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ONE-SIDED DICHOTOMIES VERSUS TWO-SIDED DICHOTOMIES: ARBITRARY GROWTH RATES

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Abstract. We obtain necessary and sufficient conditions for the existence of two-sided nonuniform exponential dichotomies with arbitrary growth rates in terms of the existence of one-sided nonuniform exponential dichotomies on the past and on the future. We consider both linear nonautonomous dynamics with discrete and continuous time, on an arbitrary Banach space.

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1. Introduction. Our main objective is to formulate necessary and sufficient conditions for the existence of two-sided nonuniform exponential dichotomies with arbitrary growth rates in terms of the existence of one-sided nonuniform exponential dichotomies on the past and on the future. We consider both a nonautonomous dynamics with discrete time defined by a sequence of linear operators A_m acting on a Banach space and a nonautonomous dynamics with continuous time induced by a linear equation $x' = A(t)x$ on a Banach space.

Results of this type were first established by Coppel [3], for uniform exponential dichotomies on a finite-dimensional space. Recently, in [1] we obtained an extension to nonuniform exponential dichotomies on a Banach space that are not necessarily

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invertible. For example, in the case of discrete time we showed that a sequence of linear operators admitting nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ also admits a nonuniform exponential dichotomy on $\mathbb{Z}$ if and only if the ambient space is the direct sum of the stable space of $\mathbb{Z}_0^+$ at the origin and the unstable space of $\mathbb{Z}_0^-$ at the origin. We emphasize that the arguments heavily rely on the relation to an admissibility property.

The main difficulty in adapting the approach in [1] to nonuniform exponential dichotomies with arbitrary growth rates is that, to the best of our knowledge, there exists no characterization in the literature in terms of an admissibility property (even in the particular case of uniform exponential dichotomies). This complication forced us to use a different approach: after using Lyapunov norms to transform the nonuniform behavior into an uniform one, we study the kernels of the projections on the future and the ranges of the projections on the past. Unsurprisingly, the condition ensuring that one-sided exponential dichotomies on future and past induce a two-sided exponential dichotomy on $\mathbb{Z}$ is the same as in the exponential case. Since (uniform and nonuniform) exponential dichotomies are particular cases of the dichotomies studied in the paper, our results give much shorter proofs of the corresponding results in [1], although at the expense of not being able to consider the general case of a noninvertible dynamics.

The main motivation for considering exponential dichotomies with arbitrary growth rates comes from results in [2]; that paper gives sufficient conditions for the existence of those exponential dichotomies when some associated Lyapunov exponents are nonzero. We note that with this general notion of an exponential dichotomy one is able to consider dynamics for which all classical Lyapunov exponents are zero or infinite.

Our results should be compared with work of Pliss [4] for a nonautonomous dynamics with discrete time on a finite-dimensional space, in which the condition "the ambient space is the direct sum of the stable space of $\mathbb{Z}_0^+$ at the origin and the unstable space of $\mathbb{Z}_0^-$ at the origin" is replaced by "the stable space of $\mathbb{Z}_0^+$ at the origin and the unstable space of $\mathbb{Z}_0^-$ at the origin generate the ambient space". Under the latter condition, the existence of exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ in general does not guarantee the existence of an exponential dichotomy on $\mathbb{Z}$, but instead only a property that is weaker than admissibility. In this property, "existence and uniqueness of bounded solutions" is replaced by "existence of bounded solutions" $(x_n)_{n\in\mathbb{Z}}$ of the equation

$$x_{n+1} - A_n x_n = y_{n+1} \quad \text{for } n \in \mathbb{Z}$$

for each bounded perturbation $(y_n)_{n\in\mathbb{Z}}$.

2. Discrete time. Let $B(X)$ be the set of all bounded linear operators acting on a Banach space $X = (X, \|\cdot\|)$. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators in $B(X)$, we define

$$\mathcal{A}(n,m) = \begin{cases} A_{n-1} \cdots A_m & \text{if } n > m, \\ \text{Id} & \text{if } n = m, \\ A_n^{-1} \cdots A_{m-1}^{-1} & \text{if } m < n \end{cases}$$

for $n, m \in \mathbb{Z}$. Now let $I \in \{\mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}_0^-\}$, where

$$\mathbb{Z}_0^+ = \{n \in \mathbb{Z} : n \geq 0\} \quad \text{and} \quad \mathbb{Z}_0^- = \{n \in \mathbb{Z} : n \leq 0\}.$$

Given an increasing function $\rho \colon \mathbb{Z} \to \mathbb{Z}$ with $\rho(0) = 0$, we say that $(A_m)_{m\in\mathbb{Z}}$ has a *ρ-nonuniform (exponential) dichotomy on I* if:

1. there exist projections $P_m \in B(X)$ for $m \in I$ satisfying
$$\mathcal{A}(n, m)P_m = P_n\mathcal{A}(n, m) \quad \text{for } n, m \in I; \tag{1}$$
2. there exist $\lambda, D > 0$ and $\varepsilon \geq 0$ such that for $n, m \in I$ we have
$$\|\mathcal{A}(n, m)P_m\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(m)|} \quad \text{for } n \geq m \tag{2}$$
and
$$\|\mathcal{A}(n, m)Q_m\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(m)|} \quad \text{for } n \leq m, \tag{3}$$
where $Q_m = \mathrm{Id} - P_m$.

One can easily verify that

$$\operatorname{Im} P_m \setminus \{0\} = \left\{v \in X \setminus \{0\} : \limsup_{n\to\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n, m)v\|_n < 0\right\}$$

for $m \geq 0$ (whenever $\mathbb{Z}_0^+ \subset I$) and

$$\operatorname{Im} Q_m \setminus \{0\} = \left\{v \in X \setminus \{0\} : \liminf_{n\to-\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n, m)v\|_n > 0\right\}$$

for $m \leq 0$ (whenever $\mathbb{Z}_0^- \subset I$).

THEOREM 1. *A sequence $A = (A_m)_{m\in\mathbb{Z}}$ of invertible linear operators in $B(X)$ has a ρ-nonuniform dichotomy on $\mathbb{Z}$ if and only if there exists projections P_m^+ for $m \geq 0$ and projections P_m^- for $m \leq 0$ such that:*

1. *A has a ρ-nonuniform dichotomy on $\mathbb{Z}_0^+$ with projections P_m^+;*
2. *A has a ρ-nonuniform dichotomy on $\mathbb{Z}_0^-$ with projections P_m^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

Proof. Clearly, properties 1–3 hold if the sequence $(A_m)_{m\in\mathbb{Z}}$ has a ρ-nonuniform dichotomy on $\mathbb{Z}$.

Now assume that properties 1–3 hold. In particular, there exist $\lambda, D > 0$ and $\varepsilon \geq 0$ such that

$$\begin{aligned}
\|\mathcal{A}(n, m)P_m^+\| &\leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon\rho(m)} && \text{for } n \geq m \geq 0,\\
\|\mathcal{A}(n, m)Q_m^+\| &\leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon\rho(m)} && \text{for } 0 \leq n \leq m,\\
\|\mathcal{A}(n, m)P_m^-\| &\leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(m)|} && \text{for } 0 \geq n \geq m,\\
\|\mathcal{A}(n, m)Q_m^-\| &\leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(m)|} && \text{for } n \leq m \leq 0,
\end{aligned} \tag{4}$$

where $Q^+_m = \mathrm{Id} - P^+_m$ and $Q^-_m = \mathrm{Id} - P^-_m$. For each $m \in \mathbb{Z}$, we define a norm $\|\cdot\|_m$ on X by

$$\|x\|_m = \begin{cases} \|x\|^+_m & \text{if } m \ge 0, \\ \|x\|^-_m & \text{if } m < 0, \end{cases} \tag{5}$$

where

$$\|x\|^+_m = \sup_{n \ge m} \big(\|\mathcal{A}(n,m) P^+_m x\| e^{\lambda(\rho(n)-\rho(m))} \big) + \sup_{0 \le n \le m} \big(\|\mathcal{A}(n,m) Q^+_m x\| e^{\lambda(\rho(m)-\rho(n))} \big)$$

and

$$\|x\|^-_m = \sup_{0 \ge n \ge m} \big(\|\mathcal{A}(n,m) P^-_m x\| e^{\lambda(\rho(n)-\rho(m))} \big) + \sup_{n \le m} \big(\|\mathcal{A}(n,m) Q^-_m x\| e^{\lambda(\rho(m)-\rho(n))} \big).$$

By (4), for each $x \in X$ we have

$$\|x\| \le \|x\|^+_m \le D e^{\varepsilon \rho(m)} \|x\| \quad \text{for } m \ge 0 \tag{6}$$

and

$$\|x\| \le \|x\|^-_m \le D e^{\varepsilon |\rho(m)|} \|x\| \quad \text{for } m \le 0. \tag{7}$$

Hence,

$$\|x\| \le \|x\|_m \le D e^{\varepsilon |\rho(m)|} \|x\| \quad \text{for } m \in \mathbb{Z},\ x \in X. \tag{8}$$

Moreover, for each $x \in X$ we have

$$\begin{aligned} \|\mathcal{A}(n,m) P^+_m x\|^+_n &\le e^{-\lambda(\rho(n)-\rho(m))} \|x\|^+_m \quad \text{for } n \ge m \ge 0, \\ \|\mathcal{A}(n,m) Q^+_m x\|^+_n &\le e^{-\lambda(\rho(m)-\rho(n))} \|x\|^+_m \quad \text{for } 0 \le n \le m, \\ \|\mathcal{A}(n,m) P^-_m\|^-_n &\le e^{-\lambda(\rho(n)-\rho(m))} \|x\|^-_m \quad \text{for } 0 \ge n \ge m, \\ \|\mathcal{A}(n,m) Q^-_m x\|^-_n &\le e^{-\lambda(\rho(m)-\rho(n))} \|x\|^-_m \quad \text{for } n \le m \le 0. \end{aligned} \tag{9}$$

On the other hand, it follows from (1) and property 3 that

$$X = \operatorname{Im} P^+_m \oplus \mathcal{A}(m,0) \operatorname{Ker} P^-_0 \quad \text{for } m \ge 0. \tag{10}$$

Let $\tilde{P}^+_m \colon X \to \operatorname{Im} P^+_m$ and $\tilde{Q}^+_m \colon X \to \mathcal{A}(m,0) \operatorname{Ker} P^-_0$ be the projections associated with the decomposition in (10).

Lemma 1. *There exist $\lambda', D' > 0$ such that for each $x \in X$ we have*

$$\|\mathcal{A}(n,m) \tilde{P}^+_m x\|^+_n \le D' e^{-\lambda'(\rho(n)-\rho(m))} \|x\|^+_m \quad \text{for } n \ge m \ge 0 \tag{11}$$

and

$$\|\mathcal{A}(n,m) \tilde{Q}^+_m x\|^+_n \le D' e^{-\lambda'(\rho(m)-\rho(n))} \|x\|^+_m \quad \text{for } 0 \le n \le m.$$

Proof of the lemma. Since $\operatorname{Im} P^+_m = \operatorname{Im} \tilde{P}^+_m$, we have

$$P^+_m \tilde{P}^+_m = \tilde{P}^+_m \quad \text{and} \quad \tilde{P}^+_m P^+_m = P^+_m.$$

Hence,

$$P_m^+ - \tilde{P}_m^+ = P_m^+(P_m^+ - \tilde{P}_m^+) = (P_m^+ - \tilde{P}_m^+)Q_m^+$$

and it follows from (6) and (9) that

$$\begin{aligned}\|\mathcal{A}(n,0)(P_0^+ - \tilde{P}_0^+)v\|_n^+ &= \|\mathcal{A}(n,0)P_0^+(P_0^+ - \tilde{P}_0^+)v\|_n^+\\ &\le e^{-\lambda\rho(n)}\|(P_0^+ - \tilde{P}_0^+)v\|_0^+\\ &= e^{-\lambda\rho(n)}\|(P_0^+ - \tilde{P}_0^+)Q_0^+v\|_0^+\\ &\le De^{-\lambda\rho(n)}\|P_0^+ - \tilde{P}_0^+\|\cdot\|Q_0^+v\|_0^+\\ &= De^{-\lambda\rho(n)}\|P_0^+ - \tilde{P}_0^+\|\cdot\|\mathcal{A}(0,m)\mathcal{A}(m,0)Q_0^+v\|_0^+\\ &= De^{-\lambda\rho(n)}\|P_0^+ - \tilde{P}_0^+\|\cdot\|\mathcal{A}(0,m)Q_m^+\mathcal{A}(m,0)v\|_0^+\\ &\le De^{-\lambda(\rho(n)+\rho(m))}\|P_0^+ - \tilde{P}_0^+\|\cdot\|\mathcal{A}(m,0)v\|_m^+\end{aligned}$$

for $m, n \in \mathbb{Z}_0^+$ and $v \in X$. Since

$$\begin{aligned}\mathcal{A}(n,m)(P_m^+ - \tilde{P}_m^+) &= \mathcal{A}(n,m)(P_m^+ - \tilde{P}_m^+)Q_m^+\\ &= (P_n^+ - \tilde{P}_n^+)\mathcal{A}(n,m)Q_m^+\\ &= (P_n^+ - \tilde{P}_n^+)\mathcal{A}(n,0)Q_0^+\mathcal{A}(0,m)Q_m^+\\ &= \mathcal{A}(n,0)(P_0^+ - \tilde{P}_0^+)Q_0^+\mathcal{A}(0,m)Q_m^+\\ &= \mathcal{A}(n,0)(P_0^+ - \tilde{P}_0^+)\mathcal{A}(0,m)Q_m^+,\end{aligned}$$

we obtain

$$\begin{aligned}\|\mathcal{A}(n,m)\tilde{P}_m^+v\|_n^+ &\le \|\mathcal{A}(n,m)P_m^+v\| + \|\mathcal{A}(n,m)(P_m^+ - \tilde{P}_m^+)v\|_n^+\\ &= \|\mathcal{A}(n,m)P_m^+v\|_n^+ + \|\mathcal{A}(n,0)(P_0^+ - \tilde{P}_0^+)\mathcal{A}(0,m)Q_m^+v\|\|_n^+\\ &\le e^{-\lambda(\rho(n)-\rho(m))}\|v\|_m^+ + De^{-\lambda(\rho(n)-\rho(m))}\|P_0^+ - \tilde{P}_0^+\|\cdot\|Q_m^+v\|_m^+\\ &= D'e^{-\lambda(\rho(n)-\rho(m))}\|v\|_m^+\end{aligned}$$

for $n \ge m$, $v \in X$ and some constant $D' > 0$. Similarly,

$$\begin{aligned}\|\mathcal{A}(n,m)\tilde{Q}_m^+v\|_n^+ &\le \|\mathcal{A}(n,m)Q_m^+v\|_n^+ + \|\mathcal{A}(n,m)(P_m^+ - \tilde{P}_m^+)v\|_n^+\\ &= \|\mathcal{A}(n,m)Q_m^+v\|_n^+ + \|\mathcal{A}(n,0)(P_0^+ - \tilde{P}_0^+)\mathcal{A}(0,m)Q_m^+v\|_n^+\\ &\le e^{-\lambda(\rho(m)-\rho(n))}\|v\|_m^+ + De^{-\lambda(\rho(m)-\rho(n))}\|P_0^+ - \tilde{P}_0^+\|\cdot\|Q_m^+v\|_m^+\\ &= D'e^{-\lambda(\rho(m)-\rho(n))}\|v\|_m^+\end{aligned}$$

for $n \le m$ and $v \in X$. □

It follows also from property 3 that

$$X = \mathcal{A}(m,0)\operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_m^- \quad \text{for } m \le 0. \tag{12}$$

Let $\tilde{P}_m^-$ and $\tilde{Q}_m^-$ be the projections associated with this decomposition. Finally, the following version of Lemma 1 can be obtained using similar arguments.

Lemma 2. *There exist $\lambda'', D'' > 0$ such that for each $x \in X$ we have*

$$\|\mathcal{A}(n,m)\tilde{P}_m^- x\|_n^- \le D'' e^{-\lambda''(\rho(n)-\rho(m))}\|x\|_m^- \quad \text{for } m \le n \le 0 \tag{13}$$

and

$$\|\mathcal{A}(n,m)\tilde{Q}_m^- x\|_n^+ \le D'' e^{-\lambda''(\rho(m)-\rho(n))}\|x\|_m^- \quad \text{for } n \le m \le 0.$$

It follows from (10) and (12) that $\tilde{P}_0^+ = \tilde{P}_0^-$ and $\tilde{Q}_0^+ = \tilde{Q}_0^-$. Now let

$$P_n = \begin{cases} \tilde{P}_n^+ & \text{if } n \ge 0, \\ \tilde{P}_n^- & \text{if } n < 0, \end{cases}$$

and $Q_n = \mathrm{Id} - P_n$. For $m, n \in \mathbb{Z}$ with $n \ge 0 > m$, it follows from (6), (7), (11) and (13) that

$$\begin{aligned} \|\mathcal{A}(m,n)P_n x\|_m &= \|\mathcal{A}(m,0)\tilde{P}_0^+ \mathcal{A}(0,n)\tilde{P}_n^- x\|_m^+ \\ &\le D' e^{-\lambda'\rho(m)}\|\mathcal{A}(0,n)\tilde{P}_n^- x\|_0^+ \\ &\le DD' e^{-\lambda'\rho(m)}\|\mathcal{A}(0,n)\tilde{P}_n^- x\| \\ &\le DD' e^{-\lambda'\rho(m)}\|\mathcal{A}(0,n)\tilde{P}_n^- x\|_0^- \\ &\le DD'D'' e^{-\lambda'\rho(m)} e^{\lambda''\rho(n)}\|x\|_n^- \\ &\le K e^{-a(\rho(m)-\rho(n))}\|x\|_n, \end{aligned}$$

for $x \in X$, where $K = DD'D'' > 0$ and $a = \max\{\lambda', \lambda''\} > 0$. Together with (11) and (13), this shows that there exist $b, N > 0$ such that

$$\|\mathcal{A}(m,n)P_n x\|_m \le N e^{-b(\rho(m)-\rho(n))}\|x\|_n \quad \text{for } x \in X,\ m \ge n$$

and it follows from (8) that

$$\|\mathcal{A}(m,n)P_n x\| \le DN e^{-b(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}\|x\| \quad \text{for } x \in X,\ m \ge n. \tag{14}$$

One can show in a similar manner that

$$\|\mathcal{A}(m,n)P_n x\| \le DN e^{-b(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}\|x\| \quad \text{for } x \in X,\ m \le n. \tag{15}$$

By (14) and (15), the sequence $(A_m)_{m\in\mathbb{Z}}$ has a ρ-nonuniform dichotomy. □

Now we consider the case of strong nonuniform exponential dichotomies. Given $I = \{\mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}_0^-\}$ and an increasing function $\rho\colon \mathbb{Z} \to \mathbb{Z}$ with $\rho(0) = 0$, we say that a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators in $B(X)$ has a *ρ-strong nonuniform (exponential) dichotomy on I* if:

1. there exist projections $P_m \in B(X)$ for $m \in I$ satisfying (1);
2. there exist $\lambda, \mu, D > 0$ and $\varepsilon \ge 0$ such that, in addition to inequalities (2) and (3), for $n, m \in I$ we have

$$\|\mathcal{A}(n,m)P_m\| \le D e^{-\mu(\rho(n)-\rho(m))+\varepsilon|\rho(m)|} \quad \text{for } n \le m$$

and

$$\|\mathcal{A}(n,m)Q_m\| \le D e^{-\mu(\rho(m)-\rho(n))+\varepsilon|\rho(m)|} \quad \text{for } n \ge m.$$

THEOREM 2. *A sequence $A=(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators in $B(X)$ has a ρ-strong nonuniform dichotomy on $\mathbb{Z}$ if and only if there exists projections P_m^+ for $m\ge 0$ and projections P_m^- for $m\le 0$ such that:*

1. *A has a ρ-strong nonuniform dichotomy on $\mathbb{Z}_0^+$ with projections P_m^+;*
2. *A has a ρ-strong nonuniform dichotomy on $\mathbb{Z}_0^-$ with projections P_m^-;*
3. *$X=\operatorname{Im}P_0^+\oplus\operatorname{Ker}P_0^-$.*

The proof of Theorem 2 is entirely analogous to the proof of Theorem 1, by defining norms $\|\cdot\|_m$ on X as in (5), where

$$\begin{aligned}\|x\|_m^+ = \max\Big\{ &\sup_{n\ge m}\big(\|\mathcal{A}(n,m)P_m^+x\|e^{\lambda(\rho(n)-\rho(m))}\big),\\ &\sup_{0\le n\le m}\big(\|\mathcal{A}(n,m)P_m^+x\|e^{\mu(\rho(n)-\rho(m))}\big)\Big\}\\ &+\max\Big\{\sup_{0\le n\le m}\big(\|\mathcal{A}(n,m)Q_m^+x\|e^{\lambda(\rho(m)-\rho(n))}\big),\\ &\sup_{n\ge m}\big(\|\mathcal{A}(n,m)Q_m^+x\|e^{\mu(\rho(m)-\rho(n))}\big)\Big\}\end{aligned}$$

and

$$\begin{aligned}\|x\|_m^- = \max\Big\{ &\sup_{0\ge n\ge m}\big(\|\mathcal{A}(n,m)P_m^-x\|e^{\lambda(\rho(n)-\rho(m))}\big),\\ &\sup_{n\le m}\big(\|\mathcal{A}(n,m)P_m^+x\|e^{\mu(\rho(n)-\rho(m))}\big)\Big\}\\ &+\max\Big\{\sup_{n\le m}\big(\|\mathcal{A}(n,m)Q_m^-x\|e^{\lambda(\rho(m)-\rho(n))}\big),\\ &\sup_{0\ge n\ge m}\big(\|\mathcal{A}(n,m)P_m^-x\|e^{\mu(\rho(m)-\rho(n))}\big)\Big\}.\end{aligned}$$

3. Continuous time. In this section we obtain corresponding results to those in Section 2 for continuous time. We continue to denote by $B(X)$ the set of all bounded linear operators acting on a Banach space X. A family $\mathcal{T}=(T(t,\tau))_{t,\tau\in\mathbb{R},\,t\ge\tau}$ of invertible linear operators in $B(X)$ is said to be an *evolution family* if:

1. $T(t,t)=\mathrm{Id}$ for $t\in\mathbb{R}$;
2. $T(t,s)T(s,\tau)=T(t,\tau)$ for $t,s,\tau\in\mathbb{R}$.

Given $I\in\{\mathbb{R},\mathbb{R}_0^+,\mathbb{R}_0^-\}$ and an increasing function $\rho\colon\mathbb{R}\to\mathbb{R}$ with $\rho(0)=0$, we say that an evolution family $\mathcal{T}$ has a *ρ-nonuniform (exponential) dichotomy on I* if:

1. there exist projections $P_t \in B(X)$ for $t \in I$ satisfying

$$P_t T(t,\tau) = T(t,\tau) P_\tau \quad \text{for } t \ge \tau; \tag{16}$$

2. there exist $\lambda, D > 0$ and $\varepsilon \ge 0$ such that for $t, \tau \in I$ we have

$$\|T(t,\tau)P_\tau\| \le De^{-\lambda(\rho(t)-\rho(\tau))+\varepsilon|\rho(\tau)|} \quad \text{for } t \ge \tau \tag{17}$$

and

$$\|T(t,\tau)Q_\tau\| \le De^{-\lambda(\rho(\tau)-\rho(t))+\varepsilon|\rho(\tau)|} \quad \text{for } t \le \tau, \tag{18}$$

where $Q_\tau = \mathrm{Id} - P_\tau$.

THEOREM 3. *An evolution family $\mathcal{T}$ of invertible linear operators in $B(X)$ has a ρ-nonuniform dichotomy on $\mathbb{R}$ if and only if there exist projections P_t^+ for $t \ge 0$ and projections P_t^- for $t \le 0$ such that:*

1. *$\mathcal{T}$ has a ρ-nonuniform dichotomy on $\mathbb{R}_0^+$ with projections P_t^+;*
2. *$\mathcal{T}$ has a ρ-nonuniform dichotomy on $\mathbb{R}_0^-$ with projections P_t^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

Proof. As in the proof of Theorem 1, it is sufficient to show that if properties 1–3 hold, then the evolution family has a ρ-nonuniform dichotomy on $\mathbb{R}$. Namely, taking the same constants λ, D and ε for the dichotomies on $\mathbb{R}_0^+$ and on $\mathbb{R}_0^-$, we define

$$\|x\|_t^+ = \sup_{\tau \ge t} \left(\|T(\tau,t)P_t^+ x\| e^{\lambda(\rho(\tau)-\rho(t))}\right) + \sup_{0 \le \tau \le t} \left(\|T(\tau,t)Q_t^+ x\| e^{\lambda(\rho(t)-\rho(\tau))}\right)$$

for $t \ge 0$ and $x \in X$, and

$$\|x\|_t^- = \sup_{0 \ge \tau \ge t} \left(\|T(\tau,t)P_t^- x\| e^{\lambda(\rho(\tau)-\rho(t))}\right) + \sup_{\tau \le t} \left(\|T(\tau,t)Q_t^- x\| e^{\lambda(\rho(t)-\rho(\tau))}\right)$$

for $t \le 0$ and $x \in X$. Finally, let

$$\|x\|_t = \begin{cases} \|x\|_t^+ & \text{if } t \ge 0, \\ \|x\|_t^- & \text{if } t < 0. \end{cases} \tag{19}$$

As in the proof of Theorem 1, one can show that for each $x \in X$,

$$\|x\| \le \|x\|_t^+ \le De^{\varepsilon\rho(t)}\|x\| \quad \text{for } t \ge 0$$

and

$$\|x\| \le \|x\|_t^- \le De^{\varepsilon|\rho(t)|}\|x\| \quad \text{for } t \le 0.$$

Hence,

$$\|x\| \le \|x\|_t \le De^{\varepsilon|\rho(t)|}\|x\| \quad \text{for } t \in \mathbb{R}.$$

Moreover, for each $x \in X$ we have

$$\begin{aligned}
\|T(t,\tau)P_\tau^+ x\|_t^+ &\le e^{-\lambda(\rho(t)-\rho(\tau))}\|x\|_\tau^+ \quad \text{for } t \ge \tau \ge 0,\\
\|T(t,\tau)P_\tau^+ x\|_t^+ &\le e^{-\lambda(\rho(\tau)-\rho(t))}\|x\|_\tau^+ \quad \text{for } t \ge \tau \ge 0,\\
\|T(t,\tau)P_\tau^- x\|_t^- &\le e^{-\lambda(\rho(t)-\rho(\tau))}\|x\|_t^- \quad \text{for } 0 \ge t \ge \tau,\\
\|T(t,\tau)Q_\tau^- x\|_t^- &\le e^{-\lambda(\rho(\tau)-\rho(t))}\|x\|_\tau^- \quad \text{for } 0 \ge t \ge \tau.
\end{aligned}$$

Now we observe that it follows from property 3 that

$$X = \operatorname{Im} P_t^+ \oplus T(t,0)\operatorname{Ker} P_0^- \quad \text{for } t \ge 0. \tag{20}$$

Let $\tilde{P}_t^+$ and $\tilde{Q}_t^+$ be the projections associated with this decomposition. Proceeding as in the proof of Lemma 1 and using the former inequalities, we obtain the following statement.

Lemma 3. *There exist $D', \lambda' > 0$ such that for each $x \in X$ we have*

$$\|T(\tau,t)\tilde{P}_t^+ x\|_\tau^+ \le D' e^{-\lambda'(\rho(\tau)-\rho(t))}\|x\|_t^+ \quad \text{for } \tau \ge t \ge 0$$

and

$$\|T(\tau,t)\tilde{Q}_t^+ x\|_\tau^+ \le D' e^{-\lambda'(\rho(t)-\rho(\tau))}\|x\|_t^+ \quad \text{for } 0 \le \tau \le t.$$

It follows also from property 3 that

$$X = T(t,0)\operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_t^- \quad \text{for } t \le 0. \tag{21}$$

Denoting by $\tilde{P}_t^-$ and $\tilde{Q}_t^-$ the projections associated with this decomposition, we have the following version of Lemma 3.

Lemma 4. *There exist $D'', \lambda'' > 0$ such that for each $x \in X$ we have*

$$\|T(\tau,t)\tilde{P}_t^- x\|_\tau^- \le D'' e^{-\lambda''(\rho(\tau)-\rho(t))}\|x\|_t^- \quad \text{for } t \le \tau \le 0$$

and

$$\|T(\tau,t)\tilde{Q}_t^- x\|_\tau^+ \le D'' e^{-\lambda''(\rho(t)-\rho(\tau))}\|x\|_t^- \quad \text{for } \tau \le t \le 0.$$

It follows from (20) and (21) that $\tilde{P}_0^+ = \tilde{P}_0^-$ and $\tilde{Q}_0^+ = \tilde{Q}_0^-$. Now let

$$P_\tau = \begin{cases} \tilde{P}_\tau^+ & \text{if } \tau \ge 0,\\ \tilde{P}_\tau^- & \text{if } \tau < 0, \end{cases}$$

and $Q_\tau = \operatorname{Id} - P_\tau$. Now one can proceed as in the proof of Theorem 1 to show that the evolution family $\mathcal{T}$ has a ρ-nonuniform dichotomy. □

We also consider the case of strong nonuniform exponential dichotomies. We say that an evolution family $\mathcal{T}$ of invertible linear operators in $B(X)$ has a *ρ-strong nonuniform (exponential) dichotomy on I* if:

1. there exist projections $P_t \in B(X)$ for $t \in I$ satisfying (16);
2. there exist $\lambda, \mu, D > 0$ and $\varepsilon \ge 0$ such that, in addition to inequalities (17) and (18), for $\tau, t \in I$ we have
$$\|T(\tau,t)P_t\| \le De^{-\mu(\rho(\tau)-\rho(t))+\varepsilon|\rho(t)|} \quad \text{for } \tau \le t$$
and
$$\|T(\tau,t)Q_t\| \le De^{-\mu(\rho(t)-\rho(\tau))+\varepsilon|\rho(t)|} \quad \text{for } \tau \ge t,$$
where $Q_t = \mathrm{Id} - P_t$.

Theorem 4. *An evolution family $\mathcal{T}$ of invertible linear operators in $B(X)$ has a ρ-strong nonuniform dichotomy on $\mathbb{R}$ if and only if there exist projections P_t^+ for $t \ge 0$ and projections P_t^- for $t \le 0$ such that:*

1. *$\mathcal{T}$ has a ρ-strong nonuniform dichotomy on $\mathbb{R}_0^+$ with projections P_t^+;*
2. *$\mathcal{T}$ has a ρ-strong nonuniform dichotomy on $\mathbb{R}_0^-$ with projections P_t^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

The proof of Theorem 4 is entirely analogous to the proof of Theorem 3, by defining norms $\|\cdot\|_t$ on X as in (19), where

$$\begin{aligned}\|x\|_t^+ &= \max\left\{\sup_{\tau\ge t}\big(\|T(\tau,t)P_t^+x\|e^{\lambda(\rho(\tau)-\rho(t))}\big), \sup_{0\le\tau\le t}\big(\|T(\tau,t)P_t^+x\|e^{\mu(\rho(\tau)-\rho(t))}\big)\right\}\\ &\quad+\max\left\{\sup_{\tau\ge t}\big(\|T(\tau,t)Q_t^+x\|e^{\mu(\rho(t)-\rho(\tau))}\big), \sup_{0\le\tau\le t}\big(\|T(\tau,t)Q_t^+x\|e^{\lambda(\rho(t)-\rho(\tau))}\big)\right\}\end{aligned}$$

and

$$\begin{aligned}\|x\|_t^- &= \max\left\{\sup_{0\ge\tau\ge t}\big(\|T(\tau,t)P_t^-x\|e^{\lambda(\rho(\tau)-\rho(t))}\big), \sup_{\tau\le t}\big(\|T(\tau,t)P_t^-x\|e^{\mu(\rho(\tau)-\rho(t))}\big)\right\},\\ &\quad+\max\left\{\sup_{0\ge\tau\ge t}\big(\|T(\tau,t)Q_t^-x\|e^{\mu(\rho(t)-\rho(\tau))}\big), \sup_{\tau\le t}\big(\|T(\tau,t)Q_t^-x\|e^{\lambda(\rho(t)-\rho(\tau))}\big)\right\}.\end{aligned}$$

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