/2 - \alpha$, and its derivative is everywhere ≤ 1, it is easy to find the probability measure ν on $\mathbb{S}^1$ for which $\int_{\mathbb{S}^1} \varphi(x)d\nu$ is minimal, given $d_1(\nu, \lambda) \leq \gamma$. It is attained by a measure ν for which $d_1(\nu, \lambda) = \gamma$, for which we "tranport" the measure λ from $x \in \mathbb{S}^1$ to $x = 1/2 - \alpha$, starting from points for which $\varphi(x) - \varphi(1/2 - \alpha)$ is larger. Thus the measure ν coincides with λ on the set $[0, 1/2 - \alpha) \cup [1/2 - \alpha + \varepsilon^*, 1)$, and in addition $\nu(\{1/2 - \alpha\}) = \varepsilon^*$. Here

$$\int_{\alpha}^{\alpha+\varepsilon^*} d(x, 1/2 - \alpha)dx = \varepsilon,$$

where $d(x, y)$ is the distance on $\mathbb{S}^1$. Thus

$$\varepsilon = \begin{cases} (\varepsilon^*)^2/2 & \varepsilon^* \leq 1/2 \\ \varepsilon^* - (\varepsilon^*)^2/2 - 1/4 & 1/2 \leq \varepsilon^* \leq 1. \end{cases} \tag{27}$$

Now in the case $\varepsilon^* \leq 1/2 + \alpha - \beta\tau$,

$$\begin{aligned} v(\rho) &\geq \int_{\mathbb{S}^1} \varphi(x)d\nu \\ &= \int_0^{1/2-\alpha} (-x)dx - \varepsilon^*(1/2 - \alpha) + \int_{1/2-\alpha+\varepsilon^*}^{1-\beta\tau} \beta\tau dx + \int_{1-\beta\tau}^{1} (1 - x)dx. \end{aligned}$$

Careful evaluation yields

$$v(\rho) \geq \alpha - \varepsilon^* + \frac{1}{2}(\varepsilon^*)^2 - \frac{1}{2}\left(\frac{1}{2} + \alpha - \beta\tau - \varepsilon^*\right)^2. \tag{28}$$

Similarly, if $\varepsilon^* \geq 1/2 + \alpha - \beta\tau$,

$$\begin{aligned} v(\rho) &\geq \int_{\mathbb{S}^1} \varphi(x)d\nu \\ &= \int_0^{1/2-\alpha} (-x)dx - \varepsilon^*(1/2 - \alpha) + \int_{1/2-\alpha+\varepsilon^*}^{1} (1 - x)dx \\ &= \alpha - \varepsilon^*(1/2 - \alpha) - \int_{1/2-\alpha}^{1/2-\alpha+\varepsilon^*} (1 - x)dx \\ &= \alpha - \varepsilon^* + \frac{1}{2}(\varepsilon^*)^2. \end{aligned} \tag{29}$$

From (27) we easily deduce that for $\varepsilon^* \leq 1$, $\varepsilon^* \leq 2\varepsilon^{1/2}$. Inserting this in (28) and (29) (and noting that in the case $\varepsilon^* \geq 1$ the claim trivially holds as $\alpha \leq 1/2$), we complete the proof. ☐

Finally, the following will enable us to extend the bound to rational ρ. Recall the definition of $\gamma_{\rho,\tau}$ in (5).

Lemma 6.4. *For all $\tau > 0$, The function $\rho \mapsto \gamma_{\rho,\tau}$ is continuous at irrational ρ, and upper semi-continuous at rational ρ.*

Proof. Note first that by definition, to calculate $\gamma_{\rho,\tau}$, it suffices to check finitely many convergents with denominators $q_n < 1 + \sqrt{\tau}/C_\rho$ (we get it by comparing the expression with the case $q_1 = 1$)

Recall the well-known property of convergents [12]: the set of all real numbers for which a rational p/q is a convergent is an open interval $(p'/q', p''/q'')$, for some rationals $p'/q' < p/q < p''/q''$. Thus the finite set of convergents over which we minimize to calculate $\gamma_{\rho,\tau}$ changes only at finitely many rational points, and in these points the set becomes "smaller". We conclude that $\rho \mapsto \gamma_{\rho,\tau}$ is a piece-wise continuous, with discrete discontinuities possible only at boundaries of intervals $p'/q', p''/q''$ corresponding to convergents p/q with denominators $q < 1 + \sqrt{\tau}/C_\rho$. As at any such rational discontinuity point ρ_0 the set over which we minimize becomes "smaller", we obviously have

$$\gamma_{\rho_0,\tau} \geq \limsup_{\rho \to \rho_0} \gamma_{\rho,\tau}.$$

□

7. Proof of Theorem 2.1

Assume ρ is irrational. Inserting (21) into Lemma 6.1, and omitting the $\delta^+\delta^-$ term in the denominator of (21), we get

$$d_1(\mu^*(\tau), \lambda) \leq \frac{(\delta^+)^2 q}{2\sqrt{3}(\delta^-)^3 \tau} + \frac{3}{4q},$$

as long as p/q is a convergent of ρ (p, q relatively prime). We now find q_n from the sequence of convergents so that it is minimal, and obtain

$$d_1(\mu^*(\tau), \lambda) \leq \frac{1}{4}\gamma_{\rho,\tau}^2.$$

Inserting now $\varepsilon = \gamma_{\rho,\tau}^2/4$ in Lemma 6.3, we complete the proof for irrational ρ.

Consider rational ρ. Lemma 6.4 implies that the right-hand side of (6) is lower-semi continuous whenever ≥ 0 (i.e. whenever the bound is non-trivial). The speed $v(\rho)$ is, however, continuous by Proposition 4.1. As (6) holds for all irrational ρ, we conclude that it must hold for all rational ρ as well.

Remark 7.1. *There are several straightforward ways to improve the bound (6). For example, the constant next to the leading term in (26) could be $\sqrt{2}$ instead of 2 for small ε (see the proof of Lemma 6.3). Furthermore, the discussion in the last paragraph of the proof of Theorem 2.1 shows that we can replace $\gamma_{\rho,\tau}$ with*

$$\gamma'_{\rho,\tau} := \limsup_{\tau' \to \tau} \gamma_{\rho,\tau'},$$

and so by Lemma 6.4 potentially improve the bound for some rational ρ.

8. Discussion

Our lower bounds on the transport speed $v(\rho)$, as well as the heuristic discussion on the frequency optimizing the transport speed (8), leave a number of unanswered questions which may be addressed. We list only some of them.

1. *Numerics.* It would be interesting to test all our bounds, and (8) in particular, numerically.
2. *Upper bounds on transport speed.* For a concrete shape of V (i.e. by incorporating some information on its derivative in the complement of $[a, b] \mod 1$), the methods outlined here could be used to obtain an upper bound on $v(\rho)$. One could thus estimate $v(\rho)$ with a known error.
3. *Weakening assumptions.* It should be possible to weaken our assumptions on V, and in particular (4). To do it, perhaps one could combine our approach with a more detailed information of the dynamics of the unforced Frenkel-Kontorova model and its set of equilibria (i.e. the set of orbits of the associated area-preserving twist maps [11]), using also some ideas from [9], such as the abstract notion of asymmetricity of the potential V.

(4) *Strengthening bounds for rational ρ.* We estimated our bounds for rational mean spacings ρ by approximating them with irrational ones, which is clearly inefficient, especially for small rationals. This could be improved by rewriting estimates in Section 6 (in particular Lemma 6.1) for rational ρ. As this would substantially increase the length of the paper, we omitted it, but could clearly be done. (See also Remark 7.1 for straightforward improvements.)

(5) *Ratchet models with random forcing.* Perhaps the most important possible application is to extend the techniques and results here also to Frenkel-Kontorova models with random forces. This would more closely model realistic thermodynamical examples [19]. All our techniques should extend to that case too (in particular monotonicity of the dynamics, dynamics on the space of probability measures), so rigorous results may be possible, and would surely be important.

Acknowledgement

We thank A. Dujella for help with facts related to continued fractions. This work was supported by the Croatian Science Foundation grant No IP-2014-09-2285.

References

[1] L. Ambrosio, N. Gigli and G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, Birkhäuser, 2005.
[2] S. Aubry and P. Y. Le Daeron, The discrete Frenkel-Kontorova model and its extensions, Phys. D **8** (1983), 381-422.
[3] C. Baesens, R. S. Mackay, Gradient dynamics of tilted Frenkel-Kontorova models, Nolinearity **11** (1998), 949-964.
[4] C. Baesens, Spatially extended systems with monotone dynamics (continuous time). *Dynamics of coupled map lattices aand of related spatially extended systems*, 241-263, Lecture Notes in Phys. 671, Springer (2005).
[5] V. Bangert, Mather sets for twist geodesics on tori, Dynamics Reported **1** (1988), 1-55.
[6] J. J. Mazo, F. Falo and L. M. Floria, Stability of metastable structures in dissipative ac dynamics of the Frenkel-Kontorova model, Phys. Rev. B 52 (1995), 6451-6457.
[7] L. M. Floria and J. J. Mazo, Dissipative dynamics of the Frenkel-Kontorova model, Advances in Physics 45 (1996), 505-598.
[8] L. Floria, F. Falo, P. Martinez and J. Mazo, Transport of modulated phases by pumping, Europhys. Lett. **60** (2002), 174-180.
[9] L. M. Floria, C. Baesens and J. Gomez-Gardenez, The Frenkel-Kontorova model, *Dynamics of coupled map lattices aand of related spatially extended systems*, 241-263, Lecture Notes in Phys. 671, Springer (2005).
[10] B. Hu, W. Qin and Z. Zheng, Rotation number of the overdamped Frenkel-Kontorova model with ac-driving, Physica D, 208 (2005), 172-190.
[11] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1995.
[12] A. Ya. Khinchin, Continued Fractions, Courier Dover Publications, 1997.
[13] J. N. Mather, Existence of quasi-periodic orbits for twist diffeomorphisms of the annulus, Topology **21** (1982), 457-467.
[14] J. Mather, Minimal measures, Comm. Math. Helvetici **64** (1989), 375-394.
[15] G. A. Misyavichyus, Estimate of the remainder term in the limit theorem for the denominators of continued fractions, Litovsk. Mat. Sb. **21** (1981), 63-74.
[16] Wen-Xin Qin, Existence and modulation of uniform sliding states in driven and overdamped particle chains, Comm. Math. Physics **311** (2011), 513-538.
[17] Wen-Xin Qin, Existence of dynamical hull functions with two variables for the ac-driven Frenkel-Kontorova model, J. Differ. Equations 255 (2013), 3472-3490.
[18] B. Rabar, Dynamics of non-autonomous Frenkel-Kontorova model, PhD thesis, University of Zagreb, 2015 (in Croatian).
[19] P. Reimann, Brownian motors: noisy transport far from equilibrium, Phys. Rep. **361** (2002).
[20] S. Slijepčević, Extended gradient systems: dimension one, Disc. Cont. Dyn. Sys. **6** (2000),503-516.
[21] S. Slijepčević, The energy flow of discrete extended gradient systems, Nonlinearity 26 (2013), 2051-2079.
[22] S. Slijepčević, The Aubry-Mather theorem for driven generalized elastic chains Discrete and Continuous Dynamical Systems 34 (2014), 2983-3011.
[23] S. Slijepčević, Stability of synchronization in dissipatively driven Frenkel-Kontorova models, Chaos 25 (2015), 083108.
[24] H. L. Smith, *Monotone dynamical systems*, Mathematical Surveys and Monographs, Vol. 41, AMS, Providence, 1996.

Department of Mathematics, Bijenička 30, Zagreb, Croatia
E-mail address: `brabar@math.hr,slijepce@math.hr`