Period Math Hung
DOI 10.1007/s10998-016-0178-4

A note on the theorem of Johnson, Palmer and Sell

Davor Dragičević[1]

Abstract The well-known theorem of Johnson, Palmer and Sell asserts that the endpoints of the Sacker–Sell spectrum of a given cocycle $\mathcal{A}$ over a topological dynamical system (M, f) are realized as Lyapunov exponents with respect to some ergodic invariant probability measure for *f*. The main purpose of this note is to give an alternative proof of this theorem which uses a more recent and independent result of Cao which formulates sufficient conditions for the uniform hyperbolicity of a given cocyle $\mathcal{A}$ in terms of the nonvanishing of Lyapunov exponents for $\mathcal{A}$. We also discuss the possibility of obtaining positive results related to the stability of the Sacker–Sell spectra under the perturbations of the cocycle $\mathcal{A}$.

Keywords Sacker–Sell spectrum · Lyapunov exponents · Invariant measures · Stability

Mathematics Subject Classification Primary 37C40 · 37C60

1 Introduction

In their landmark paper [10], Sacker and Sell introduced the notion of (what is now called) the Sacker–Sell spectrum for cocycles over topological dynamical systems and they described all possible structures of the spectrum. Furthermore, they indicated a strong relationship between their spectral theory and the theory of Lyapunov exponents which plays a central role in the stability of dynamical systems. A deeper connection between those two theories was discovered in a remarkable paper [7] where the authors proved that the endpoints of the Sacker–Sell spectrum of a given cocycle are realized as Lyapunov exponents of the cocycle with respect to some ergodic probability measure which is invariant for the base on which the cocycle acts. The arguments in [7] heavily rely on the new proof of the celebrated Oseledets

✉ Davor Dragičević
d.dragicevic@unsw.edu.au

1 School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia

Published online: 12 November 2016

multiplicative ergodic theorem presented in the same paper and on the study of the induced flow on the projective bundle.

The main purpose of this note is to derive the theorem of Johnson, Palmer and Sell as a simple consequence of a more recent and independent result of Cao [5] who gave a beautiful criterion for the uniform hyperbolicity of cocycles in terms of nonvanishing Lyapunov exponents (see Theorem 3.1). We note that in general the cocycles with nonzero Lyapunov exponents exibit only a weaker form of the hyperbolicity which is known in the litrature as nonuniform hyperbolicity. We refer to [3] for the detailed exposition of this theory which goes back to the landmark works of Oseledets [8] and Pesin [9].

Furthermore, using the recent results on the continuity of Lyapunov exponents, we obtain some simple consequences regarding the stability of the Sacker–Sell spectrum with respect to perturbations of the cocycle. Although those observations are far from satisfactory, we hope that they could lead to new directions in the research on the stability of spectrum.

2 Sacker–Sell spectrum

Let M be a compact topological space and let $f\colon M \to M$ be a homeomorphism. We denote by GL_d the space of all invertible operators acting on $\mathbb{R}^d$. A continuous map $\mathcal{A}\colon M \times \mathbb{Z} \to GL_d$ is said to be a *cocycle* over f if:

1. $\mathcal{A}(x, 0) = \text{Id}$ for each $x \in M$;
2. $\mathcal{A}(x, n+m) = \mathcal{A}(f^n(x), m)\mathcal{A}(x, n)$ for every $x \in M$ and $n, m \in \mathbb{Z}$.

A map $A\colon M \to GL_d$ given by $A(x) = \mathcal{A}(x, 1)$ is called a *generator* of a cocycle $\mathcal{A}$.

Let $\mathcal{A}$ be a cocycle over f. We say that $\mathcal{A}$ is *uniformly hyperbolic* if:

1. There exists a family of projections $P(x)$, $x \in M$ satisfying
$$A(x)P(x) = P(f(x))A(x), \quad x \in M; \tag{2.1}$$
2. There exist $D, \lambda > 0$ such that for each $x \in M$ and $n \geq 0$
$$\|\mathcal{A}(x, n)P(x)\| \leq De^{-\lambda n} \tag{2.2}$$
and
$$\|\mathcal{A}(x, -n)(\text{Id} - P(x))\| \leq De^{-\lambda n}. \tag{2.3}$$

Let $\mathcal{A}$ be a cocycle over f and $a \in \mathbb{R}$. We define a map $\mathcal{A}_a\colon X \times \mathbb{Z} \to GL_d$ by $\mathcal{A}_a(x, n) = e^{-an}\mathcal{A}(x, n)$. Clearly, $\mathcal{A}_a$ is also a cocycle over f. For a cocycle $\mathcal{A}$ over f we recall that the Sacker and Sell spectrum of $\mathcal{A}$ is given by

$$\Sigma = \Sigma(\mathcal{A}) = \left\{ a \in \mathbb{R} : \text{the cocycle } \mathcal{A}_a \text{ is not uniformly hyperbolic} \right\}.$$

The following theorem was established in [10].

Theorem 2.1 *There exist numbers*

$$a_1 \leq b_1 < a_2 \leq b_2 < \ldots a_k \leq b_k, \quad 1 \leq k \leq d$$

such that
$$\Sigma = [a_1, b_1] \cup \ldots \cup [a_k, b_k]. \tag{2.4}$$

Furthermore, for each $i \in \{1, \ldots, k\}$ there is a family $W_i(x)$, $x \in M$ of subspaces of $\mathbb{R}^d$ such that:

1. *for each* $i \in \{1, \dots, k\}$ *the map* $x \mapsto W_i(x)$ *is continuous;*
2. *for each* $x \in M$ *and* $i \in \{1, \dots, k\}$ *we have that* $A(x)W_i(x) = W_i(f(x))$*;*
3. *for* $i \neq j$ *we have that* $W_i(x) \cap W_j(x) = \{0\}$ *for each* $x \in M$*;*
4. $\mathbb{R}^d = W_1(x) + \dots + W_k(x)$ *for each* $x \in M$*;*
5. *for every* $i \in \{1, \dots, k\}$ *and* $v \in W_i(x) \setminus \{0\}$ *we have that*

$$\limsup_{n \to +\infty} \frac{1}{n} \log \|\mathcal{A}(x, n)v\| \in [a_i, b_i].$$

We note that in [2], the authors have developed a version of Sacker–Sell theory with respect to a much weaker notion of hyperbolicity.

3 Nonvanishing of Lyapunov exponents and uniform hyperbolicity

We recall that a Borel measurable set $B \subset M$ has *total probability* if $\mu(B) = 1$ for every f-invariant Borel probability measure μ. The following theorem is due to Cao [5].

Theorem 3.1 *Let* $\mathcal{A}$ *be a cocycle over* f *and assume that there exist families of subspaces* $E^s(x)$*,* $x \in M$ *and* $E^u(x)$*,* $x \in M$ *such that:*

1. *the mappings* $x \mapsto E^s(x)$ *and* $x \mapsto E^u(x)$ *are continuous;*
2. $\mathbb{R}^d = E^s(x) \oplus E^u(x)$ *for each* $x \in M$ *and this splitting is* $\mathcal{A}$*-invariant;*
3. $\mathcal{A}$ *has negative Lyapunov exponents in the* $E^s(x)$ *direction and positive Lyapunov exponents in the* $E^u(x)$ *direction on a set of total probability.*

Then, the cocycle $\mathcal{A}$ *is uniformly hyperbolic.*

We note that the above theorem is proved in [5] in the particular case when $\mathcal{A}$ is a derivative cocycle. However, by repeating the same arguments one can easily obtain the version stated here.

4 Theorem of Johnson, Palmer and Sell

The following theorem is due to Johnson et al. 1 [7]. We give a short proof that uses Theorem 3.1.

Theorem 4.1 *Let* $\mathcal{A}$ *be a cocycle over* f*. Then, each boundary point of* Σ *is a Lyapunov exponent of a cocycle* $\mathcal{A}$ *with respect to some* f*-invariant ergodic probability measure.*

Proof Let Σ be given by (2.4) for the cocycle $\mathcal{A}$. We are going to prove that b_j for $j \in \{1, \dots, k\}$ is a Lyapunov exponent of a cocycle $\mathcal{A}$ with respect to some f-invariant probability measure μ (the argument for a_j is similar). Then the standard argument using the ergodic decomposition of μ shows that μ can be chosen to be ergodic (see [7]).

Assume the opposite, that is that b_j is not a Lyapunov exponent of a cocycle $\mathcal{A}$ with respect to f-invariant probability measure. We consider the cocycle $\mathcal{A}_{b_j}$. Obviously, the Lyapunov exponents of the cocycle $\mathcal{A}_{b_j}$ with respect to some f-invariant probability measure μ are given by $-b_j + \lambda$, where λ is a Lyapunov exponent of $\mathcal{A}$ with respect to μ.

Let

$$E^s(x) = W_1(x) + \dots + W_j(x) \quad \text{and} \quad E^u(x) = W_{j+1}(x) + \dots + W_k(x),$$

where $W_i(x)$ are given by Theorem 2.1 applied to the cocycle $\mathcal{A}$. It follows from Theorem 2.1 that subspaces $E^s(x)$ and $E^u(x)$ depend continuously on x and that they form an $\mathcal{A}$-invariant splitting of $\mathbb{R}^d$. Moreover, the last assertion in Theorem 2.1 together with our assumption that b_j is not a Lyapunov exponent of a cocycle $\mathcal{A}$ with respect to some f-invariant probability measure, implies that the cocycle $\mathcal{A}_{b_j}$ has negative Lyapunov exponents in the $E^s(x)$ direction and positive Lyapunov exponents in the $E^u(x)$ direction on a set of total probability given by the Oseledets multiplicative ergodic theorem (see [3,7,8]). By Theorem 3.1, we have that $\mathcal{A}_{b_j}$ is uniformly hyperbolic and hence $b_j \notin \Sigma$ which yields a contradiction. □

5 A small step towards stability of spectrum

In the Sect. 2, we have described the spectrum of a cocycle $\mathcal{A}$ over the whole base M. On the other hand, one can associate a spectrum to each cocycle over some f-invariant subset $\Lambda \subset M$. In [10], Sacker and Sell have studied the stability of the spectrum under the perturbations of Λ. In this section, we are interested in the behaviour of the spectrum under the perturbations of the cocycle.

The famous theorem of Mañé and Bochi [4] asserts that in an arbitrary C^0-neighborhood of every cocycle which is not uniformly hyperbolic one can find a cocycle whose Lyapunov exponents are all zero. Together with Theorem 4.1 we conclude that even in a simple setting when the dynamical system (M, f) is uniquely ergodic, we don't have any positive stability result for the Sacker and Sell spectrum under C^0-perturbations. Indeed, by perturbing the original cocycle the spectrum can reduce to a single point.

On the other hand, recent results dealing with the continuity of Lyapunov exponents (see [6] and the references therein) indicate a possibility that there might exist positive results regarding the stability of the Sacker and Sell spectrum. The point of the departure from the discussion in the previous paragraph would be that we require additional regularity conditions (both for the original cocycle and the perturbation) as well as relatively simple base (M, f).

We now describe the setting in [6]. Let $M = \mathbb{T}$ be a circle and let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a strong Diophantine condition (see [6]). Furthermore, let $f: M \to M$ be given by $f(x) = (x + \omega)(\mathrm{mod}\ 1)$. Finally, by C^ω we denote the space of all analytic cocycles over f with values in GL_d (identifying the cocycle with its generator). The main result of [6] states that in this setting the Lyapunov exponents are continuous functions of a cocycle.

Corollary 5.1 *Let $A \in C^\omega$, $\varepsilon > 0$ and assume that a is a boundary point of $\Sigma(A)$. Then, there exists a neighborhood $\mathcal{U}$ of A in C^ω such that for each $B \in \mathcal{U}$ there exists a point $b \in \Sigma(B)$ such that $|a - b| < \varepsilon$.*

Proof Let μ be the Lebesgue measure on M which is the unique invariant measure for f. It follows from Theorem 4.1 that a is a Lyapunov exponent of $\mathcal{A}$ with respect to μ. Let $\mathcal{U}$ be a neighborhood of A such that for each cocycle $B \in \mathcal{U}$ there exists a Lyapunov exponent b of B with respect to μ such that $|b - a| < \varepsilon$. The existence of such neighborhood $\mathcal{U}$ of A follows from the results in [6]. Then, the last statement of Theorem 2.1 implies that b belongs to $\Sigma(B)$. □

For some weaker results on the stability of Lyapunov exponents, we refer to [1].

Acknowledgements I would like to thank Gary Froyland for carefully reading this manuscript and for many useful comments. Furthermore, I am thankful to my collaborators Luis Barreira and Claudia Valls for introducing me to Sacker–Sell spectral theory. The author was supported by an Australian Research Council Discovery

Project DP150100017 and in part by Croatian Science Foundation under the project IP-2014-09-2285. The paper is dedicated to the memory of P. R. Nelson (1958–2016).

References

1. L. Barreira, D. Dragičević, C. Valls, Tempered exponential dichotomies and Lyapunov exponents for perturbations. Commun. Contemp. Math. **18**(1550058), 16 (2016)
2. L. Barreira, D. Dragičević, C. Valls, Strong nonuniform spectrum for arbitrary growth rates, Commun. Contemp. Math. (in press). doi:10.1142/S0219199716500085
3. L. Barreira, Ya. Pesin, *Nonuniform Hyperbolicity, Encyclopedia of Mathematics and its Application* (Cambridge University Press, Cambridge, 2007)
4. J. Bochi, Genericity of zero Lyapunov exponents. Ergod. Theory Dyn. Syst. **22**, 1667–1696 (2002)
5. Y. Cao, Nonzero Lyapunov exponents and uniform hyperbolicity. Nonlinearity **16**, 1473–1479 (2003)
6. P. Duarte, S. Klein, Continuity of the Lyapunov exponents for quasiperiodic cocycles. Commun. Math. Phys. **332**, 1113–1166 (2014)
7. R. Johnson, K. Palmer, G. Sell, Ergodic properties of linear dynamical systems. SIAM J. Math. Anal. **18**, 1–33 (1987)
8. V. Oseledets, A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems. Trans. Mosc. Math. Soc. **19**, 197–221 (1968)
9. Ya. Pesin, Families of invariant manifolds corresponding to nonzero characteristic exponents. Math. USSR-Izvest. **10**, 1261–1305 (1976)
10. R. Sacker, G. Sell, A spectral theory for linear differential systems. J. Differ. Equ. **27**, 320–358 (1978)