A VERSION OF A THEOREM OF PLISS FOR NONUNIFORM AND NONINVERTIBLE DICHOTOMIES

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Abstract. For a dynamics on the whole line, both for discrete and continuous time, we extend a result of Pliss that gives a characterization of the notion of a trichotomy in various directions. More precisely, the result gives a characterization in terms of an admissibility property in the whole line (namely, the existence of bounded solutions of a linear dynamics under any nonlinear bounded perturbation) of the existence of a trichotomy, that is, of exponential dichotomies in the future and in the past, together with a certain transversality condition at time zero. In particular, we consider arbitrary linear operators acting on a Banach space as well as sequences of norms instead of a single norm, which allows considering the general case of a nonuniform exponential behavior.

1. Introduction

Our work is a contribution to what is usually called the admissibility theory of differential equations and dynamical systems. The theory essentially started with seminal work of Perron in [17] where he shows that the stability or the conditional stability of a linear differential equation can be deduced from the study of some nonlinear perturbations. More precisely, the conditional stability along what is then the stable space of a linear differential equation $x' = A(t)x$ on a finite-dimensional space can be deduced from what is usually called an *admissibility property* of the equation: originally this meant the existence of bounded solutions of the perturbed equation

$$x' = A(t)x + f(t)$$

on $\mathbb{R}_0^+$ for any bounded continuous perturbation f. A corresponding study for discrete time was initiated by Li in [13]. In particular, it was proved by Maizel' in [14] (for an integrally bounded coefficient matrix) and by Coppel in [7] (in the general case) that the admissibility property on $\mathbb{R}_0^+$ implies that the linear equation admits an exponential dichotomy. For the description of some early contributions we refer the reader to the books [15, 9, 8] (for equations on finite-dimensional spaces) and [12] (for equations on infinite-dimensional spaces). Related results for discrete time were obtained by Coffman and Schäffer in [6]. For more recent references, we refer the reader to [5, 11, 21].

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Our main interest in the paper is the class of dynamics on the whole line, both for discrete and continuous time. For simplicity of the exposition, here in the introduction we consider only the case of discrete time and of (uniform) exponential dichotomies. More precisely, we consider the equation

$$x_{n+1} - A_n x_n = y_{n+1}, \quad n \in \mathbb{Z}, \tag{1}$$

where $(A_n)_{n\in\mathbb{Z}}$ is a sequence of invertible bounded linear operators acting on a Banach space X and $(y_n)_{n\in\mathbb{Z}}$ is a bounded sequence in this space. The following result or, more precisely, an analogous version for continuous time, was established by Pliss [20] in the finite-dimensional setting.

Theorem 1. *For a finite-dimensional space X the following properties are equivalent:*

1. *for each bounded sequence $(y_n)_{n\in\mathbb{Z}}$, there exists a bounded solution $(x_n)_{n\in\mathbb{Z}}$ of equation (1);*
2. *the sequences $(A_m)_{m\ge 0}$ and $(A_m)_{m\le 0}$ admit exponential dichotomies, and the stable space of the first is transverse to the unstable space of the second.*

See Section 2 for the notion of a (uniform) exponential dichotomy. Two spaces are said to be *transverse* if their sum is the whole space. We refer the reader to [18, 21] for details concerning Pliss' result, which in particular plays an important role in recent work connecting structural stability with the shadowing property (see [19, 21]).

Our main objective is to obtain a generalization of Theorem 1 in various directions:

1. we consider arbitrary Banach spaces instead of a finite-dimensional space and linear operators A_n that are not necessarily invertible;
2. we consider sequences of norms instead of a single norm, which relates naturally to the study of a nonuniform exponential behavior.

We emphasize that the class of nonuniform exponential dichotomies is much larger than the class of uniform exponential dichotomies, particularly in the context of ergodic theory (see [4] for details). For example, almost all trajectories with nonzero Lyapunov exponents on any compact energy level of a Hamiltonian system have a differential that admits a nonuniform exponential dichotomy, although in general not a uniform exponential dichotomy.

Theorem 1 should be compared with a related type of results in which to the first property we add the requirement that the solution is unique. In particular, we have the following result (see for example [10]).

Theorem 2. *The following properties are equivalent:*

1. *for each bounded sequence $(y_n)_{n\in\mathbb{Z}}$, there exists a* ***unique*** *bounded solution $(x_n)_{n\in\mathbb{Z}}$ of equation (1);*
2. *the sequences $(A_m)_{m\ge 0}$ and $(A_m)_{m\le 0}$ admit exponential dichotomies, and the stable space of the first together with the unstable space of the second form a* ***direct sum****;*
3. *the sequence $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy.*

We note that due to what is marked in bold, none of Theorems 1 and 2 is a consequence of the other.

A priori it may seem that property 1 in Theorem 2 is somewhat more natural due to the characterization given by property 3. However, it exists already in the literature a corresponding notion that is equivalent to properties 1 and 2 in Theorem 1, namely that of a *trichotomy* (the name is somewhat inconvenient since in the context of the hyperbolicity theory it would be appropriate to use it in connection to the existence of a partially hyperbolic behavior). The notion of a trichotomy is a generalization of the notion of an exponential dichotomy in which the exponential behaviors into the future and into the past need not agree at the origin, although they still satisfy a certain compatibility condition. Namely, we say that a sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *trichotomy* if the sequences $(A_m)_{m\ge 0}$ and $(A_m)_{m\le 0}$ admit exponential dichotomies, say with projections P_m^+ and P_m^-, and

$$P_0^+P_0^- = P_0^-P_0^+ = P_0^+.$$

This type of behavior appeared for the first time in Coppel's book [8]. In the case of discrete time the notion was considered by Papaschinopoulos in [16].

Using the notion of a trichotomy one can rephrase Theorem 1 as follows (see Section 3 for details).

Theorem 3. *For a finite-dimensional space X the following properties are equivalent:*

1. *for each bounded sequence $(y_n)_{n\in\mathbb{Z}}$, there exists a bounded solution $(x_n)_{n\in\mathbb{Z}}$ of equation (1);*
2. *the sequences $(A_m)_{m\ge 0}$ and $(A_m)_{m\le 0}$ admit exponential dichotomies, and the stable space of the first is transverse to the unstable space of the second;*
3. *the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a trichotomy.*

Theorem 3 is a simple consequence of Theorem 6 that considers the more general case of a nonuniform exponential behavior.

2. Nonuniform exponential dichotomies

Let $X = (X, \|\cdot\|)$ be a Banach space and let $B(X)$ be the set of all bounded linear operators acting on X. Moreover, let $I \in \{\mathbb{Z}_0^+, \mathbb{Z}_0^-, \mathbb{Z}\}$, where

$$\mathbb{Z}_0^+ = \{n \in \mathbb{Z} : n \ge 0\} \quad \text{and} \quad \mathbb{Z}_0^- = \{n \in \mathbb{Z} : n \le 0\}.$$

Given a sequence $(A_m)_{m\in I}$ in $B(X)$, we define

$$\mathcal{A}(n,m) = \begin{cases} A_{n-1}\cdots A_m & \text{if } n > m, \\ \text{Id} & \text{if } n = m \end{cases}$$

for $n, m \in I$ with $n \ge m$. We say that $(A_m)_{m\in I}$ admits a *nonuniform exponential dichotomy on I* if:

1. there exist projections $P_m \in B(X)$ for $m \in I$ satisfying
$$\mathcal{A}(n,m)P_m = P_n\mathcal{A}(n,m) \quad \text{for } n \ge m \tag{2}$$
such that each map
$$\mathcal{A}(n,m)|\operatorname{Ker} P_m \colon \operatorname{Ker} P_m \to \operatorname{Ker} P_n \tag{3}$$
is one-to-one and onto;

2. there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that for $n, m \in I$ we have
$$\|\mathcal{A}(n,m)P_m\| \leq De^{-\lambda(n-m)+\varepsilon|m|} \tag{4}$$
for $n \geq m$ and
$$\|\mathcal{A}(n,m)Q_m\| \leq De^{-\lambda(m-n)+\varepsilon|m|} \tag{5}$$
for $n \leq m$, where $Q_m = \mathrm{Id} - P_m$ and
$$\mathcal{A}(n,m) = (\mathcal{A}(m,n)|\operatorname{Ker} P_n)^{-1} : \operatorname{Ker} P_m \to \operatorname{Ker} P_n$$
for $n < m$.

Moreover, we say that $(A_m)_{m\in I}$ admits a *(uniform) exponential dichotomy on I* if it admits a nonuniform exponential dichotomy on I with $\varepsilon = 0$. To our best knowledge, the latter notion was first considered by Henry in [10] (in the general noninvertible case and both for discrete and continuous time).

Incidentally, one could also consider the more general situation when the map $\mathcal{A}(n,m)|\operatorname{Ker} P_m$ in (3) is one-to-one, but not necessarily onto. A corresponding notion was first considered by Aulbach and Kalkbrenner in [1] (when $\varepsilon = 0$). We note that they give an example showing that it is in general impossible to give a characterization of that notion in terms of an admissibility property, that is, in terms of the perturbations and solutions of equation (1).

Now we consider a family of norms $\|\cdot\|_m$ on X for $m \in I$. We say that $(A_m)_{m\in I}$ admits a *(uniform) exponential dichotomy on I with respect to the norms $\|\cdot\|_m$* if there exist projections $P_m \in B(X)$ for $m \in I$ satisfying property 1 and there exist constants $\lambda, D > 0$ such that
$$\|\mathcal{A}(n,m)P_m x\|_n \leq De^{-\lambda(n-m)}\|x\|_m$$
for $n \geq m$ and $x \in X$, and
$$\|\mathcal{A}(n,m)Q_m x\|_n \leq De^{-\lambda(m-n)}\|x\|_m$$
for $n \leq m$ and $x \in X$.

Given a family of norms $\|\cdot\|_m$ on X for $m \in \mathbb{Z}$, we denote by Y_∞ the set of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X$ such that
$$\|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}}\|x_n\|_n < +\infty.$$
It is easy to verify that $(Y_\infty, \|\cdot\|_\infty)$ is a Banach space. Moreover, let
$$\mathcal{S} = \left\{x \in X : \sup_{n\geq 0}\|\mathcal{A}(n,0)x\|_n < +\infty\right\}$$
and let $\mathcal{U}$ be the set of all $x \in X$ for which there exists a sequence $(z_n)_{n\leq 0}$ such that
$$z_0 = x, \quad z_n = A_{n-1}z_{n-1} \text{ for } n \leq 0 \tag{6}$$
and $\sup_{n\leq 0}\|z_n\|_n < +\infty$. Clearly, $\mathcal{S}$ and $\mathcal{U}$ are (linear) subspaces of X.

Now we present the main result of this section. We recall that a subspace $Y \subset X$ is said to be *complemented* if there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$.

Theorem 4. *For a sequence $(A_m)_{m\in\mathbb{Z}}$ in $B(X)$, the following statements are equivalent:*

1. (a) $(A_m)_{m\ge 0}$ *admits a nonuniform exponential dichotomy on* $\mathbb{Z}_0^+$ *with projections* P_m^+;
   (b) $(A_m)_{m\le 0}$ *admits a nonuniform exponential dichotomy on* $\mathbb{Z}_0^-$ *with projections* P_m^-;
   (c)
$$X = \operatorname{Im} P_0^+ + \operatorname{Ker} P_0^-. \tag{7}$$
2. *there exists a sequence of norms* $\|\cdot\|_m$ *on* X *for* $m \in \mathbb{Z}$ *such that:*
   (a) *for each* $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$, *there exists* $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$ *such that*
$$x_{n+1} - A_n x_n = y_{n+1} \quad \text{for } n \in \mathbb{Z}; \tag{8}$$
   (b) *the subspaces* $\mathcal{S}$ *and* $\mathcal{U}$ *are complemented;*
   (c) *there exist* $C > 0$ *and* $\varepsilon \ge 0$ *such that*
$$\|x\| \le \|x\|_n \le Ce^{\varepsilon|n|}\|x\| \quad \text{for } x \in X \text{ and } n \in \mathbb{Z}. \tag{9}$$

Proof. Assume that property 1 holds. Without loss of generality, one can assume that the constants in the notion of a nonuniform exponential dichotomy are the same for both dichotomies (on $\mathbb{Z}_0^+$ and on $\mathbb{Z}_0^-$). Hence, there exist constants $D, \lambda > 0$ and $\varepsilon \ge 0$ such that

$$\begin{aligned}
\|\mathcal{A}(n,m)P_m^+\| &\le De^{-\lambda(n-m)+\varepsilon m} \quad \text{for } n \ge m \ge 0,\\
\|\mathcal{A}(n,m)Q_m^+\| &\le De^{-\lambda(m-n)+\varepsilon m} \quad \text{for } 0 \le n \le m,\\
\|\mathcal{A}(n,m)P_m^-\| &\le De^{-\lambda(n-m)+\varepsilon|m|} \quad \text{for } 0 \ge n \ge m,\\
\|\mathcal{A}(n,m)Q_m^-\| &\le De^{-\lambda(m-n)+\varepsilon|m|} \quad \text{for } n \le m \le 0,
\end{aligned} \tag{10}$$

where $Q_m^+ = \mathrm{Id} - P_m^+$ and $Q_m^- = \mathrm{Id} - P_m^-$. For each $n \in \mathbb{Z}$ and $x \in X$, let

$$\|x\|_n = \begin{cases} \|x\|_n^+ & \text{if } n \ge 0,\\ \|x\|_n^- & \text{if } n < 0,\end{cases}$$

where

$$\|x\|_m^+ = \sup_{n\ge m}\left(\|\mathcal{A}(n,m)P_m^+x\|e^{\lambda(n-m)}\right) + \sup_{0\le n\le m}\left(\|\mathcal{A}(n,m)Q_m^+x\|e^{\lambda(m-n)}\right)$$

and

$$\|x\|_m^- = \sup_{0\ge n\ge m}\left(\|\mathcal{A}(n,m)P_m^-x\|e^{\lambda(n-m)}\right) + \sup_{n\le m}\left(\|\mathcal{A}(n,m)Q_m^-x\|e^{\lambda(m-n)}\right).$$

We have

$$\|x\|_m^+ \ge \|P_m^+x\| + \|Q_m^+x\| \ge \|P_m^+x + Q_m^+x\| = \|x\|$$

and

$$\|x\|_m^- \ge \|P_m^-x\| + \|Q_m^-x\| \ge \|P_m^-x + Q_m^-x\| = \|x\|.$$

On the other hand, using (10) we obtain

$$\|x\|_m^+ \le De^{\varepsilon m}\|x\| + De^{\varepsilon m}\|x\| = 2De^{\varepsilon m}\|x\|$$

and

$$\|x\|_m^- \le De^{\varepsilon|m|}\|x\| + De^{\varepsilon|m|}\|x\| = 2De^{\varepsilon|m|}\|x\|.$$

Therefore,

$$\|x\| \le \|x\|_n \le 2De^{\varepsilon|n|}\|x\| \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X.$$

This shows that (9) holds with $C = 2D$. Moreover,

$$\begin{aligned}
\|\mathcal{A}(n,m)P_m^+ x\|_n^+ &\le e^{-\lambda(n-m)}\|x\|_m^+ \quad \text{for } n \ge m \ge 0,\\
\|\mathcal{A}(n,m)Q_m^+ x\|_n^+ &\le e^{-\lambda(m-n)}\|x\|_m^+ \quad \text{for } 0 \le n \le m,\\
\|\mathcal{A}(n,m)P_m^- x\|_n^- &\le e^{-\lambda(n-m)}\|x\|_m^- \quad \text{for } 0 \ge n \ge m,\\
\|\mathcal{A}(n,m)Q_m^- x\|_n^- &\le e^{-\lambda(m-n)}\|x\|_m^- \quad \text{for } n \le m \le 0.
\end{aligned} \tag{11}$$

Lemma 1. *We have* $\operatorname{Im} P_0^+ = \mathcal{S}$.

Proof of the lemma. It follows from (11) that

$$\sup_{m\ge 0}\|\mathcal{A}(m,0)x\|_m^+ < +\infty \tag{12}$$

for $x \in \operatorname{Im} P_0^+$. Now take $x \in X$ satisfying (12). Since $x = P_0^+x + Q_0^+x$, it follows from (11) that

$$\begin{aligned}
\sup_{m\ge 0}\|\mathcal{A}(m,0)Q_0^+x\|_m^+ &= \sup_{m\ge 0}\|\mathcal{A}(m,0)(x - P_0^+x)\|_m^+\\
&\le \sup_{m\ge 0}\|\mathcal{A}(m,0)x\|_m^+ + \sup_{m\ge 0}\|\mathcal{A}(m,0)P_0^+x\|_m^+ < +\infty.
\end{aligned}$$

On the other hand, again by (11), for $m \ge 0$ we have

$$\|Q_0^+x\|_0^+ = \|\mathcal{A}(0,m)\mathcal{A}(m,0)Q_0^+x\|_0^+ \le e^{-\lambda m}\|\mathcal{A}(m,0)Q_0^+x\|_m^+.$$

Letting $m \to \infty$ we obtain $Q_0^+x = 0$ and so $x = P_0^+x \in \operatorname{Im} P_0^+$. □

Lemma 2. *We have* $\operatorname{Im} Q_0^- = \mathcal{U}$.

Proof of the lemma. Clearly, $x \in \mathcal{U}$ for each $x \in \operatorname{Im} Q_0^-$. Now take $x \in \mathcal{U}$ and write $z_n = P_n^- z_n + Q_n^- z_n$ for $n \le 0$ (with z_n as in (6)). By (2) we have

$$\mathcal{A}(0,n)P_n^- z_n = P_0^- x \quad \text{and} \quad \mathcal{A}(0,n)Q_n^- z_n = Q_0^- x$$

for $n \le 0$. Hence, it follows from (11) that $\sup_{n\le 0}\|P_n^- z_n\|_n^- < +\infty$. On the other hand, again by (11), for $n \le 0$ we have

$$\|P_0^- x\|_0^- = \|\mathcal{A}(0,n)P_n^- z_n\|_n^- \le e^{\lambda n}\|P_n^- z_n\|_n^-.$$

Letting $n \to -\infty$ we obtain $P_0^- x = 0$ and so $x = Q_0^- x \in \operatorname{Im} Q_0^-$. □

It follows from Lemmas 1 and 2 that the spaces $\mathcal{S}$ and $\mathcal{U}$ are complemented. Now we consider the space

$$Y^+ = \left\{\mathbf{x} = (x_n)_{n\ge 0} \subset X : \sup_{n\ge 0}\|x_n\|_n < +\infty\right\}.$$

Lemma 3. *For each* $\mathbf{y} = (y_n)_{n\ge 0} \in Y^+$ *with* $y_0 = 0$*, there exists* $\mathbf{x} = (x_n)_{n\ge 0} \in Y^+$ *with* $x_0 \in \operatorname{Im} Q_0^-$ *such that*

$$x_{n+1} - A_n x_n = y_{n+1} \quad \textit{for } n \ge 0. \tag{13}$$

Proof of the lemma. For each $n \ge 0$, let

$$x_n^* = \sum_{k=0}^{n} \mathcal{A}(n,k)P_k^+ y_k - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k)Q_k^+ y_k.$$

It follows from (11) that

$$\sum_{k=0}^{n} e^{-\lambda(n-k)}\|y_k\|_k^+ + \sum_{k=n+1}^{\infty} e^{-\lambda(k-n)}\|y_k\|_k^+ \le \frac{1+e^{-\lambda}}{1-e^{-\lambda}} \sup_{k\ge 0}\|y_k\|_k^+$$

for $n \ge 0$ and so $\mathbf{x}^* = (x_n^*)_{n\ge 0} \in Y^+$. By (7), one can write $x_0^* = x_0' + x_0''$, with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. Now let

$$x_n = x_n^* - \mathcal{A}(n,0)x_0' \quad \text{for } n \ge 0.$$

Then $\mathbf{x} = (x_n)_{n\ge 0} \in Y^+$ and $x_0 \in \operatorname{Im} Q_0^-$. Moreover, one can easily verify that (13) holds. □

Take $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$ with $y_n = 0$ for $n \le 0$. By Lemma 3, there exists $\mathbf{x}^* = (x_n^*)_{n\ge 0} \in Y^+$ such that $x_0^* \in \operatorname{Im} Q_0^-$ and

$$x_{n+1}^* - A_n x_n^* = y_{n+1} \quad \text{for } n \ge 0.$$

Let

$$x_n = \begin{cases} x_n^* & \text{if } n \ge 0, \\ \mathcal{A}(n,0)x_0^* & \text{if } n < 0. \end{cases} \tag{14}$$

Clearly, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$ and (8) holds.

We also consider the space

$$Y^- = \left\{ \mathbf{x} = (x_n)_{n\le 0} \subset X : \sup_{n\le 0}\|x_n\|_n < +\infty \right\}.$$

Lemma 4. *For each $\mathbf{y} = (y_n)_{n\le 0} \in Y^-$, there exists $\mathbf{x} = (x_n)_{n\le 0} \in Y^-$ with $x_0 \in \operatorname{Im} P_0^+$ such that*

$$x_{n+1} - A_n x_n = y_{n+1} \quad \textit{for } n \le -1. \tag{15}$$

Proof of the lemma. For each $n \le 0$, let

$$x_n^* = -\sum_{k=n+1}^{0} \mathcal{A}(n,k)Q_k^- y_k + \sum_{k=-\infty}^{n} \mathcal{A}(n,k)P_k^- y_k.$$

It follows from (11) that $\mathbf{x}^* = (x_n^*)_{n\le 0} \in Y^-$. By (7), one can write $x_0^* = x_0' + x_0''$, with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. Let

$$x_n = x_n^* - \mathcal{A}(n,0)x_0'' \quad \text{for } n \le 0.$$

Then $\mathbf{x} = (x_n)_{n\le 0} \in Y^-$ and $x_0 \in \operatorname{Im} P_0^+$. Moreover, one can easily verify that (15) holds. □

Take $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$ with $y_n = 0$ for $n > 0$. By Lemma 4, there exists $\mathbf{x}^* = (x_n^*)_{n\le 0} \in Y^-$ such that $x_0^* \in \operatorname{Im} P_0^+$ and

$$x_{n+1}^* - A_n x_n^* = y_{n+1} \quad \text{for } n \le -1.$$

Let

$$x_n = \begin{cases} x_n^* & \text{if } n \le 0, \\ \mathcal{A}(n,0)x_0^* & \text{if } n > 0. \end{cases} \tag{16}$$

Clearly, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$ and (8) holds.

Now we observe that each sequence $\mathbf{y} \in Y_\infty$ can be written in the form $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$, with $\mathbf{y}^1, \mathbf{y}^2 \in Y_\infty$ such that $y_n^1 = 0$ for $n \le 0$ and $y_n^2 = 0$ for

$n > 0$. We obtain a solution of (8) by adding the sequences in (14) and (16). This establishes property 2 in the theorem.

Now we assume that property 2 holds. Since $\mathcal{S}$ and $\mathcal{U}$ are complemented, there exist closed subspaces Z_1 and Z_2 of X such that

$$X = \mathcal{S} \oplus Z_1 = \mathcal{U} \oplus Z_2.$$

Lemma 5. *For each $(y_n)_{n\ge 0} \in Y^+$ with $y_0 = 0$, there exists a unique $(x_n)_{n\ge 0} \in Y^+$ such that $x_0 \in Z_1$ and (13) holds.*

Proof of the lemma. Since property 2 holds, there exists $(x^*_n)_{n\ge 0} \in Y^+$ such that

$$x^*_{n+1} - A_n x^*_n = y_{n+1} \quad \text{for } n \ge 0.$$

Write $x^*_0 = y + z$, with $y \in \mathcal{S}$ and $z \in Z_1$. Let

$$x_n = x^*_n - \mathcal{A}(n, 0)y, \quad n \ge 0.$$

It follows from the definition of $\mathcal{S}$ that $(x_n)_{n\ge 0} \in Y^+$ and $x_0 = z \in Z_1$.

In order to establish the uniqueness of the solution, take $(x'_n)_{n\ge 0} \in Y^+$ such that $x'_0 \in Z_1$ and (13) holds. Then

$$x_n - x'_n = \mathcal{A}(n, 0)(x_0 - x'_0) \quad \text{for } n \ge 0$$

and so $x_0 - x'_0 \in \mathcal{S}$. On the other hand, we also have $x_0 - x'_0 \in Z_1$, which shows that $x_0 = x'_0$ (since $X = \mathcal{S} \oplus Z_1$). Therefore, $x_n = x'_n$ for $n \ge 0$. □

The proof of the following lemma is completely analogous.

Lemma 6. *For each $(y_n)_{n\le 0} \in Y^-$, there exists a unique $(x_n)_{n\le 0} \in Y^-$ such that $x_0 \in Z_2$ and (15) holds.*

By Lemmas 5 and 6 together with Theorem 2 in [3], the sequence $(A_m)_{m\in\mathbb{Z}}$ admits nonuniform exponential dichotomies with respect to sequence of norms $\|\cdot\|_n$ both on $\mathbb{Z}^+_0$ and $\mathbb{Z}^-_0$. Denoting the corresponding projections respectively by P^+_m for $m \ge 0$ and P^-_m for $m \le 0$, we have $\operatorname{Im} P^+_0 = \mathcal{S}$ and $\operatorname{Ker} P^-_0 = \mathcal{U}$. Hence, there exist constants $\lambda, D > 0$ such that

$$\begin{aligned}
\|\mathcal{A}(n, m)P^+_m x\|_n &\le De^{-\lambda(n-m)}\|x\|_m \quad \text{for } n \ge m \ge 0,\\
\|\mathcal{A}(n, m)Q^+_m\|_n &\le De^{-\lambda(m-n)}\|x\|_m \quad \text{for } 0 \le n \le m,\\
\|\mathcal{A}(n, m)P^-_m x\|_n &\le De^{-\lambda(n-m)}\|x\|_m \quad \text{for } 0 \ge n \ge m,\\
\|\mathcal{A}(n, m)Q^-_m x\|_n &\le De^{-\lambda(m-n)}\|x\|_m \quad \text{for } n \le m \le 0,
\end{aligned}$$

where $Q^+_m = \mathrm{Id} - P^+_m$ and $Q^-_m = \mathrm{Id} - P^-_m$. By (9), we conclude that

$$\begin{aligned}
\|\mathcal{A}(n, m)P^+_m\| &\le CDe^{-\lambda(n-m)+\varepsilon m} \quad \text{for } n \ge m \ge 0,\\
\|\mathcal{A}(n, m)Q^+_m\| &\le CDe^{-\lambda(m-n)+\varepsilon m} \quad \text{for } 0 \le n \le m,\\
\|\mathcal{A}(n, m)P^-_m\| &\le CDe^{-\lambda(n-m)+\varepsilon|m|} \quad \text{for } 0 \ge n \ge m,\\
\|\mathcal{A}(n, m)Q^-_m\| &\le CDe^{-\lambda(m-n)+\varepsilon|m|} \quad \text{for } n \le m \le 0,
\end{aligned}$$

and so the sequence $(A_m)_{m\in\mathbb{Z}}$ admits nonuniform exponential dichotomies both on $\mathbb{Z}^+_0$ and $\mathbb{Z}^-_0$. It remains to show that (7) holds. Take $v \in X$ and consider the sequence $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ with $y_0 = v$ and $y_n = 0$ for $n \ne 0$. Clearly, $\mathbf{y} \in Y_\infty$. Hence, there exists $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ such that (8) holds. In particular,

$x_n = \mathcal{A}(n,0)x_0$ for $n \geq 0$ and $A_{-1}x_{-1} = \mathcal{A}(0,n)x_n$ for $n \leq -1$. Therefore, $x_0 \in \mathcal{S} = \operatorname{Im} P_0^+$, $A_{-1}x_{-1} \in \mathcal{U} = \operatorname{Ker} P_0^-$ and so $v \in \operatorname{Im} P_0^+ + \operatorname{Ker} P_0^-$. This completes the proof of the theorem. □

3. Nonuniform exponential trichotomies

It turns out that for linear operators acting on a finite-dimensional space, property 1 in Theorem 4 is equivalent to the notion of a nonuniform exponential trichotomy. In this section we always assume that the space X is finite-dimensional and that the operators A_n are invertible.

We say that a sequence $(A_n)_{n\in\mathbb{Z}}$ admits a *nonuniform exponential trichotomy* if:

1. $(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections P_m^+;
2. $(A_m)_{m\leq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with projections P_m^-;
3. $$P_0^- = P_0^- P_0^+ = P_0^+ P_0^-. \tag{17}$$

Theorem 5. *Conditions* (7) *and* (17) *can be interchanged in the notion of a nonuniform exponential trichotomy, up to the eventual need of considering different projections.*

Proof. Assume that (17) holds. Then $\operatorname{Im} P_0^- \subset \operatorname{Im} P_0^+$. Hence,

$$\dim \operatorname{Im} P_0^+ + \dim \operatorname{Ker} P_0^- = n + \dim \operatorname{Im} P_0^+ - \dim \operatorname{Im} P_0^- \geq n,$$

and so (7) holds. Now assume that (7) holds. Since X is finite-dimensional, one can choose subspaces $Z_1 \subset \operatorname{Ker} P_0^-$ and $Z_2 \subset \operatorname{Im} P_0^+$ such that

$$X = \operatorname{Im} P_0^+ \oplus Z_1 \quad \text{and} \quad X = Z_2 \oplus \operatorname{Ker} P_0^-.$$

Let $\tilde{P}_0^+ \colon X \to \operatorname{Im} P_0^+$ and $\tilde{P}_0^- \colon X \to Z_2$ be the projections associated, respectively, to each decomposition.

Lemma 7. *Assume that the sequence $(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections P_m. If P_m', for $m \in \mathbb{Z}_0^+$, are projections such that*

$$P_m' \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n' \quad \textit{for } m,n \in \mathbb{Z}_0^+ \quad \textit{and} \quad \operatorname{Im} P_0 = \operatorname{Im} P_0', \tag{18}$$

then $(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with respect to the projections P_m'.

Proof of the lemma. By the second equality in (18), we have $P_0 P_0' = P_0'$ and $P_0' P_0 = P_0$. Hence,

$$P_0 - P_0' = P_0(P_0 - P_0') = (P_0 - P_0')Q_0.$$

It follows from (4) and (5) that

$$\begin{aligned}
\|\mathcal{A}(n,0)(P_0-P_0')v\| &= \|\mathcal{A}(n,0)P_0(P_0-P_0')v\| \\
&\le De^{-\lambda n}\|(P_0-P_0')v\| \\
&= De^{-\lambda n}\|(P_0-P_0')Q_0v\| \\
&\le De^{-\lambda n}\|P_0-P_0'\|\cdot\|Q_0v\| \\
&= De^{-\lambda n}\|P_0-P_0'\|\cdot\|\mathcal{A}(0,m)\mathcal{A}(m,0)Q_0v\| \\
&= De^{-\lambda n}\|P_0-P_0'\|\cdot\|\mathcal{A}(0,m)Q_m\mathcal{A}(m,0)v\| \\
&\le D^2e^{-\lambda n-\lambda m+\varepsilon m}\|P_0-P_0'\|\cdot\|\mathcal{A}(m,0)v\|
\end{aligned}$$

for each $m,n\in\mathbb{Z}_0^+$ and $v\in X$. Therefore,

$$\begin{aligned}
\|\mathcal{A}(n,m)P_m'v\| &\le \|\mathcal{A}(n,m)P_mv\|+\|\mathcal{A}(n,m)(P_m-P_m')v\| \\
&= \|\mathcal{A}(n,m)P_mv\|+\|\mathcal{A}(n,0)(P_0-P_0')\mathcal{A}(0,m)v\| \\
&\le De^{-\lambda(n-m)+\varepsilon m}\|v\|+D^2e^{-\lambda(n-m)+\varepsilon m}\|P_0-P_0'\|\cdot\|v\| \\
&= D'e^{-\lambda(n-m)+\varepsilon m}\|v\|
\end{aligned}$$

for $n\ge m$, where

$$D'=D+D^2\|P_0-P_0'\|.$$

Similarly,

$$\begin{aligned}
\|\mathcal{A}(n,m)Q_m'v\| &\le \|\mathcal{A}(n,m)Q_mv\|+\|\mathcal{A}(n,m)(P_m-P_m')v\| \\
&= \|\mathcal{A}(n,m)Q_mv\|+\|\mathcal{A}(n,0)(P_0-P_0')\mathcal{A}(0,m)v\| \\
&\le De^{-\lambda(m-n)+\varepsilon m}\|v\|+D^2e^{-\lambda(m-n)+\varepsilon m}\|P_0-P_0'\|\cdot\|v\| \\
&= D'e^{-\lambda(m-n)+\varepsilon m}\|v\|
\end{aligned}$$

for $n\le m$, where $Q_m'=\mathrm{Id}-P_m'$. We conclude that the sequence $(A_m)_{m\ge 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with respect to the projections P_m'. □

The following result can be obtained in an analogous manner to that in the proof of Lemma 7.

Lemma 8. *Assume that the sequence $(A_m)_{m\le 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with projections P_m. If P_m', for $m\in\mathbb{Z}_0^-$, are projections such that*

$$P_m'\mathcal{A}(m,n)=\mathcal{A}(m,n)P_n' \quad \text{for } m,n\in\mathbb{Z}_0^- \quad \text{and} \quad \operatorname{Ker}P_0=\operatorname{Ker}P_0',$$

then $(A_m)_{m\le 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with respect to the projections P_m'.

It follows from Lemmas 7 and 8 that the sequence $(A_m)_{m\ge 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections $\tilde{P}_m^+$ and that the sequence $(A_m)_{m\le 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with projections $\tilde{P}_m^-$, where

$$\tilde{P}_m^+=\mathcal{A}(m,0)\tilde{P}_0^+\mathcal{A}(0,m) \quad \text{and} \quad \tilde{P}_m^-=\mathcal{A}(m,0)\tilde{P}_0^-\mathcal{A}(0,m).$$

Finally, it is easy to verify that $\tilde{P}_0^+$ and $\tilde{P}_0^-$ satisfy (17). □

The following result is now a direct consequence of Theorems 4 and 5.

Theorem 6. *For a sequence* $(A_m)_{m\in\mathbb{Z}}$ *of invertible* $d\times d$ *matrices, the following statements are equivalent:*

1. $(A_m)_{m\in\mathbb{Z}}$ *admits a nonuniform exponential trichotomy;*
2. *there exists a sequence of norms* $\|\cdot\|_m$ *on* $\mathbb{R}^d$ *for* $m\in\mathbb{Z}$ *such that:*
   (a) *for each* $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_\infty$*, there exists* $\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\in Y_\infty$ *such that*
   $$x_{n+1}-A_nx_n=y_{n+1}\quad \text{for } n\in\mathbb{Z};$$
   (b) *there exist* $C>0$ *and* $\varepsilon\ge 0$ *such that*
   $$\|x\|\le\|x\|_n\le Ce^{\varepsilon|n|}\|x\|\quad \text{for } x\in\mathbb{R}^d \text{ and } n\in\mathbb{Z}. \tag{19}$$

Now we use Theorem 6 to establish the robustness of the notion of a nonuniform exponential trichotomy.

Theorem 7. *Let* $(A_m)_{m\in\mathbb{Z}}$ *and* $(B_m)_{m\in\mathbb{Z}}$ *be two sequences of invertible* $d\times d$ *matrices such that:*

1. $(A_m)_{m\in\mathbb{Z}}$ *admits a nonuniform exponential trichotomy;*
2. *there exists* $c>0$ *such that*
$$\|A_m-B_m\|\le ce^{-\varepsilon|m|}\quad \text{for } m\in\mathbb{Z}. \tag{20}$$

If c *is sufficiently small, then the sequence* $(B_m)_{m\in\mathbb{Z}}$ *also admits a nonuniform exponential trichotomy.*

Proof. Let $\|\cdot\|_n$, for $n\in\mathbb{Z}$, be a sequence of norms as in Theorem 6. We claim that for each $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_\infty$, there exists $\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\in Y_\infty$ such that
$$x_{n+1}-B_nx_n=y_{n+1}\quad \text{for } n\in\mathbb{Z}. \tag{21}$$
It follows from (19) and (20) that
$$\|(A_{n-1}-B_{n-1})x\|_n\le cD\|x\|_n\quad \text{for } n\in\mathbb{Z} \text{ and } x\in\mathbb{R}^d,$$
where $D=Ce^{\varepsilon}$. Take $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_\infty$. By Theorem 6, there exists $\mathbf{x}^0=(x_n^0)_{n\in\mathbb{Z}}\in Y_\infty$ such that
$$x_{n+1}^0-A_nx_n^0=y_{n+1}\quad \text{for } n\in\mathbb{Z}.$$
Moreover, it follows from the proofs of Lemmas 3 and 4 that $\mathbf{x}^0$ can be chosen so that
$$\|\mathbf{x}^0\|_\infty\le\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\|\mathbf{y}\|_\infty.$$
Now we proceed by induction. Assume that we have constructed a sequence $\mathbf{x}^{i-1}=(x_n^{i-1})_{n\in\mathbb{Z}}\in Y_\infty$ satisfying
$$\|\mathbf{x}^{i-1}\|_\infty\le(cD)^{i-1}\left(\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\right)^i\|\mathbf{y}\|_\infty.$$
Let
$$y_n^i=(B_{n-1}-A_{n-1})x_{n-1}^{i-1},\quad n\in\mathbb{Z}.$$
Then $\mathbf{y}^i=(y_n^i)_{n\in\mathbb{Z}}\in Y_\infty$ and
$$\|\mathbf{y}^i\|\le\left(cD\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\right)^i\|\mathbf{y}\|_\infty.$$

By Theorem 6, there exists $\mathbf{x}^i = (x_n^i)_{n\in\mathbb{Z}} \in Y_\infty$ such that

$$x_{n+1}^i - A_n x_n^i = y_{n+1}^i \quad \text{for } n \in \mathbb{Z},$$

satisfying

$$\|\mathbf{x}^i\|_\infty \le \frac{1+e^{-\lambda}}{1-e^{-\lambda}}\|\mathbf{y}^i\|_\infty \le (cD)^i \bigg(\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\bigg)^{i+1} \|\mathbf{y}\|_\infty.$$

Finally, let $\mathbf{x} = \sum_{i=0}^\infty \mathbf{x}^i$. It follows from the above estimates that for c so small such that

$$cD\frac{1+e^{-\lambda}}{1-e^{-\lambda}} < 1,$$

the sequence $\mathbf{x}$ is well-defined. Moreover, we have

$$\begin{aligned} x_{n+1} - A_n x_n &= \sum_{i=0}^\infty (x_{n+1}^i - A_n x_n^i) \\ &= y_{n+1} + \sum_{i=1}^\infty y_{n+1}^i \\ &= y_{n+1} + \sum_{i=1}^\infty (B_n - A_n) x_n^{i-1} \\ &= y_{n+1} + (B_n - A_n) x_n, \end{aligned}$$

which shows that (21) holds. The conclusion of the theorem follows now directly from Theorem 6. □

4. The case of continuous time

We continue to denote by $B(X)$ the set of all bounded linear operators acting on a Banach space X. Let $I \in \{\mathbb{R}, \mathbb{R}_0^+, \mathbb{R}_0^-\}$. A family $T(t,\tau)$, for $t,\tau \in I$ with $t \ge \tau$, of linear operators in $B(X)$ is called an *evolution family on I* if:

1. $T(t,t) = \mathrm{Id}$ for $t \in I$;
2. $T(t,s)T(s,\tau) = T(t,\tau)$ for $t,s,\tau \in I$ with $t \ge s \ge \tau$;
3. for each $t,\tau \in I$ and $x \in X$, the maps

   $$(-\infty, t] \cap I \ni s \mapsto T(t,s)x \quad \text{and} \quad [\tau, \infty) \cap I \ni s \mapsto T(s,\tau)x$$

   are continuous.

We say that an evolution family $T(t,\tau)$ on I admits an *nonuniform exponential dichotomy on I* if:

1. there exist projections $P_t \in B(X)$ for $t \in I$ satisfying

   $$P_t T(t,\tau) = T(t,\tau) P_\tau \quad \text{for } t \ge \tau$$

   such that each map

   $$T(t,\tau)|\,\mathrm{Ker}\, P_\tau \colon \mathrm{Ker}\, P_\tau \to \mathrm{Ker}\, P_t$$

   is invertible;

2. there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that for $t, \tau \in I$ we have
$$\|T(t,\tau)P_\tau\| \leq De^{-\lambda(t-\tau)+\varepsilon|\tau|} \tag{22}$$
for $t \geq \tau$ and
$$\|T(t,\tau)Q_\tau\| \leq De^{-\lambda(\tau-t)+\varepsilon|\tau|} \tag{23}$$
for $t \leq \tau$, where $Q_\tau = \mathrm{Id} - P_\tau$ and
$$T(t,\tau) = (T(\tau,t)|\operatorname{Ker} P_t)^{-1}\colon \operatorname{Ker} P_\tau \to \operatorname{Ker} P_t$$
for $t < \tau$.

Given a family of norms $\|\cdot\|_t$ on X for $t \in \mathbb{R}$, we consider the spaces
$$Y_\infty = \left\{ x\colon \mathbb{R} \to X \text{ continuous} : \|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_t < +\infty \right\}$$
and
$$Y_1 = \left\{ x\colon \mathbb{R} \to X \text{ measurable} : \|x\|_1 := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|x(\tau)\|_\tau \, d\tau < +\infty \right\}.$$
Then $(Y_\infty, \|\cdot\|_\infty)$ and $(Y_1, \|\cdot\|_1)$ are Banach spaces (in the case of Y_1 identifying functions that are equal Lebesgue-almost everywhere). Moreover, let
$$\mathcal{S} = \left\{ x \in X : \sup_{t\geq 0} \|T(t,0)x\|_t < +\infty \right\}$$
and let $\mathcal{U}$ be the set of all $v \in X$ for which there exists a continuous function $x\colon (-\infty, 0] \to X$ such that $x(0) = v$, $x(t) = T(t,\tau)x(\tau)$ for $t \in [\tau, 0]$ and $\sup_{t\leq 0} \|x(t)\|_t < +\infty$. Clearly, $\mathcal{S}$ and $\mathcal{U}$ are subspaces of X.

The following result is a version of Theorem 1 for evolution families.

Theorem 8. *For an evolution family $T(t,\tau)$ in $B(X)$, the following statements are equivalent:*

1. (a) *$T(t,\tau)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projections P_t^+;*
   (b) *$T(t,\tau)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^-$ with projections P_t^-;*
   (c)
$$X = \operatorname{Im} P_0^+ + \operatorname{Ker} P_0^-. \tag{24}$$
2. *there exists a family of norms $\|\cdot\|_t$ on X for $t \in \mathbb{R}$ such that:*
   (a) *for each $y \in Y_1$, there exists $x \in Y_\infty$ such that*
$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\,ds \quad \text{for } t \geq \tau; \tag{25}$$
   (b) *the subspaces $\mathcal{S}$ and $\mathcal{U}$ are complemented;*
   (c) *there exist $C > 0$ and $\varepsilon \geq 0$ such that*
$$\|x\| \leq \|x\|_t \leq Ce^{\varepsilon|t|}\|x\| \quad \text{for } x \in X \text{ and } t \in \mathbb{R}. \tag{26}$$

Proof. We first assume that property 1 holds. For each $t \geq 0$ and $x \in X$, let
$$\|x\|_t^+ = \sup_{\tau\geq t} \left(\|T(\tau,t)P_t^+ x\| e^{\lambda(\tau-t)} \right) + \sup_{0\leq\tau\leq t} \left(\|T(\tau,t)Q_t^+ x\| e^{\lambda(t-\tau)} \right).$$

It follows from (22) and (23) that

$$\|x\| \le \|x\|_t^+ \le De^{\varepsilon t}\|x\| \quad \text{for } t \ge 0 \text{ and } x \in X. \tag{27}$$

Moreover,

$$\|T(t,\tau)P_\tau^+ x\|_t^+ \le e^{-\lambda(t-\tau)}\|x\|_\tau^+ \tag{28}$$

for $t \ge \tau \ge 0$ and $x \in X$, and

$$\|T(t,\tau)P_\tau^+ x\|_t^+ \le e^{-\lambda(\tau-t)}\|x\|_\tau^+ \tag{29}$$

for $0 \le t \le \tau$ and $x \in X$.

Analogously, for each $t \le 0$ and $x \in X$, let

$$\|x\|_t^- = \sup_{0 \ge \tau \ge t}\big(\|T(\tau,t)P_t^- x\|e^{\lambda(\tau-t)}\big) + \sup_{\tau \le t}\big(\|T(\tau,t)Q_t^- x\|e^{\lambda(t-\tau)}\big).$$

It follows from (22) and (23) that

$$\|x\| \le \|x\|_t^- \le De^{\varepsilon|t|}\|x\| \quad \text{for } t \le 0 \text{ and } x \in X. \tag{30}$$

Moreover,

$$\|T(t,\tau)P_\tau^- x\|_t^- \le e^{-\lambda(t-\tau)}\|x\|_t^- \tag{31}$$

for $0 \ge t \ge \tau$ and $x \in X$, and

$$\|T(t,\tau)Q_\tau^- x\|_t^- \le e^{-\lambda(\tau-t)}\|x\|_\tau^- \tag{32}$$

for $t \le \tau \le 0$ and $x \in X$.

In addition, one can show that

$$s \mapsto \|T(s,t)x\|_s^+ \text{ is continuous on } [t,+\infty)$$

for $t \ge 0$ and $x \in X$, and that

$$s \mapsto \|T(s,t)x\|_s^- \text{ is continuous on } [t,0]$$

for $t \le 0$ and $x \in X$. Finally, for each $t \in \mathbb{R}$ and $x \in X$, let

$$\|x\|_t = \begin{cases} \|x\|_t^+ & \text{if } t \ge 0, \\ \|x\|_t^- & \text{if } t < 0. \end{cases}$$

It follows from (27) and (30) that

$$\|x\| \le \|x\|_t \le 2De^{\varepsilon|t|}\|x\| \quad \text{for } x \in X \text{ and } t \in \mathbb{R}.$$

This shows that (26) holds with $C = 2D$. The following lemma can be obtained in an analogous manner to that in the proofs of Lemmas 1 and 2.

Lemma 9. *We have*

$$\operatorname{Im} P_0^+ = \mathcal{S} \quad \text{and} \quad \operatorname{Im} Q_0^- = \mathcal{U}.$$

It particular, it follows from Lemma 9 that $\mathcal{S}$ and $\mathcal{U}$ are complemented.

Now we introduce auxiliary spaces. Let

$$Y_\infty^+ = \left\{x\colon \mathbb{R}_0^+ \to X \text{ continuous} : \sup_{t \ge 0}\|x(t)\|_t < +\infty\right\}$$

and

$$Y_1^+ = \left\{x\colon \mathbb{R}_0^+ \to X \text{ measurable} : \sup_{t \ge 0}\int_t^{t+1}\|x(\tau)\|_\tau\, d\tau < +\infty\right\}.$$

Lemma 10. *For each $y \in Y_1^+$, there exists $x \in Y_\infty^+$ with $x(0) \in \operatorname{Im} Q_0^-$ such that*

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\,ds \quad \text{for } t \ge \tau \ge 0. \tag{33}$$

Proof of the lemma. Take $y \in Y_1^+$ and extend it to a function $y\colon \mathbb{R} \to X$ by letting $y(t) = 0$ for $t < 0$. Moreover, for $t \ge 0$, let

$$x_1^*(t) = \int_0^t T(t,\tau)P_\tau y(\tau)\,d\tau \quad \text{and} \quad x_2^*(t) = \int_t^\infty T(t,\tau)Q_\tau y(\tau)\,d\tau.$$

It follows from (28) that

$$\begin{aligned}
\|x_1^*(t)\|_t &\le \int_{-\infty}^t \|T(t,\tau)P_\tau^+ y(\tau)\|_t\,d\tau \\
&\le \int_{-\infty}^t e^{-\lambda(t-\tau)}\|y(\tau)\|_\tau\,d\tau \\
&= \sum_{m=0}^\infty \int_{t-m-1}^{t-m} e^{-\lambda(t-\tau)}\|y(\tau)\|_\tau\,d\tau \\
&\le \sum_{m=0}^\infty e^{-\lambda m}\int_{t-m-1}^{t-m} \|y(\tau)\|_\tau\,d\tau \\
&\le \frac{1}{1-e^{-\lambda}} \sup_{t\ge 0}\int_t^{t+1} \|y(\tau)\|_\tau\,d\tau
\end{aligned}$$

for $t \ge 0$. Similarly, by (29),

$$\|x_2^*(t)\|_t \le \frac{1}{1-e^{-\lambda}} \sup_{t\ge 0}\int_t^{t+1} \|y(\tau)\|_\tau\,d\tau$$

for $t \ge 0$. The estimates also show that $x_1^*(t)$ and $x_2^*(t)$ are well-defined. Now let $x^*(t) = x_1^*(t) - x_2^*(t)$. Clearly, $\sup_{t\ge 0}\|x^*(t)\|_t < +\infty$. For $t \ge \tau \ge 0$, we have

$$\begin{aligned}
x^*(t) &= \int_\tau^t T(t,s)y(s)\,ds - \int_\tau^t T(t,s)P_s^+ y(s)\,ds - \int_\tau^t T(t,s)Q_s^+ y(s)\,ds \\
&\quad + \int_0^t T(t,s)P_s^+ y(s)\,ds - \int_t^\infty T(t,s)Q_s^+ y(s)\,ds \\
&= \int_\tau^t T(t,s)y(s)\,ds + \int_0^\tau T(t,s)P_s^+ y(s)\,ds - \int_\tau^\infty T(t,s)Q_s^+ y(s)\,ds \\
&= \int_\tau^t T(t,s)y(s)\,ds + T(t,\tau)x^*(\tau)
\end{aligned}$$

and so identity (33) holds with x replaced by x^*. In particular, this implies that x^* is continuous and so $x^* \in Y_\infty^+$. By (24), one can write $x^*(0) = x_0' + x_0''$, with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. We define a function $x\colon \mathbb{R}_0^+ \to X$ by

$$x(t) = x^*(t) - T(t,0)x_0'$$

for $t \ge 0$. Then $x \in Y_\infty^+$, $x(0) \in \operatorname{Im} Q_0^-$ and (33) holds. □

Take $y \in Y_1$ with $y(t) = 0$ for $t < 0$. By Lemma 10, there exists $x^* \in Y_\infty^+$ such that (33) holds and $x^*(0) \in \operatorname{Im} Q_0^-$. Let

$$x(t) = \begin{cases} x^*(t) & \text{if } t \ge 0, \\ T(t,0)x^*(0) & \text{if } t < 0. \end{cases} \tag{34}$$

Clearly, $x \in Y_\infty$ and (25) holds.

Similarly, let

$$Y_\infty^- = \left\{ x \colon \mathbb{R}_0^- \to X \text{ continuous} : \sup_{t \le 0} \|x(t)\|_t^- < +\infty \right\}$$

and

$$Y_1^- = \left\{ x \colon \mathbb{R}_0^- \to X \text{ measurable} : \sup_{t \le 0} \int_{t-1}^t \|x(\tau)\|_\tau^- \, d\tau < +\infty \right\}.$$

Lemma 11. *For each $y \in Y_1^-$, there exists $x \in Y_\infty^-$ with $x(0) \in \operatorname{Im} P_0^+$ such that*

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\, ds \quad \text{for } 0 \ge t \ge \tau. \tag{35}$$

Proof of the lemma. Take $y \in Y_1^-$. For $t \le 0$, let

$$x^*(t) = -\int_t^0 T(t,\tau)Q_\tau^- y(\tau)\, d\tau + \int_{-\infty}^t T(t,\tau)P_\tau^- y(\tau)\, d\tau.$$

It follows from inequalities (31) and (32) that $x^*(t)$ is well-defined and that $\sup_{t \le 0} \|x^*(t)\|_t^- < +\infty$. Moreover, one can easily verify that identity (35) holds with x replaced by x^*. By (24), one can write $x^*(0) = x_0' + x_0''$, with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. We define a function $x \colon \mathbb{R}_0^- \to X$ by

$$x(t) = x^*(t) - T(t,0)x_0''$$

for $t \le 0$. Then $x \in Y^-$, $x(0) \in \operatorname{Im} P_0^+$ and (35) holds. ☐

Take $y \in Y_1$ with $y(t) = 0$ for $t \ge 0$. By Lemma 11, there exists $x^* \in Y_\infty^-$ such that (35) holds and $x^*(0) \in \operatorname{Im} P_0^+$. Let

$$x(t) = \begin{cases} x^*(t) & \text{if } t \le 0, \\ T(t,0)x^*(0) & \text{if } t > 0. \end{cases} \tag{36}$$

Clearly, $x \in Y$ and (25) holds.

Finally, we observe that each function $y \in Y_1$ can be written in the form $y = y^1 + y^2$, with $y^1, y^2 \in Y_1$ such that $y^1(t) = 0$ for $t \le 0$ and $y^2(t) = 0$ for $t > 0$. We obtain a solution of (25) by adding the solutions in (34) and (36). This establishes property 2 in the theorem.

Now we establish the converse. Assume that property 2 holds. Since $\mathcal{S}$ and $\mathcal{U}$ are complemented, there exists closed subspaces Z_1 and Z_2 of X such that

$$X = \mathcal{S} \oplus Z_1 = U \oplus Z_2.$$

The following lemma can be obtained in an analogous manner to that in the proofs of Lemmas 5 and 6.

Lemma 12. *For each $y \in Y_1^+$, there exists a unique $x \in Y_\infty^+$ such that $x(0) \in Z_1$ and* (33) *holds. Moreover, for each $y \in Y_1^-$, there exists a unique $x \in Y_\infty^-$ such that $x(0) \in Z_2$ and* (35) *holds.*

By Lemma 12 together with Theorem 2 in [2], the evolution family $T(t,\tau)$ admits exponential dichotomies with respect to a family of norms $\|\cdot\|_t$ both on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$.

Proceeding as in the proof of Theorem 4, we find that the evolution family $T(t,\tau)$ admits nonuniform exponential dichotomies both on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$, say with projections, respectively, P_t^+ for $t \ge 0$ and P_t^- for $t \le 0$. Moreover,

$$\operatorname{Im} P_0^+ = \mathcal{S} \quad \text{and} \quad \operatorname{Ker} P_0^- = \mathcal{U}.$$

It remains to establish (24). Take $v \in X$ and define $y\colon \mathbb{R} \to X$ by $y(t) = T(t,0)v\chi_{[0,1]}(t)$ for $t \in \mathbb{R}$. Clearly, $y \in Y_1$ and so there exists $x \in Y_\infty$ such that (25) holds. In particular, we have $x(t) = T(t,\tau)x(\tau)$ for $0 \ge t \ge \tau$ and

$$x(t) = T(t,0)(x(0)+v)$$

for $t \ge 1$. Hence, $x(0) \in \operatorname{Ker} P_0^-$ and $x(0)+v \in \operatorname{Im} P_0^+$, which shows that $v \in \operatorname{Im} P_0^+ + \operatorname{Ker} P_0^-$. ☐

In a similar manner to that in Section 3, for linear operators acting on a finite-dimensional space, property 1 in Theorem 8 is equivalent to the notion of a nonuniform exponential trichotomy (for continuous time). The arguments are analogous to those in Section 3 and so we omit them.

References

1. B. Aulbach and J. Kalkbrenner, *Exponential forward splitting for noninvertible difference equations*, Comput. Math. Appl. **42** (2001), 743–754.
2. L. Barreira, D. Dragičević and C. Valls, *Admissibility on the half line for evolution families*, J. Anal. Math., to appear.
3. L. Barreira, D. Dragičević and C. Valls, *Nonuniform hyperbolicity and one-sided admissibility*, preprint.
4. L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity*, Encyclopedia of Mathematics and its Applications 115, Cambridge University Press, 2007.
5. C. Chicone and Yu. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Mathematical Surveys and Monographs 70, Amer. Math. Soc., 1999.
6. C. Coffman and J. Schäffer, D*ichotomies for linear difference equations*, Math. Ann. **172** (1967), 139–166.
7. W. Coppel, *On the stability of ordinary differential equations*, J. London Math. Soc. **39** (1964), 255–260.
8. W. Coppel, *Dichotomies in Stability Theory*, Lect. Notes. in Math. 629, Springer, 1978.
9. Ju. Dalec′kiĭ and M. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs 43, Amer. Math. Soc., 1974.
10. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes in Math. 840, Springer, 1981.
11. N. Huy, *Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line*, J. Funct. Anal. **235** (2006), 330–354.
12. B. Levitan and V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, 1982.
13. T. Li, *Die Stabilitätsfrage bei Differenzengleichungen*, Acta Math. **63** (1934), 99–141.
14. A. Maizel', *On stability of solutions of systems of differential equations*, Trudi Ural'skogo Politekhnicheskogo Instituta, Mathematics **51** (1954), 20–50.

15. J. Massera and J. Schäffer, *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics 21, Academic Press, 1966.
16. G. Papaschinopoulos, *On exponential trichotomy of linear difference equations*, Appl. Anal. **40** (1991), 89–109.
17. O. Perron, *Die Stabilit ”atsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.
18. S. Pilyugin, *Generalizations of the notion of hyperbolicity*, J. Difference Equ. Appl. **12** (2006), 271–282.
19. S. Pilyugin and S. Tikhomirov, *Lipschitz shadowing implies structural stability*, Nonlinearity **23** (2010), 2509–2515.
20. V. Pliss, *Bounded solutions of inhomogeneous linear systems of differential equations*, in Problems of Asymptotic Theory of Nonlinear Oscillations, Naukova Dumka, Kiev, 1977, pp. 168–173 (in Russian).
21. D. Todorov, *Generalizations of analogs of theorems of Maizel and Pliss and their application in shadowing theory*, Discrete Contin. Dyn. Syst. **33** (2013), 4187–4205.

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