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Results in Mathematics

Nonuniform Spectrum on the Half Line and Perturbations

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Abstract. For a one-sided nonautonomous dynamics defined by a sequence of invertible matrices, we develop a spectral theory (in the sense of Sacker and Sell) for the notion of a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. We emphasize that this notion is ubiquitous in the context of ergodic theory, unlike the notion of a uniform exponential dichotomy. In particular, we show that each Lyapunov exponent belongs to one interval of the spectrum. We also consider a class of sufficiently small nonlinear perturbations of a linear dynamics satisfying a nonuniform bounded growth condition and we show that each solution is either eventually zero or the Lyapunov exponents belong to one interval of the spectrum.

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1. Introduction

For a one-sided nonautonomous dynamics defined by a sequence of invertible $d \times d$ matrices $(A_m)_{m \in \mathbb{N}}$, we consider the notion of a *nonuniform exponential dichotomy with an arbitrarily small nonuniform part*. Our main objective is to develop a spectral theory (in the sense of Sacker and Sell) with respect to this notion and to study its relation to the theory of Lyapunov exponents. The original Sacker–Sell spectrum (see [14]) was introduced for linear cocycles over a flow (or, equivalently, for linear skew-product flows) with respect to the notion of a uniform exponential dichotomy. The underlying ideas where

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later used by Siegmund [15] and Aulbuch and Siegmund [2] to develop corresponding spectral theories, respectively, for nonautonomous linear differential and difference equations. We emphasize that in all these works the spectrum is computed with respect to the notion of a *uniform* exponential dichotomy.

The notion of a nonuniform exponential dichotomy with an arbitrarily small nonuniform part contains the notion of a uniform exponential dichotomy as a very special case. Moreover, the notion is ubiquitous in the context of smooth ergodic theory. Indeed, let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism preserving a probability measure μ. This means that

$$\mu(f(A)) = \mu(A)$$

for every measurable set $A \subset \mathbb{R}^d$. Then the trajectory of μ-almost every point $x \in \mathbb{R}^d$ with *nonzero* Lyapunov exponents gives rise to a sequence of invertible matrices

$$A_m = d_{f^m(x)} f, \quad m \in \mathbb{N}$$

that admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. In addition, if the diffeomorphism f is not uniformly hyperbolic, then the above construction gives examples of trajectories that admit a nonuniform exponential dichotomy with an arbitrarily small nonuniform part but not a uniform exponential dichotomy. We refer to [3] for details and for many examples of diffeomorphisms with those properties. In particular, if Σ' is the spectrum introduced in [1], that is, the set of all $a \in \mathbb{R}$ for which the sequence $(e^{-a}A_m)_{m\in\mathbb{N}}$ admits a uniform exponential dichotomy, and Σ is the set of all $a \in \mathbb{R}$ for which the sequence $(e^{-a}A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part, then $\Sigma \subset \Sigma'$ and in general $\Sigma \neq \Sigma'$.

Now we briefly summarize the results of the paper. In Sect. 3, we give a complete description of the structure of Σ. In particular, we show that Σ can be the empty set, the whole $\mathbb{R}$ or a finite union

$$I_1 \cup [a_2, b_2] \cup \cdots \cup [a_{k-1}, b_{k-1}] \cup I_k,$$

where $I_1 = [a_1, b_1]$ or $I_1 = (-\infty, b_1]$ and $I_k = [a_k, b_k]$ or $I_k = [a_k, +\infty)$, for some finite numbers

$$a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$$

and some integer $k \le d$. Moreover, under a certain nonuniform bounded growth condition (see condition (17)), the spectrum Σ is compact and for each vector $v \in \mathbb{R}^d\setminus\{0\}$, both Lyapunov exponents

$$\lambda^-(v) = \liminf_{n\to+\infty} \frac{1}{n} \log\|\mathcal{A}_n v\| \quad \text{and} \quad \lambda^+(v) = \limsup_{n\to+\infty} \frac{1}{n} \log\|\mathcal{A}_n v\|,$$

where $\mathcal{A}_n = A_n \cdots A_1$, belong to same interval I_i. In addition, if the sequence $(A_m)_{m\in\mathbb{N}}$ is Lyapunov regular, in which case $\lambda^-(v) = \lambda^+(v)$ for every $v \in$

$\mathbb{R}^d \backslash \{0\}$, then Σ is simply the finite set formed by all values of the Lyapunov exponents (see Sect. 5).

In Sect. 4 we show that our original linear dynamics and the nonlinear dynamics obtained from a small nonlinear perturbation essentially share the same asymptotic properties. More precisely, given a sequence of continuous functions $f_n \colon \mathbb{R}^d \to \mathbb{R}^d$, we consider the dynamics

$$x(n+1) = A_n x(n) + f_n(x(n)), \quad n \in \mathbb{N}. \tag{1}$$

Under a certain growth assumption on the functions f_n (see conditions (20) and (21)), we show that each solution $(x(n))_{n\in\mathbb{N}}$ of (1) is either eventually zero or its Lyapunov exponents

$$\lambda^- = \liminf_{n\to+\infty} \frac{1}{n} \log\|x(n)\| \quad \text{and} \quad \lambda^+ = \limsup_{n\to+\infty} \frac{1}{n} \log\|x(n)\|$$

belong to the same interval I_i. In particular, when the sequence $(A_m)_{m\in\mathbb{N}}$ is Lyapunov regular, each solution of (1) is either eventually zero or the numbers λ^- and λ^+ coincide and are equal to some Lyapunov exponent of the linear dynamics defined by $(A_m)_{m\in\mathbb{N}}$. For perturbations of the equation $x' = Ax$, an equivalent result was obtained by Coppel [7]. Former results were obtained by Perron [11], Lettenmeyer [9] and Hartman and Wintner [8]. For perturbations of an autonomous delay equation see Pituk [12,13] (for finite-dimensional spaces and finite delay) and Matsui, Matsunaga and Murakami [10] (for Banach spaces and infinite delay). Related results for autonomous difference equations were obtained by Coffman [6].

2. Preliminaries

We first introduce the notion of a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $d \times d$ matrices. For each $m, n \in \mathbb{N}$ we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1} \cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ A_m^{-1} \cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a *nonuniform exponential dichotomy with an arbitrarily small nonuniform part* if there exist projections P_m for $m \in \mathbb{N}$ satisfying

$$P_m \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n \quad \text{for } m, n \in \mathbb{N}, \tag{2}$$

a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)+\varepsilon n} \quad \text{for } m \ge n \tag{3}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)+\varepsilon n} \quad \text{for } m \le n, \tag{4}$$

where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{N}$.

The following result shows that the images of the projections P_m are uniquely determined, that is, are independent of the actual projections P_m that are taken in the notion of a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. We note that the same does not happen to the images of the projections Q_m.

Proposition 1. *For each $n \in \mathbb{N}$, we have*

$$\mathrm{Im}\, P_n = \left\{ v \in \mathbb{R}^d : \sup_{m \geq n} \|\mathcal{A}(m,n)v\| < +\infty \right\}.$$

Proof. It follows from (3) that

$$\sup_{m \geq n} \|\mathcal{A}(m,n)v\| < +\infty \tag{5}$$

for $v \in \mathrm{Im}\, P_n$. Now take a vector $v \in \mathbb{R}^d$ satisfying (5). Since $v = P_n v + Q_n v$, it follows from (3) that

$$\sup_{m \geq n} \|\mathcal{A}(m,n)Q_n v\| < +\infty. \tag{6}$$

On the other hand, by (4), for $m \geq n$ we have

$$\|Q_n v\| = \|\mathcal{A}(n,m)\mathcal{A}(m,n)Q_n v\| \leq De^{-\lambda(m-n)+\varepsilon m}\|\mathcal{A}(m,n)Q_n v\|$$

and so,

$$\frac{1}{D}e^{\lambda(m-n)-\varepsilon m}\|Q_n v\| \leq \|\mathcal{A}(m,n)Q_n v\|.$$

Hence, whenever $Q_n v \neq 0$, taking $\varepsilon < \lambda$ we obtain

$$\sup_{m \geq n} \|\mathcal{A}(m,n)Q_n v\| = +\infty.$$

But this contradicts to (6). Therefore, $Q_n v = 0$ and so $v \in \mathrm{Im}\, P_n$. This completes the proof of the proposition. □

The following statement specifies the freedom that is allowed when choosing the projections P_m in (2) or, equivalently, the images of the projections Q_m.

Proposition 2. *Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part with respect to projections P_m. Moreover, let P'_m, for $m \in \mathbb{N}$, be projections such that*

$$P'_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P'_n \quad \textit{for } m,n \in \mathbb{N}. \tag{7}$$

Then $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part with respect to the projections P'_m if and only if $\mathrm{Im}\, P_1 = \mathrm{Im}\, P'_1$.

Proof. If $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part with respect to the projections P'_m, then it follows directly from Proposition 1 that

$$\operatorname{Im} P'_1 = \left\{ v \in \mathbb{R}^d : \sup_{m\geq 1} \|\mathcal{A}(m,1)v\| < +\infty \right\} = \operatorname{Im} P_1.$$

Now assume that $\operatorname{Im} P_1 = \operatorname{Im} P'_1$. Then

$$P_1 P'_1 = P'_1 \quad \text{and} \quad P'_1 P_1 = P_1.$$

In particular,

$$P_1 - P'_1 = P_1(P_1 - P'_1) = (P_1 - P'_1)Q_1$$

and so it follows from (3) and (4) that

$$\begin{aligned}
\|\mathcal{A}(n,1)(P_1 - P'_1)v\| &= \|\mathcal{A}(n,1)P_1(P_1 - P'_1)v\| \\
&\leq e^{\varepsilon} D e^{-\lambda(n-1)} \|(P_1 - P'_1)v\| \\
&= e^{\varepsilon+\lambda} D e^{-\lambda n} \|(P_1 - P'_1)Q_1 v\| \\
&\leq e^{\varepsilon+\lambda} D e^{-\lambda n} \|P_1 - P'_1\| \cdot \|Q_1 v\| \\
&= e^{\varepsilon+\lambda} D e^{-\lambda n} \|P_1 - P'_1\| \cdot \|\mathcal{A}(1,m)\mathcal{A}(m,1)Q_1 v\| \\
&= e^{\varepsilon+\lambda} D e^{-\lambda n} \|P_1 - P'_1\| \cdot \|\mathcal{A}(1,m)Q_m\mathcal{A}(m,1) v\| \\
&\leq e^{\varepsilon+2\lambda} D^2 e^{-\lambda n-\lambda m+\varepsilon m} \|P_1 - P'_1\| \cdot \|\mathcal{A}(m,1) v\|
\end{aligned}$$

for $m, n \in \mathbb{N}$ and $v \in X$. Therefore,

$$\begin{aligned}
\|\mathcal{A}(n,m)P'_m v\| &\leq \|\mathcal{A}(n,m)P_m v\| + \|\mathcal{A}(n,m)(P_m - P'_m)v\| \\
&= \|\mathcal{A}(n,m)P_m v\| + \|\mathcal{A}(n,1)(P_1 - P'_1)\mathcal{A}(1,m)v\| \\
&\leq D e^{-\lambda(n-m)+\varepsilon m}\|v\| + e^{\varepsilon+2\lambda} D^2 e^{-\lambda(n-m)+\varepsilon m} \|P_1 - P'_1\| \cdot \|v\| \\
&= D' e^{-\lambda(n-m)+\varepsilon m}\|v\|
\end{aligned}$$

for $n \geq m$, where

$$D' = D + D^2 e^{\varepsilon+2\lambda}\|P_1 - P'_1\|.$$

Similarly, letting $Q'_m = \operatorname{Id} - P'_m$ we obtain

$$\begin{aligned}
\|\mathcal{A}(n,m)Q'_m v\| &\leq \|\mathcal{A}(n,m)Q_m v\| + \|\mathcal{A}(n,m)(P_m - P'_m)v\| \\
&= \|\mathcal{A}(n,m)Q_m v\| + \|\mathcal{A}(n,1)(P_1 - P'_1)\mathcal{A}(1,m)v\| \\
&\leq D e^{-\lambda(m-n)+\varepsilon m}\|v\| + e^{\varepsilon+2\lambda} D^2 e^{-\lambda(m-n)+\varepsilon m} \|P_1 - P'_1\| \cdot \|v\| \\
&= D' e^{-\lambda(m-n)+\varepsilon m}\|v\|
\end{aligned}$$

for $n \leq m$. This shows that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part with respect to the projections P'_m. □

The following result is a simple consequence of Proposition 2.

Proposition 3. *Assume that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part with respect to projections P_m. Moreover, let $Y \subset \mathbb{R}^d$ be a subspace such that*

$$\mathbb{R}^d = \operatorname{Im} P_1 \oplus Y. \tag{8}$$

Then the projections

$$P'_m = \mathcal{A}(m,1)P'_1\mathcal{A}(1,m),$$

where P'_1 and $Id - P'_1$ are those obtained from the decomposition in (8), satisfy (7) as well as $\operatorname{Im} P'_1 = \operatorname{Im} P_1$, and so the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part with respect to the projections P'_m.

3. Nonuniform Spectrum

Now we introduce the main object of our work. Given a sequence $(A_m)_{m\in\mathbb{N}}$ of invertible $d \times d$ matrices, its *nonuniform spectrum* is the set Σ of all numbers $a \in \mathbb{R}$ such that the sequence $(e^{-a}A_m)_{m\in\mathbb{N}}$ does not admit a nonuniform exponential dichotomy with an arbitrarily small nonuniform part.

For each $a \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$S_a(n) = \left\{v \in \mathbb{R}^d : \sup_{m\ge n} \left(e^{-a(m-n)}\|\mathcal{A}(m,n)v\|\right) < +\infty\right\}.$$

Moreover, let $S_{-\infty}(n) = \{0\}$ and $S_\infty(n) = \mathbb{R}^d$. Clearly, each set $S_a(n)$ is a linear space and

$$A_n S_a(n) = S_a(n+1)$$

for $a \in [-\infty, +\infty]$ and $n \in \mathbb{N}$. In particular, the numbers $\dim S_a(n)$ are independent of n. We denote their common value by $\dim S_a$. We also note that if $a < b$, then $S_a(n) \subset S_b(n)$.

Proposition 4. *The set $\Sigma \subset \mathbb{R}$ is closed. Moreover, for $a \in \mathbb{R}\backslash\Sigma$ we have*

$$S_a(n) = S_b(n)$$

for all $n \in \mathbb{Z}$ and b in some open neighborhood of a.

Proof. Take $a \in \mathbb{R}\backslash\Sigma$. Then there exist projections P_n for $n \in \mathbb{N}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-a(m-n)}\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)+\varepsilon n} \quad \text{for } m \ge n$$

and

$$\|e^{-a(m-n)}\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)+\varepsilon n} \quad \text{for } m \le n.$$

For each $b \in \mathbb{R}$ we have

$$\|e^{-b(m-n)}\mathcal{A}(m,n)P_n\| \le De^{-(\lambda-a+b)(m-n)+\varepsilon n} \quad \text{for } m \ge n$$

and

$$\|e^{-b(m-n)}\mathcal{A}(m,n)Q_n\| \le De^{-(\lambda+a-b)(n-m)+\varepsilon n} \quad \text{for} m \le n.$$

This implies that $b \in \mathbb{R}\setminus\Sigma$ whenever $|a-b| < \lambda$ and in particular the set Σ is closed. Moreover, it follows from Proposition 1 that

$$S_b(n) = S_a(n) = \operatorname{Im} P_n$$

for $n \in \mathbb{N}$ and b as above. □

The following result gives a complete description of the structure of the nonuniform spectrum.

Theorem 5. *For a sequence $(A_m)_{m\in\mathbb{N}}$ of invertible $d \times d$ matrices:*

1. *either $\Sigma = \emptyset$, $\Sigma = \mathbb{R}$ or*

$$\Sigma = I_1 \cup [a_2, b_2] \cup \cdots \cup [a_{p-1}, b_{p-1}] \cup I_p, \tag{9}$$

   where $I_1 = [a_1, b_1]$ or $I_1 = (-\infty, b_1]$ and $I_p = [a_p, b_p]$ or $I_p = [a_p, +\infty)$ for some numbers

$$a_1 \le b_1 < a_2 \le b_2 < \cdots < a_p \le b_p \quad \text{and} \quad p \le d;$$

2. *when (9) holds, taking any numbers*

$$\rho_i \in (b_i, a_{i+1}) \quad \text{for } i = 1, \ldots, p-1$$

   and

$$\delta > 0, \quad \rho_0 = \inf \Sigma - \delta, \quad \rho_p = \sup \Sigma + \delta \tag{10}$$

   we have:

   (a) *for each $n \in \mathbb{N}$ the spaces $S_{\rho_i}(n)$, for $i = 0, \ldots, p-1$, are independent of $\delta, \rho_1, \ldots, \rho_{p-1}$;*

   (b) *for each $i = 1, \ldots, p$ and $v \in S_{\rho_i}(1)\setminus S_{\rho_{i-1}}(1)$ we have*

$$\liminf_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,1)v\| \ge a_i$$

   and

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,1)v\| \le b_i,$$

   where $-\infty \le a_i \le b_i \le +\infty$ are the endpoints of I_i.

Proof. We first establish an auxiliary result.

Lemma 1. *For each $a_1, a_2 \in \mathbb{R}\setminus\Sigma$ with $a_1 < a_2$, the following statements are equivalent:*

1. *$S_{a_1}(n) = S_{a_2}(n)$ for some $n \in \mathbb{N}$ (and so for all $n \in \mathbb{N}$);*
2. *$[a_1, a_2] \subset \mathbb{R}\setminus\Sigma$.*

Proof of the lemma. Assume that $S_{a_1}(n) = S_{a_2}(n)$ for all $n \in \mathbb{N}$. It follows from Propositions 1 and 2 that the sequences $e^{-a_1}A_m$ and $e^{-a_2}A_m$ admit nonuniform exponential dichotomies with an arbitrarily small nonuniform part with respect to the same sequence of projections P_m. Hence, there exist $\lambda_1, \lambda_2 > 0$ and for each $\varepsilon > 0$ constants $D_1 = D_1(\varepsilon), D_2 = D_2(\varepsilon) > 0$ such that for $i = 1, 2$ we have

$$\|e^{-a_i(m-n)}\mathcal{A}(m,n)P_n\| \le D_i e^{-\lambda_i(m-n)+\varepsilon n} \quad \text{for } m \ge n \tag{11}$$

and

$$\|e^{-a_i(m-n)}\mathcal{A}(m,n)Q_n\| \le D_i e^{-\lambda_i(n-m)+\varepsilon n} \quad \text{for } m \le n. \tag{12}$$

For each $a \in [a_1, a_2]$, by (11) we have

$$\|e^{-a(m-n)}\mathcal{A}(m,n)P_n\| \le D_1 e^{-\lambda_1(m-n)+\varepsilon n} \quad \text{for } m \ge n$$

and similarly, by (12),

$$\|e^{-a(m-n)}\mathcal{A}(m,n)Q_n\| \le D_2 e^{-\lambda_2(n-m)+\varepsilon n} \quad \text{for } m \le n.$$

Taking the constants $\lambda = \min\{\lambda_1, \lambda_2\}$ and $D = \max\{D_1, D_2\}$, we conclude that $[a_1, a_2] \subset \mathbb{R}\backslash\Sigma$.

Now we assume that $[a_1, a_2] \subset \mathbb{R}\backslash\Sigma$ and we proceed by contradiction. Namely, assume that, in addition, $S_{a_1}(n) \ne S_{a_2}(n)$ for some $n \in \mathbb{N}$. Let

$$b = \inf\big\{a \in \mathbb{R}\backslash\Sigma : S_a(n) = S_{a_2}(n) \text{ for some } n \in \mathbb{N}\big\}.$$

Since $S_{a_1}(n) \ne S_{a_2}(n)$, it follows from Proposition 4 that $a_1 < b < a_2$. We show that $b \in \Sigma$. Otherwise, we consider two possibilities: either $S_b(n) = S_{a_2}(n)$ or $S_b(n) \ne S_{a_2}(n)$. In the first case, by Proposition 4 we have $S_{b'}(n) = S_{a_2}(n)$ and $b' \in \mathbb{R}\backslash\Sigma$ for all $b' \in (b - \varepsilon, b]$ and some $\varepsilon > 0$. But this contradicts to the definition of b. In the second case, again by Proposition 4 we have $S_{b'}(n) \ne S_{a_2}(n)$ and $b' \in \mathbb{R}\backslash\Sigma$ for all $b' \in [b, b + \varepsilon)$ and some $\varepsilon > 0$ that again contradicts to the definition of b. Hence, $b \in \Sigma$ but this contradicts to the assumption that $[a_1, a_2] \subset \mathbb{R}\backslash\Sigma$. □

We proceed with the proof of the theorem. By Proposition 4, the set Σ is a disjoint union of (possibly infinite) closed intervals. Assume that Σ is composed of $d + 1$ disjoint closed intervals. Then there exist $c_1, \dots, c_d \in \mathbb{R}\backslash\Sigma$ such that the intervals

$$(-\infty, c_1), (c_1, c_2), \dots, (c_{d-1}, c_d), (c_d, +\infty)$$

intersect Σ. By Lemma 1, we have

$$0 \le \dim S_{c_1} < \dim S_{c_2} < \cdots < \dim S_{c_d} \le d. \tag{13}$$

Now we show that

$$\dim S_{c_d} < d \quad \text{and} \quad \dim S_{c_1} > 0. \tag{14}$$

If $\dim S_{c_1} = 0$, then $S_{c_1}(n) = \{0\}$ for $n \in \mathbb{N}$. Since $c_1 \in \mathbb{R}\backslash\Sigma$, there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-c_1(m-n)}\mathcal{A}(m,n)\| \le De^{-\lambda(n-m)+\varepsilon n} \quad \text{for } m \le n.$$

Hence, for $b < c_1$ we have

$$\|e^{-b(m-n)}\mathcal{A}(m,n)\| \le De^{-\lambda(n-m)+\varepsilon n} \quad \text{for } m \le n.$$

This shows that $(-\infty, c_1) \subset \mathbb{R}\backslash\Sigma$, which is impossible since $(-\infty, c_1)$ intersects Σ. Now we assume that $\dim S_{c_d} = d$. Then $S_{c_d}(n) = \mathbb{R}^d$ for $n \in \mathbb{N}$. Since $c_d \in \mathbb{R}\backslash\Sigma$, there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-c_d(m-n)}\mathcal{A}(m,n)\| \le De^{-\lambda(m-n)+\varepsilon n} \quad \text{for } m \ge n.$$

Hence, for $b > c_d$ we have

$$\|e^{-b(m-n)}\mathcal{A}(m,n)\| \le De^{-\lambda(m-n)+\varepsilon n} \quad \text{for } m \ge n.$$

This shows that $(c_d, +\infty) \subset \mathbb{R}\backslash\Sigma$, which is impossible since $(c_d, +\infty)$ intersects Σ. Finally, it follows from (14) that (13) cannot hold and so there are at most d disjoint closed intervals on the right-hand side of (9). This establishes property 1 in the theorem.

On the other hand, it follows from Lemma 1 that if $\sigma_i \in (b_i, a_{i+1})$ for $i = 1, \ldots, p-1$, then $S_{\sigma_i}(n) = S_{\rho_i}(n)$ for $i = 1, \ldots, p-1$. In other words, the spaces S_{ρ_i} are independent of the choice of numbers ρ_i. The same observation applies to the spaces S_{ρ_0}. Indeed, it follows from Lemma 1 that $S_\rho(n) = S_{\inf\Sigma-\delta'}(n)$ for $\delta' > 0$. This implies that the space S_{ρ_0} is independent of the choice of δ, which establishes property $2a$.

For each $i \in \{0, 1, \ldots, p\}$, the sequence $(e^{-\rho_i}A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part, that is, there exist projections P_n for $n \in \mathbb{N}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{(\rho_i-\lambda)(m-n)+\varepsilon n} \quad \text{for } m \ge n \tag{15}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-(\lambda+\rho_i)(n-m)+\varepsilon n} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$. It follows from Proposition 1 that $S_{\rho_i}(n) = \mathrm{Im}\,P_n$ does not depend on the choice of P_n. Hence, for $x \in S_{\rho_i}(1)\backslash S_{\rho_{i-1}}(1) \subset S_{\rho_i}(1)$, we have $x \in \mathrm{Im}\,P_1$ and it follows from (15) that

$$\limsup_{n\to+\infty} \frac{1}{n}\log\|\mathcal{A}(n,1)x\| \le \rho_i - \lambda < \rho_i.$$

Finally, Lemma 1 allows one to let $\rho_i \searrow b_i$ (except for $i = p$ when I_p is unbounded) and so,

$$\limsup_{n\to+\infty} \frac{1}{n}\log\|\mathcal{A}(n,1)x\| \le b_i.$$

Similarly, since the sequence $(e^{-\rho_{i-1}}A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part, there exist projections $\tilde{P}_n$ satisfying (2), a constant $\tilde{\lambda} > 0$ and for each $\varepsilon > 0$ a constant $\tilde{D} = \tilde{D}(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)\tilde{P}_n\| \le \tilde{D}e^{(\rho_{i-1}-\tilde{\lambda})(m-n)+\varepsilon n} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)\tilde{Q}_n\| \le \tilde{D}e^{-(\tilde{\lambda}+\rho_{i-1})(n-m)+\varepsilon n} \quad \text{for } m \le n, \tag{16}$$

where $\tilde{Q}_n = \mathrm{Id} - \tilde{P}_n$. By Proposition 3, one can assume that $x \in \mathrm{Im}\,\tilde{Q}_1$ and thus, it follows from (16) that

$$\rho_{i-1} - \varepsilon < \tilde{\lambda} + \rho_{i-1} - \varepsilon \le \liminf_{n\to+\infty} \frac{1}{n} \log\|\mathcal{A}(n,1)x\|.$$

Letting $\varepsilon \to 0$ and $\rho_{i-1} \nearrow a_i$ (except for $i = 1$ when I_1 is unbounded), we obtain

$$\liminf_{n\to+\infty} \frac{1}{n} \log\|\mathcal{A}(n,1)x\| \ge a_i.$$

This completes the proof of the theorem. □

4. Lyapunov Exponents and Nonuniform Spectrum

In this section we consider the particular case when the dynamics defined by a sequence $(A_m)_{m\in\mathbb{N}}$ satisfies a nonuniform bounded growth condition. Namely, we assume that there exists $\mu > 0$ so that for each $\varepsilon > 0$ there exists $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)\| \le De^{\mu|m-n|+\varepsilon n} \tag{17}$$

for $m, n \in \mathbb{N}$. This implies that $\Sigma \subset [-\mu, \mu]$ and so, in view of Theorem 5, the nonuniform spectrum is given by

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_p, b_p] \tag{18}$$

for some $p \le d$. Moreover, $S_{\rho_0}(n) = \{0\}$ and $S_{\rho_p}(n) = \mathbb{R}^d$ (see (10)). Hence, again by Theorem 5, for each $v \in \mathbb{R}^d$ there exists $i \in \{1, \ldots, p\}$ such that

$$a_i \le \liminf_{k\to+\infty} \frac{1}{k} \log\|\mathcal{A}(k,1)v\| \le \limsup_{k\to+\infty} \frac{1}{k} \log\|\mathcal{A}(k,1)v\| \le b_i.$$

Now we consider the nonlinear dynamics

$$x(k+1) = A_k x(k) + f_k(x(k)) \quad k \in \mathbb{N}, \tag{19}$$

where each function $f_k \colon \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfies

$$\|f_k(x)\| \le \gamma_k\|x\|, \quad x \in \mathbb{R}^d, \tag{20}$$

for some sequence $(\gamma_k)_{k\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$\lim_{k\to+\infty} e^{\delta k}\gamma_k = 0 \quad \text{for some } \delta > 0. \tag{21}$$

Under these assumptions, the following result shows that the nonlinear dynamics in (19) essentially shares the nonuniform spectrum of the unperturbed linear dynamics. More precisely, each Lyapunov exponent of the nonlinear dynamics also belongs to some interval $[a_i, b_i]$.

Theorem 6. *Assume that conditions (17), (18) and (20), (21) hold. Then for any solution $(x(k))_{k\in\mathbb{N}}$ of (19) one of the following alternatives holds:*

1. *$x(k) = 0$ for any sufficiently large k;*
2. *there exists $i \in \{1, \dots, p\}$ such that*

$$a_i \leq \liminf_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \leq \limsup_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \leq b_i.$$

Proof. We begin with some auxiliary results.

Lemma 2. *We have*

$$\limsup_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \leq b_p.$$

Proof of the lemma. Take $d > b_p$. Then there exist $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m, n)\| \leq De^{(d-\lambda)(m-n)+\varepsilon n} \quad \text{for } m \geq n. \tag{22}$$

It follows from (19), (20) and (22) that

$$\begin{aligned}\|x(m)\| &\leq De^{(d-\lambda)(m-n)+\varepsilon n}\|x(n)\| + D' \sum_{k=n}^{m-1} e^{(d-\lambda)(m-k)+\varepsilon k}\gamma_k\|x(k)\| \\ &= De^{(d-\lambda)(m-n)+\varepsilon n}\|x(n)\| \\ &\quad + D'e^{(d-\lambda)(m-n)} \sum_{k=n}^{m-1} e^{-(d-\lambda)(k-n)+\varepsilon k}\gamma_k\|x(k)\|,\end{aligned}$$

where $D' = De^{\varepsilon-d+\lambda}$. Therefore,

$$e^{-(d-\lambda)(m-n)}\|x(m)\| \leq De^{\varepsilon n}\|x(n)\| + D' \sum_{k=n}^{m-1} e^{-(d-\lambda)(k-n)}\|x(k)\|e^{\varepsilon k}\gamma_k$$

for $m \geq n$. One can use induction to show that

$$\|x(m)\| \leq D\|x(n)\|e^{(d-\lambda)(m-n)+\varepsilon n}e^{\sum_{k=n}^{m-1} D'e^{\varepsilon k}\gamma_k} \tag{23}$$

for $m \geq n$. Taking $\varepsilon \leq \delta$, it follows from (21) that

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|x(m)\| \leq d - \lambda < d.$$

Finally, letting $d \searrow b_p$ we obtain the desired result. □

Lemma 3. *Assume that the first alternative in the theorem does not hold. Then*

$$\liminf_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \geq a_1.$$

Proof of the lemma. We first note that $x(1) \neq 0$, since otherwise it would follow from (23) that $x(n) = 0$ for all $n \in \mathbb{N}$. Now take $d < a_1$. Then there exist $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)\| \leq De^{-(d+\lambda)(n-m)+\varepsilon n} \quad \text{for } m \leq n.$$

Proceeding in a similar manner to that in the proof of Lemma 2, we find that

$$\|x(m)\| \leq D\|x(n)\|e^{-(d+\lambda)(n-m)+\varepsilon n}e^{\sum_{k=m}^{n-1} D'e^{\varepsilon k}\gamma_k}$$

for $m \leq n$, where $D' = De^{\varepsilon - d + \lambda}$. Hence,

$$\liminf_{k \to +\infty} \frac{1}{k} \log\|x(k)\| \geq d + \lambda - \varepsilon > d - \varepsilon.$$

Letting $\varepsilon \to 0$ and then $d \nearrow a$ we obtain the conclusion of the lemma. □

Now take $c \in \mathbb{R}\backslash\Sigma$. Then the sequence $(e^{-c}A_m)_{m\in\mathbb{N}}$ admits a nonuniform dichotomy with an arbitrarily small nonuniform part and so there exist projections P_m for $m \in \mathbb{N}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \leq De^{(c-\lambda)(m-n)+\varepsilon n} \quad \text{for } m \geq n \tag{24}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \leq De^{-(\lambda+c)(n-m)+\varepsilon n} \quad \text{for } m \leq n. \tag{25}$$

By Propositions 1 and 4, the projections P_m depend only on the connected component of $\mathbb{R}\backslash\Sigma$ to which c belongs. For each $n \in \mathbb{N}$, we consider the norm

$$\begin{aligned}\|x\|_n = &\sup_{m \geq n} \left(\|\mathcal{A}(m,n)P_n x\|e^{-(c-\lambda)(m-n)}\right) \\ &+ \sup_{m \leq n} \left(\|\mathcal{A}(m,n)Q_n x\|e^{(\lambda+c)(n-m)}\right).\end{aligned}$$

It follows readily from (24) and (25) that

$$\|x\| \leq \|x\|_n \leq 2De^{\varepsilon n}\|x\| \tag{26}$$

for $n \in \mathbb{N}$ and $x \in X$. Moreover, we have the following bounds.

Lemma 4. *For each $x \in \mathbb{R}^d$ we have*

$$\|\mathcal{A}(m,n)P_n x\|_m \leq e^{(c-\lambda)(m-n)}\|x\|_n \quad \textit{for } m \geq n \tag{27}$$

and

$$\|\mathcal{A}(m,n)Q_n x\|_m \leq e^{-(\lambda+c)(n-m)}\|x\|_n \quad \textit{for } m \leq n. \tag{28}$$

Proof of the lemma. We have

$$\begin{aligned}\|\mathcal{A}(m,n)P_nx\|_m &= \sup_{k\geq m}\big(\|\mathcal{A}(k,m)P_m\mathcal{A}(m,n)P_nx\|e^{-(c-\lambda)(k-m)}\big)\\ &= \sup_{k\geq m}\big(\|\mathcal{A}(k,n)P_nx\|e^{-(c-\lambda)(k-m)}\big)\\ &= e^{(c-\lambda)(m-n)}\sup_{k\geq m}\big(\|\mathcal{A}(k,n)P_nx\|e^{-(c-\lambda)(k-n)}\big)\\ &\leq e^{(c-\lambda)(m-n)}\|x\|_n\end{aligned}$$

for $m\geq n$ and so (27) holds. The proof of (28) is completely analogous. □

We write

$$x_P(k)=P_kx(k) \quad\text{and}\quad x_Q(k)=Q_kx(k).$$

It follows from (2) and (19) that

$$x_P(k+1)=A_kx_P(k)+P_{k+1}f_k(x(k))$$

and

$$x_Q(k+1)=A_kx_Q(k)+Q_{k+1}f_k(x(k))$$

for $k\in\mathbb{N}$. By (20), (24), (26) and (27), we obtain

$$\begin{aligned}\|x_P(k+1)\|_{k+1} &\leq e^{c-\lambda}\|x_P(k)\|_k+\|P_{k+1}f_k(x(k))\|_{k+1}\\ &\leq e^{c-\lambda}\|x_P(k)\|_k+2De^{\varepsilon(k+1)}\|P_{k+1}f_k(x(k))\|\\ &\leq e^{c-\lambda}\|x_P(k)\|_k+2D^2e^{2\varepsilon(k+1)}\|f_k(x(k))\|\\ &\leq e^{c-\lambda}\|x_P(k)\|_k+2D^2e^{2\varepsilon(k+1)}\gamma_k\|x(k)\|\\ &\leq e^{c-\lambda}\|x_P(k)\|_k+2D^2e^{2\varepsilon(k+1)}\gamma_k\big(\|x_P(k)\|_k+\|x_Q(k)\|_k\big).\end{aligned}$$

Hence,

$$\|x_P(k+1)\|_{k+1}\leq e^{c-\lambda}\|x_P(k)\|_k+D'\eta_k\big(\|x_P(k)\|_k+\|x_Q(k)\|_k\big) \tag{29}$$

for $k\in\mathbb{N}$, where

$$D'=2D^2e^{2\varepsilon} \quad\text{and}\quad \eta_k=e^{2\varepsilon k}\gamma_k.$$

Similarly, by (20), (25), (26) and (28), we obtain

$$\|x_Q(k+1)\|_{k+1}\geq e^{c+\lambda}\|x_Q(k)\|_k-D'\eta_k\big(\|x_P(k)\|_k+\|x_Q(k)\|_k\big) \tag{30}$$

for $k\in\mathbb{N}$.

Lemma 5. *Either*

$$\|x_Q(k)\|_k\leq\|x_P(k)\|_k \quad \textit{for any sufficiently large } k \tag{31}$$

or

$$\|x_P(k)\|_k<\|x_Q(k)\|_k \quad \textit{for any sufficiently large } k. \tag{32}$$

Proof of the lemma. It follows from (29) and (30) that

$$\|x_P(k+1)\|_{k+1} \le (e^{c-\lambda} + D'\eta_k)\|x_P(k)\|_k + D'\eta_k\|x_Q(k)\|_k \tag{33}$$

and

$$\|x_Q(k+1)\|_{k+1} \ge (e^{c+\lambda} - D'\eta_k)\|x_Q(k)\|_k - D'\eta_k\|x_P(k)\|_k. \tag{34}$$

Now we assume that (31) does not hold. Then there exists an arbitrarily large k_0 such that

$$\|x_P(k_0)\|_{k_0} < \|x_Q(k_0)\|_{k_0}.$$

We shall prove by induction that if k_0 is sufficiently large, then $\|x_P(k)\|_k < \|x_Q(k)\|_k$ for $k \ge k_0$. So, let us assume that $\|x_P(k)\|_k < \|x_Q(k)\|_k$ for some $k \ge k_0$. By (33) and (34), we obtain

$$\|x_P(k+1)\|_{k+1} \le (e^{c-\lambda} + 2D'\eta_k)\|x_Q(k)\|_k$$

and

$$\|x_Q(k+1)\|_{k+1} \ge (e^{c+\lambda} - 2D'\eta_k)\|x_Q(k)\|_k.$$

Therefore,

$$\|x_P(k+1)\|_{k+1} \le \frac{e^{c-\lambda} + 2D'\eta_k}{e^{c+\lambda} - 2D'\eta_k}\|x_Q(k+1)\|_{k+1}.$$

Taking ε sufficiently small, it follows from (21) that $\eta_k = e^{2\varepsilon k}\gamma_k \to 0$ and thus,

$$\frac{e^{c-\lambda} + 2D'\eta_k}{e^{c+\lambda} - 2D'\eta_k} \to \frac{e^{c-\lambda}}{e^{c+\lambda}} < 1.$$

Hence, if k_0 is sufficiently large, then

$$\|x_P(k+1)\|_{k+1} < \|x_Q(k+1)\|_{k+1}.$$

This completes the proof of the lemma. □

The final ingredient is the following result.

Lemma 6. *Assume that the first alternative in the theorem does not hold. Then one of the following alternatives holds:*

1.
$$\limsup_{k\to+\infty} \frac{1}{k}\log\|x(k)\| < c \tag{35}$$
and
$$\lim_{k\to+\infty} \frac{\|x_Q(k)\|_k}{\|x_P(k)\|_k} = 0; \tag{36}$$

2.
$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x(k)\| > c \tag{37}$$
and
$$\lim_{k\to+\infty} \frac{\|x_P(k)\|_k}{\|x_Q(k)\|_k} = 0. \tag{38}$$

Proof of the lemma. Assume first that (31) holds and let

$$S = \limsup_{k\to+\infty} \frac{\|x_Q(k)\|_k}{\|x_P(k)\|_k}.$$

It follows from (31) that $0 \le S \le 1$. By (29) and (31), we have

$$\|x_P(k+1)\|_{k+1} \le (e^{c-\lambda} + 2D'\eta_k)\|x_P(k)\|_k \tag{39}$$

for all large k. Hence, it follows from (30) and (39) that

$$\frac{\|x_Q(k+1)\|_{k+1}}{\|x_P(k+1)\|_{k+1}} \ge \frac{e^{c+\lambda} - D'\eta_k}{e^{c-\lambda} + 2D'\eta_k} \cdot \frac{\|x_Q(k)\|_k}{\|x_P(k)\|_k} - \frac{D'\eta_k}{e^{c-\lambda} + 2D'\eta_k}$$

for all large k. Taking ε sufficiently small and using (21), we obtain

$$\frac{e^{c+\lambda} - D'\eta_k}{e^{c-\lambda} + 2D'\eta_k} \to \frac{e^{c+\lambda}}{e^{c-\lambda}} > 1 \quad \text{and} \quad \frac{D'\eta_k}{e^{c-\lambda} + 2D'\eta_k} \to 0,$$

which implies that $S = 0$. This establishes (36). In order to prove (35), take k_0 so large such that (39) holds for $k \ge k_0$. Then

$$\|x_P(k)\|_k \le \|x_P(k_0)\|_{k_0} e^{(c-\lambda)(k-k_0)} \prod_{j=k_0}^{k} (1 + 2D'\eta_k e^{\lambda-c})$$

for $k \ge k_0$. On the other hand, if follows from (21) that

$$\frac{1}{k}\sum_{j=k_0}^{k} \log(1 + 2D'\eta_k e^{\lambda-c}) \le \frac{1}{k}\sum_{j=k_0}^{k} 2D'\eta_k e^{\lambda-c} \to 0$$

and so,

$$\limsup_{k\to+\infty} \frac{1}{k} \log\|x_P(k)\|_k \le c - \lambda < c.$$

Finally, by (26) and (31), we have

$$\limsup_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \le \limsup_{k\to+\infty} \frac{1}{k} \log(2\|x_P(k)\|_k) = \limsup_{k\to+\infty} \frac{1}{k} \log\|x_P(k)\|_k,$$

which establishes inequality (35).

Now assume that (32) holds and let

$$S = \limsup_{k\to+\infty} \frac{\|x_P(k)\|_k}{\|x_Q(k)\|_k}.$$

It follows from (32) that $0 \le S \le 1$. By (30) and (32), we have

$$\|x_Q(k+1)\|_{k+1} \ge (e^{c+\lambda} - 2D'\eta_k)\|x_Q(k)\|_k, \tag{40}$$

for all large k. Hence, it follows from (29) and (40) that

$$\frac{\|x_P(k+1)\|_{k+1}}{\|x_Q(k+1)\|_{k+1}} \le \frac{e^{c-\lambda} + D'\eta_k}{e^{c+\lambda} - 2D'\eta_k} \cdot \frac{\|x_P(k)\|_k}{\|x_Q(k)\|_k} + \frac{D'\eta_k}{e^{c+\lambda} - 2D'\eta_k}$$

for all large k. Taking ε sufficiently small and using (21), we obtain

$$\frac{e^{c-\lambda}+D'\eta_k}{e^{c+\lambda}-2D'\eta_k} \to \frac{e^{c-\lambda}}{e^{c+\lambda}} < 1 \quad \text{and} \quad \frac{D'\eta_k}{e^{c+\lambda}-2D'\eta_k} \to 0,$$

which implies that $S=0$. This establishes (38). Now take k_0 so large such that (40) holds for $k \ge k_0$. Then

$$\|x_Q(k)\|_k \ge \|x_Q(k_0)\|_{k_0} e^{(c+\lambda)(k-k_0)} \prod_{j=k_0}^{k} (1-2D'\eta_k e^{-\lambda-c})$$

for $k \ge k_0$. On the other hand, it follows from (21) that

$$\frac{1}{k}\sum_{j=k_0}^{k} \log \frac{1}{1-2D'\eta_k e^{-\lambda-c}} \le \frac{1}{k}\sum_{j=k_0}^{k} \frac{2D'\eta_k e^{-\lambda-c}}{1-2D'\eta_k e^{-\lambda-c}} \to 0$$

and so,

$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x_Q(k)\|_k \ge c+\lambda > c.$$

Finally, by (26),

$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x(k)\| \ge \liminf_{k\to+\infty} \frac{1}{k}\log\left(\frac{1}{2D}e^{-\varepsilon k}\|x_Q(k)\|_k\right) > c-\varepsilon.$$

Since ε can be taken arbitrarily small, we conclude that (37) holds. □

We proceed with the proof of the theorem. Let $(x(k))_{k\in\mathbb{N}}$ be a solution of (19) and assume that the first alternative in the theorem does not hold. Take $c_0 < a_1$, $c_p > b_p$ and $c_i \in (b_i, a_{i+1})$ for $i \in \{1,\dots,p-1\}$. It follows from Lemma 6 that for each $i \in \{0,\dots,p\}$ either

$$\limsup_{k\to+\infty} \frac{1}{k}\log\|x(k)\| < c_i$$

or

$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x(k)\| > c_i.$$

Together with Lemmas 2 and 3, this implies that there exists $i \in \{1,\dots,p\}$ such that

$$\limsup_{k\to+\infty} \frac{1}{k}\log\|x(k)\| < c_i$$

and

$$\liminf_{k\to+\infty} \frac{1}{k}\log\|x(k)\| > c_{i-1}.$$

Finally, letting $c_{i-1} \nearrow a_i$ and $c_i \searrow b_i$, we obtain

$$a_i \le \liminf_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \le \limsup_{k\to+\infty} \frac{1}{k} \log\|x(k)\| \le b_i.$$

This completes the proof of the theorem. □

5. Lyapunov Regularity

In this section we consider the nonuniform spectrum of a Lyapunov regular trajectory. We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of invertible $d \times d$ matrices is *Lyapunov regular* if there exist a decomposition

$$\mathbb{R}^d = \bigoplus_{i=1}^{s} E_i$$

and real numbers $\lambda_1 < \cdots < \lambda_s$ such that:

1. for $i = 1, \ldots, s$ and $v \in E_i \setminus \{0\}$, we have
$$\lim_{n\to+\infty} \frac{1}{n} \log\|\mathcal{A}(n,1)v\| = \lambda_i; \tag{41}$$
2. $$\lim_{n\to+\infty} \frac{1}{n} \log|\det \mathcal{A}(n,1)| = \sum_{i=1}^{s} \lambda_i \dim E_i.$$

Theorem 7. *If the sequence $(A_n)_{n\in\mathbb{N}}$ is Lyapunov regular, then*

$$\Sigma = \{\lambda_1, \ldots, \lambda_s\}.$$

Proof. Take $a \in \mathbb{R}$ such that $a \ne \lambda_i$ for $i \in \{1, \ldots, s\}$. The Lyapunov exponents associated to the sequence $(e^{-a}A_n)_{n\in\mathbb{N}}$ are the nonzero numbers $-a+\lambda_i$, for $i = 1, \ldots, s$. Now let P_1 and Q_1 be the projections associated to the decomposition

$$\mathbb{R}^d = \left(\bigoplus_{i:\lambda_i<a} E_i\right) \oplus \left(\bigoplus_{i:\lambda_i>a} E_i\right).$$

It follows from Theorem 4 in [4] that the sequence $(e^{-a}A_n)_{n\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part, with projections

$$P_n = \mathcal{A}(n,1)P_1\mathcal{A}(1,n), \quad n \in \mathbb{N}.$$

Hence, $a \notin \Sigma$ and $\Sigma \subset \{\lambda_1, \ldots, \lambda_s\}$.

For the reverse inclusion, take $i \in \{1, \ldots, s\}$ and assume that the sequence $(e^{-\lambda_i}A_n)_{n\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Then there exist projections P_m for $m \in \mathbb{N}$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ satisfying (2) such that

$$e^{-\lambda_i(m-n)}\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \ge n \tag{42}$$

and

$$e^{-\lambda_i(m-n)}\|\mathcal{A}(m,n)(\mathrm{Id}-P_n)\| \le De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \le n. \tag{43}$$

For $v \in E_i \setminus \{0\}$, it follows from (42) that

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,1)P_1 v\| \le -\lambda + \lambda_i < \lambda_i. \tag{44}$$

Hence, by (41), we have $P_1 v \ne v$. On the other hand, by (43),

$$\frac{1}{D} e^{(\lambda+\lambda_i-\varepsilon)m}\|(\mathrm{Id}-P_1)v\| \le \|\mathcal{A}(m,1)(\mathrm{Id}-P_1)v\|$$

for $m \ge 0$. Since $P_1 v \ne v$, we obtain

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,1)(\mathrm{Id}-P_1)v\| \ge \lambda + \lambda_i - \varepsilon > \lambda_i$$

for any sufficiently small $\varepsilon > 0$, which together with (44) show that

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,1)v\| > \lambda_i.$$

But this contradicts to (41). Therefore, $\lambda_i \in \Sigma$ and since i is arbitrary, we conclude that $\{\lambda_1, \ldots, \lambda_s\} \subset \Sigma$. □

The following result established in [5] can now be obtained as a direct consequence of Theorems 6 and 7.

Theorem 8. *Assume that $(A_m)_{m\in\mathbb{N}}$ is a Lyapunov regular sequence and let $(x(k))_{k\in\mathbb{N}}$ be a solution of (19), for some sequence $(f_k)_{k\in\mathbb{N}}$ satisfying properties (20) and (21). Then one of the following alternatives holds:*

1. *$x(k) = 0$ for any sufficiently large k;*
2. *there exists $i \in \{1, \ldots, s\}$ such that*

$$\lim_{k\to+\infty} \frac{1}{k} \log\|x(k)\| = \lambda_i.$$

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