Chaos, Solitons and Fractals 85 (2016) 120–127

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

# Fredholm operators and nonuniform exponential dichotomies☆

Luis Barreira[a,*], Davor Dragičević[b], Claudia Valls[a]

[a] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Lisboa 1049-001, Portugal
[b] School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia

ARTICLE INFO

*Article history:*
Received 5 July 2015
Revised 8 January 2016
Accepted 27 January 2016
Available online 19 February 2016

*MSC:*
Primary:37D99

*Keywords:*
Exponential dichotomies
Robustness
Fredholm operators

ABSTRACT

We show that the existence of a nonuniform exponential dichotomy for a one-sided sequence $(A_m)_{m \geq 0}$ of invertible $d \times d$ matrices is equivalent to the Fredholm property of a certain linear operator between spaces of bounded sequences. Moreover, for a two-sided sequence $(A_m)_{m\in\mathbb{Z}}$ we show that the existence of a nonuniform exponential dichotomy implies that a related operator $S$ is Fredholm and that if it is Fredholm, then the sequence admits nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$. We also give conditions on $S$ so that the sequence admits a nonuniform exponential dichotomy on $\mathbb{Z}$. Finally, we use the former characterizations to establish the robustness of the notion of a nonuniform exponential dichotomy.



## 1. Introduction

### 1.1. From uniform to nonuniform exponential behavior

Our main aim is to discuss the relation between the existence of a *nonuniform* exponential dichotomy for a sequence of invertible $d \times d$ matrices (see Section 2 for the definition) and the Fredholm property of a certain linear operator. Related results were first obtained by Palmer [14,15] for ordinary differential equations and *uniform* exponential dichotomies. Further results were obtained by Lin [11] for functional differential equations, by Blázquez [6], Rodrigues and Silveira [20], Zeng [23] and Zhang [24] for parabolic evolution equations, and by Chow and Leiva [7], Sacker and Sell [21] and Rodrigues and Ruas-Filho [19] for abstract evolution equations. We emphasize that all these works consider only uniform exponential dichotomies.

In comparison to the classical notion of a uniform exponential dichotomy, the notion of a nonuniform exponential dichotomy corresponds to a much weaker requirement. For example, in the context of ergodic theory almost all linear variational equations with nonzero Lyapunov exponents of a measure-preserving flow (such as any Hamiltonian flow restricted to a compact hypersurface) admit a nonuniform exponential dichotomy (see for example [4]). On the other hand, the extra exponentials in the notion of a nonuniform exponential dichotomy (see (5) and (6)) complicate a corresponding study.

In order to circumvent this difficulty we shall use Lyapunov norms (see Proposition 4). These are norms adapted to each particular dynamics with respect to which a nonuniform exponential dichotomy becomes uniform (the crucial properties of the Lyapunov norms are those in (3), (4) and (7)). The use of Lyapunov norms in the study of

☆ L.B. and C.V. were supported by FCT/Portugal through UID/MAT/04459/2013. D.D. was supported in part by an Australian Research Council Discovery Project DP150100017, Croatian Science Foundation under the Project IP-2014-09-2285 and by the University of Rijeka Research Grant 13.14.1.2.02.

* Corresponding author.
*E-mail addresses:* barreira@math.ist.utl.pt (L. Barreira), ddragicevic@math.uniri.hr (D. Dragičević), cvalls@math.ist.utl.pt (C. Valls).

http://dx.doi.org/10.1016/j.chaos.2016.01.021
0960-0779/

nonuniform hyperbolicity goes back to seminal work of Pesin [16] (see also [4,5]).

Incidentally, an alternative characterization of a nonuniform exponential behavior that does not involve constructing Lyapunov norms a priori was developed in [10] in the context of ergodic theory (when the nonuniform part of the exponential dichotomy can be made arbitrarily small). The characterization is expressed in terms of the invertibility of certain linear operators on Fréchet spaces.

### 1.2. Brief formulation of our results

As noted above, our main aim is to discuss the relation between the existence of a nonuniform exponential dichotomy for a sequence of matrices and the Fredholm property of a certain linear operator.

More precisely, we consider separately the cases of one-sided and two-sided sequences. Let

$$l^\infty = \left\{ \mathbf{x} = (x_m)_{m\geq 0} \subset \mathbb{R}^d : \sup_{m\geq 0} \|x_m\|_m < +\infty \right\}$$

for some norms $\|\cdot\|_m$ on $\mathbb{R}^d$, for $m \geq 0$, and denote by $l_0^\infty$ the set of all $\mathbf{x} \in l^\infty$ with $x_0 = 0$. Given a sequence $(A_m)_{m\geq 0}$ of invertible $d \times d$ matrices, we define a linear operator $T : \mathcal{D}(T) \to l_0^\infty$ by

$$(T\mathbf{x})_0 = 0 \quad \text{and} \quad (T\mathbf{x})_{m+1} = x_{m+1} - A_m x_m, \;\; m \geq 0,$$

in the set $\mathcal{D}(T)$ of all $\mathbf{x} \in l^\infty$ such that $T\mathbf{x} \in l_0^\infty$.

In particular, we establish the following result (see Theorems 6 and 7).

**Theorem 1.** *The sequence $(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ if and only if $T$ is a Fredholm operator for some norms $\|\cdot\|_m$ satisfying*

$$\|x\| \leq \|x\|_m \leq Ce^{\varepsilon m}\|x\|, \quad m \geq 0, \; x \in \mathbb{R}^d,$$

*for some constants $C > 0$ and $\varepsilon \geq 0$.*

In order the formulate a corresponding result for two-sided sequences, let

$$l_{\mathbb{Z}}^\infty = \left\{ \mathbf{x} = (x_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^d : \sup_{m\in\mathbb{Z}} \|x_m\|_m < +\infty \right\},$$

for some norms $\|\cdot\|_m$ on $\mathbb{R}^d$, for $m \in \mathbb{Z}$. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d \times d$ matrices, we define a linear operator $S : \mathcal{D}(S) \to l^\infty$ by

$$(S\mathbf{x})_m = x_m - A_{m-1}x_{m-1}, \;\; m \in \mathbb{Z},$$

in the set $\mathcal{D}(S)$ of all $\mathbf{x} \in l^\infty$ such that $S\mathbf{x} \in l^\infty$.

We also establish the following version of Theorem 1 for two-sided sequences of matrices (see Theorems 10 and 11).

**Theorem 2.** *The sequence $(A_m)_{m\in\mathbb{Z}}$ admits nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ if and only if $S$ is a Fredholm operator for some norms $\|\cdot\|_m$ satisfying*

$$\|x\| \leq \|x\|_m \leq Ce^{\varepsilon|m|}\|x\|, \quad m \in \mathbb{Z}, \; x \in \mathbb{R}^d,$$

*for some constants $C > 0$ and $\varepsilon \geq 0$.*

For two-sided sequences of matrices, we also show that if $S$ is a Fredholm operator and

$$R = S|_E : E \to c_0$$

is injective, where $E = S^{-1}c_0$ and

$$c_0 = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^\infty : \lim_{|n|\to+\infty} \|x_n\|_n = 0 \right\}, \tag{1}$$

then the sequence of matrices admits a nonuniform exponential dichotomy on the whole $\mathbb{Z}$ (see Theorem 11).

As an immediate consequence of Theorems 1 and 2, by considering a constant sequence of norms $\|\cdot\|_m = \|\cdot\|$ on $\mathbb{R}^d$ we obtain the following discrete-time version of results of Palmer [14,15]. However, in contrast to what happens in his work (for continuous time), we do not require any bounded growth condition for the matrices $A_m$.

**Theorem 3.** *The following properties hold:*

1. *$(A_m)_{m\geq 0}$ admits a uniform exponential dichotomy on $\mathbb{Z}_0^+$ if and only if $T$ is a Fredholm operator taking $\|\cdot\|_m = \|\cdot\|$ for $m \geq 0$;*
2. *$(A_m)_{m\in\mathbb{Z}}$ admits uniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ if and only if $S$ is a Fredholm operator taking $\|\cdot\|_m = \|\cdot\|$ for $m \in \mathbb{Z}$.*

We are not able to provide a reference for Theorem 3, but it certainly should be considered a folklore result in the area (although perhaps adding a bounded growth condition).

### 1.3. Application to robustness

In addition, we use the characterization of the existence of a nonuniform exponential dichotomy, both for one-sided and two-sided sequences of matrices, to establish the robustness of the notion (see Theorems 8 and 12). We note that the study of robustness has a long history. In particular, it was discussed by Massera and Schäffer [12], Coppel [8], and in the case of Banach spaces by Dalec′kiĭ and Kreĭn [9]. For more recent works we refer to [13,17,18] and the references therein. We refer the reader to [5] and the references therein for the study of robustness of a nonuniform exponential behavior.

As noted above, in order to circumvent the difficulty caused by the extra exponentials in the notion of a nonuniform exponential dichotomy we use Lyapunov norms. Thus, one might think that we would always need to know the Lyapunov norms a priori in order to be able to use our results. Of course this depends on the particular context, although our proof of the robustness of a nonuniform exponential dichotomy (both for one-sided and two-sided sequences of matrices) shows that sometimes there is no need whatsoever to know explicitly the Lyapunov norms in order to apply our results.

## 2. Preliminaries

Given $I \in \{\mathbb{Z}_0^+, \mathbb{Z}_0^-, \mathbb{Z}\}$, where

$$\mathbb{Z}_0^+ = \{m \in \mathbb{Z} : m \geq 0\} \quad \text{and} \quad \mathbb{Z}_0^- = \{m \in \mathbb{Z} : m \leq 0\},$$

we consider a sequence $(A_m)_{m\in I}$ of invertible $d \times d$ matrices and norms $\|\cdot\|_m$ on $\mathbb{R}^d$ for $m \in I$. For each $m, n \in I$

we define

$$\mathcal{A}(m,n)=\begin{cases}A_{m-1}\cdots A_n & \text{if } m>n,\\ \mathrm{Id} & \text{if } m=n,\\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m<n.\end{cases}$$

We say that the sequence $(A_m)_{m\in I}$ admits an *exponential dichotomy* on $I$ with respect to the norms $\|\cdot\|_m$ if:

1. There exist projections $P_m:\mathbb{R}^d\to\mathbb{R}^d$ for $m\in I$ satisfying
$$\mathcal{A}(n,m)P_m=P_n\mathcal{A}(n,m)\quad\text{for } n,m\in I; \tag{2}$$
2. There exist $\lambda$, $D>0$ such that for $m$, $n\in I$ and $x\in\mathbb{R}^d$ we have
$$\|\mathcal{A}(m,n)P_nx\|_m\le De^{-\lambda(m-n)}\|x\|_n\quad\text{for } m\ge n \tag{3}$$
and
$$\|\mathcal{A}(m,n)Q_nx\|_m\le De^{-\lambda(n-m)}\|x\|_n\quad\text{for } m\le n, \tag{4}$$
where $Q_n=\mathrm{Id}-P_n$.

Moreover, we say that the sequence $(A_m)_{m\in I}$ admits a *nonuniform exponential dichotomy* on $I$ if:

1. There exist projections $P_m:\mathbb{R}^d\to\mathbb{R}^d$ for $m\in I$ satisfying (2);
2. There exist $\lambda$, $D>0$ and $\varepsilon\ge0$ such that for $m$, $n\in I$ we have
$$\|\mathcal{A}(m,n)P_n\|\le De^{-\lambda(m-n)+\varepsilon|n|}\quad\text{for } m\ge n \tag{5}$$
and
$$\|\mathcal{A}(m,n)Q_n\|\le De^{-\lambda(n-m)+\varepsilon|n|}\quad\text{for } m\le n, \tag{6}$$
where $Q_n=\mathrm{Id}-P_n$.

The relation between the two notions is given by the following result.

**Proposition 4.** *The following properties are equivalent:*

1. *$(A_m)_{m\in I}$ admits a nonuniform exponential dichotomy on $I$;*
2. *$(A_m)_{m\in I}$ admits an exponential dichotomy on $I$ with respect to norms $\|\cdot\|_m$, for $m\in I$, satisfying*
$$\|x\|\le\|x\|_m\le Ce^{\varepsilon|m|}\|x\|,\quad m\in I,\ x\in\mathbb{R}^d \tag{7}$$
*for some constants $C>0$ and $\varepsilon\ge0$.*

## 3. One-sided exponential dichotomies

In this section we consider exponential dichotomies on $\mathbb{Z}_0^+$. Let

$$l^\infty=\left\{\mathbf{x}=(x_m)_{m\ge0}\subset\mathbb{R}^d:\|\mathbf{x}\|_\infty<+\infty\right\}, \tag{8}$$

where

$$\|\mathbf{x}\|_\infty=\sup_{m\ge0}\|x_m\|_m.$$

Then $(l^\infty,\|\cdot\|_\infty)$ is a Banach space. Moreover, let $l_0^\infty$ be the set of all $\mathbf{x}=(x_m)_{m\ge0}\in l^\infty$ with $x_0=0$. Clearly, $l_0^\infty$ is a closed subspace of $l^\infty$.

Given a sequence $(A_m)_{m\ge0}$ of invertible $d\times d$ matrices, we consider the linear operator $T:\mathcal{D}(T)\to l_0^\infty$ defined by

$$(T\mathbf{x})_0=0\quad\text{and}\quad(T\mathbf{x})_{m+1}=x_{m+1}-A_mx_m,\ \ m\ge0,$$

in the domain $\mathcal{D}(T)$ composed of the sequences $\mathbf{x}\in l^\infty$ such that $T\mathbf{x}\in l_0^\infty$.

**Proposition 5.** *The operator $T:\mathcal{D}(T)\to l_0^\infty$ is closed.*

**Proof.** Let $(\mathbf{x}^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(T)$ converging to $\mathbf{x}\in l^\infty$ such that $T\mathbf{x}^k$ converges to $\mathbf{y}\in l_0^\infty$. Then

$$x_m=\lim_{k\to\infty}x_m^k\quad\text{and}\quad y_m=\lim_{k\to\infty}(T\mathbf{x}^k)_m$$

for $m\ge0$, and so

$$\begin{aligned}x_{m+1}-A_mx_m&=\lim_{k\to+\infty}(x_{m+1}^k-A_mx_m^k)\\&=\lim_{k\to+\infty}(T\mathbf{x}^k)_{m+1}=y_{m+1}\end{aligned}$$

for $m\ge0$. This shows that $T\mathbf{x}=\mathbf{y}$ and $\mathbf{x}\in\mathcal{D}(T)$. □

For $x\in\mathcal{D}(T)$ we consider the graph norm

$$\|\mathbf{x}\|_T=\|\mathbf{x}\|_\infty+\|T\mathbf{x}\|_\infty. \tag{9}$$

Since $T$ is closed, $(\mathcal{D}(T),\|\cdot\|_T)$ is a Banach space. Moreover, the operator

$$T:(\mathcal{D}(T),\|\cdot\|_T)\to l_0^\infty$$

is bounded and from now on we denote it simply by $T$.

**Theorem 6.** *If the sequence $(A_m)_{m\ge0}$ admits an exponential dichotomy on $\mathbb{Z}_0^+$ with respect to norms $\|\cdot\|_m$, then $T$ is a Fredholm operator.*

**Proof.** We first show that $T$ is onto, which implies that the codimension of $T$ is finite. Given $\mathbf{y}=(y_m)_{m\ge0}\in l_0^\infty$, let

$$x_n=\sum_{k=0}^n\mathcal{A}(n,k)P_ky_k-\sum_{k=n+1}^\infty\mathcal{A}(n,k)Q_ky_k,\quad n\ge0.$$

It follows from (3) and (4) that $x_n$ is well defined. Moreover,

$$\begin{aligned}\|x_n\|_n&\le D\sum_{k=0}^ne^{-\lambda(n-k)}\|y_k\|_k+D\sum_{k=n+1}^\infty e^{-\lambda(k-n)}\|y_k\|_k\\&\le D\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\sup_{k\ge0}\|y_k\|_k\end{aligned}$$

for $n\ge0$ and $\mathbf{x}=(x_n)_{n\ge0}\in l^\infty$. Furthermore, we have

$$\begin{aligned}(T\mathbf{x})_{n+1}&=x_{n+1}-A_nx_n\\&=\sum_{k=0}^{n+1}\mathcal{A}(n+1,k)P_ky_k-\sum_{k=0}^n\mathcal{A}(n+1,k)P_ky_k\\&\quad-\sum_{k=n+2}^\infty\mathcal{A}(n+1,k)Q_ky_k+\sum_{k=n+1}^\infty\mathcal{A}(n+1,k)Q_ky_k\\&=P_{n+1}x_{n+1}+Q_{n+1}y_{n+1}=y_{n+1}\end{aligned}$$

for $n\ge0$. Hence, $\mathbf{x}\in\mathcal{D}(T)$ and $T\mathbf{x}=\mathbf{y}$.

In order to prove that $T$ is a Fredholm operator, it remains to show that the dimension of $\operatorname{Ker}T$ is finite. The sequence $\mathbf{x}=(x_m)_{m\ge0}$ belongs to $\operatorname{Ker}T$ if and only if $x_m=\mathcal{A}(m,0)x_0$ for $m\ge0$. Hence, the map $R:\operatorname{Ker}T\to\mathbb{R}^d$ defined by $R\mathbf{x}=x_0$ is injective. Moreover, it follows from (3) and (4) that $\operatorname{Im}R=\operatorname{Im}P_0$. Indeed,

$$\operatorname{Ker}T\subset\mathcal{D}(T)\subset l^\infty$$

and one can easily verify that the initial conditions $x_0 \in \mathbb{R}^d$ for which the sequence $x_m = \mathcal{A}(m,0)x$ is bounded are precisely those in $\operatorname{Im} P_0$. Thus, $\dim \operatorname{Ker} T = \dim \operatorname{Im} P_0 < +\infty$. □

Now we establish the converse of Theorem 6. We say that $\mathbf{x} = (x_n)_{n\ge 0}$ has *bounded support* if there exists $n \in \mathbb{N}$ such that $x_m = 0$ for $m > n$.

**Theorem 7.** *If $T$ is a Fredholm operator, then the sequence $(A_m)_{m\ge 0}$ admits an exponential dichotomy on $\mathbb{Z}_0^+$ with respect to some norms $\|\cdot\|_m$.*

**Proof.** We first prove some auxiliary results. □

**Lemma 1.** *For each $\mathbf{y} = (y_m)_{m\ge 0}$ with bounded support and $y_0 = 0$, there exists a sequence $\mathbf{x} = (x_m)_{m\ge 0}$ with bounded support such that*

$$x_{m+1} - A_m x_m = y_{m+1} \quad \text{for } m \ge 0. \tag{10}$$

**Proof of the lemma.** Take $n \in \mathbb{N}$ such that $y_m = 0$ for $m > n$ and define $x_0 = -\sum_{k=0}^{n} \mathcal{A}(0,k)y_k$. Moreover, let

$$x_m = \begin{cases} \mathcal{A}(m,0)x_0 + \displaystyle\sum_{k=0}^{m} \mathcal{A}(m,k)y_k & \text{if } 0 \le m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

Then the sequence $\mathbf{x} = (x_m)_{m\ge 0}$ satisfies (10). □

Given a Banach space $X$, we denote by $X^*$ the set of all bounded linear maps $\phi : X \to \mathbb{R}$. We recall that a bounded linear operator $A$: $X \to Y$ between Banach spaces induces the adjoint linear operator $A^*$: $Y^* \to X^*$ defined by $(A^*\phi)(x) = \phi(Ax)$ for $\phi \in Y^*$ and $x \in X$.

Now we consider the space

$$c_0 = \left\{ \mathbf{x} = (x_n)_{n\ge 0} \in l_0^\infty : \lim_{n\to+\infty} \|x_n\|_n = 0 \right\}.$$

Let $E = T^{-1}c_0$ and consider the map $S = T|_E : E \to c_0$. Moreover, let $S^* : c_0^* \to E^*$ be the adjoint operator.

**Lemma 2.** *We have* $\operatorname{Ker} S^* = \{0\}$.

**Proof of the lemma.** Take $\phi \in \operatorname{Ker} S^*$ and a sequence $\mathbf{y} \in c_0$ with bounded support. By Lemma 1, there exists $\mathbf{x} \in \mathcal{D}(T)$ such that $S\mathbf{x} = T\mathbf{x} = \mathbf{y}$. Therefore,

$$\phi(\mathbf{y}) = \phi(S\mathbf{x}) = (S^*\phi)\mathbf{x} = 0$$

for each $\mathbf{y} \in c_0$ with bounded support. Since the sequences with bounded support are dense in $c_0$, we conclude that $\phi = 0$. □

**Lemma 3.** *For each $\mathbf{y} \in c_0$, there exists $\mathbf{x} \in l^\infty$ satisfying (10).*

**Proof of the lemma.** Since $T$ is a Fredholm operator, $\operatorname{Im} T$ is closed and so the same happens to $\operatorname{Im} S$. For example by Theorem 4.6 in [22] we have

$$\{\operatorname{Im} S\}^0 := \left\{ \phi \in c_0^* : \phi(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in c_0 \right\} = \operatorname{Ker} S^*.$$

It follows from Lemma 2 that $\{\operatorname{Im} S\}^0 = \{0\}$ and thus, by the Hahn–Banach theorem we have $\operatorname{Im} S = c_0$, which yields the statement in the lemma. □

**Lemma 4.** *There exists a subspace $Z$ of $\mathbb{R}^d$ such that for each $\mathbf{y} \in c_0$, there exists a unique $\mathbf{x} \in l^\infty$ with $x_0 \in Z$ satisfying (10).*

**Proof of the lemma.** Let $Z$ be the subspace of $\mathbb{R}^d$ consisting of all vectors $x \in \mathbb{R}^d$ such that $\sup_{n\ge 0} \|\mathcal{A}(n,0)x\|_n < +\infty$. Moreover, let $Z'$ be any subspace of $\mathbb{R}^d$ such that $\mathbb{R}^d = Z \oplus Z'$. By Lemma 3, given $\mathbf{y} = (y_m)_{m\ge 0} \in c_0$, there exists $\mathbf{x} = (x_m)_{m\ge 0} \in l^\infty$ satisfying (10). Write $x_0 = y_0 + z_0$, where $y_0 \in Z$ and $z_0 \in Z'$. Now we consider the sequence $\mathbf{x}^* = (x_m^*)_{m\ge 0}$ defined by

$$x_m^* = x_m - \mathcal{A}(m,0)y_0, \quad m \ge 0.$$

Then $\mathbf{x}^* \in l^\infty$, $x_0^* \in Z'$ and (10) holds with each $x_m$ replaced by $x_m^*$.

It remains to show that $\mathbf{x}^*$ is unique. Assume that for some sequence $\mathbf{y} = (y_m)_{m\ge 0} \in c_0$ there exist $\mathbf{x}^i = (x_m^i)_{m\ge 0} \in l^\infty$ with $x_0^i \in Z'$ satisfying (10) for $i = 1, 2$. Then

$$x_m^1 - x_m^2 = \mathcal{A}(m,0)(x_0^1 - x_0^2) \quad \text{form} \ge 0$$

and thus $x_0^1 - x_0^2 \in Z$. Hence, $x_0^1 - x_0^2 \in Z \cap Z'$ and so $x_0^1 = x_0^2$. This implies that $x_m^1 = x_m^2$ for $m \ge 0$ and so $\mathbf{x}^1 = \mathbf{x}^2$. □

It follows now readily from results in [3] that the sequence $(A_m)_{m\ge 0}$ admits an exponential dichotomy on $\mathbb{Z}_0^+$ with respect to some norms $\|\cdot\|_m$.

The following result is an application of Theorems 6 and 7.

**Theorem 8.** *Let $(A_m)_{m\ge 0}$ and $(B_m)_{m\ge 0}$ be sequences of invertible $d \times d$ matrices such that $(A_m)_{m\ge 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ and*

$$\lim_{n\to\infty} \left( \|A_n - B_n\| e^{\varepsilon n} \right) = 0. \tag{11}$$

*Then the sequence $(B_m)_{m\ge 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections $P'_m$ satisfying $\dim \operatorname{Im} P_m = \dim \operatorname{Im} P'_m$ for $m \ge 0$.*

**Proof.** Since the sequence $(A_m)_{m\ge 0}$ admits a nonuniform exponential dichotomy, it follows from Proposition 4 that it also admits an exponential dichotomy with respect to some norms $\|\cdot\|_m$ satisfying (7).

On the other hand, by (11) there exists $K > 0$ such that

$$\|A_n - B_n\| \le K e^{-\varepsilon n} \quad \text{for } n \ge 0. \tag{12}$$

By (7) and (12), there exists $c > 0$ such that

$$\|(A_n - B_n)x\|_{n+1} \le c\|x\|_n \quad \text{for } x \in \mathbb{R}^d \text{ and } n \ge 0. \tag{13}$$

It follows from (13) that the linear operator $\tilde{T} : \mathcal{D}(T) \to l_0^\infty$ given by

$$(\tilde{T}\mathbf{x})_0 = 0 \quad \text{and} \quad (\tilde{T}\mathbf{x})_{m+1} = x_{m+1} - B_m x_m, \ m \ge 0,$$

is well defined and bounded. □

**Lemma 5.** *$\tilde{T}$ is a Fredholm operator and $\operatorname{ind} T = \operatorname{ind} \tilde{T}$.*

**Proof of the lemma.** We define an operator $C : \mathcal{D}(T) \to l_0^\infty$ by

$$(C\mathbf{x})_0 = 0 \quad \text{and} \quad (C\mathbf{x})_{m+1} = (B_m - A_m)x_m, \ m \ge 0.$$

In order to show that $C$ is compact, for each $n \in \mathbb{N}$ we define an operator $C^n : \mathcal{D}(T) \to l_0^\infty$ by

$$(C^n\mathbf{x})_m = (B_{m-1} - A_{m-1})x_{m-1} \quad \text{for } m = 1, \dots, n$$

and $(C^n\mathbf{x})_m = 0$ otherwise. Clearly, $C^n$ is compact for each $n$. Moreover, it follows from (7) that there exists $K > 0$ such that

$$\begin{aligned}\|(C - C^n)\mathbf{x}\|_\infty &= \sup_{m>n} \|(B_{m-1} - A_{m-1})x_{m-1}\|_m \\ &\le K \sup_{m>n}\left(e^{\varepsilon m}\|(B_{m-1} - A_{m-1})x_{m-1}\|\right) \\ &\le \gamma_n\|\mathbf{x}\|_T,\end{aligned}$$

where

$$\gamma_n = K \sup_{m>n}\left(e^{\varepsilon m}\|B_{m-1} - A_{m-1}\|\right).$$

By (11) we have $\lim_{n\to+\infty}\gamma_n = 0$ and so the operator $C$ is compact. On the other hand, by Theorem 6, $T$ is a Fredholm operator and since $\tilde{T} = T + C$ with $C$ compact, we conclude that $\tilde{T}$ is also a Fredholm operator, and that $\operatorname{ind} T = \operatorname{ind} \tilde{T}$.

By Theorem 7 together with Lemma 5, the sequence $(B_m)_{m\ge 0}$ admits an exponential dichotomy with respect to the norms $\|\cdot\|_m$. Hence, by Proposition 4, it also admits a nonuniform exponential dichotomy. Moreover, since $\operatorname{ind} T = \operatorname{ind} \tilde{T}$, the projections associated to the two sequences of matrices have the same ranks. □

Finally, we briefly discuss the particular case of exponential stability. We say that the sequence $(A_m)_{m\ge 0}$ is *exponentially* stable with respect to the norms $\|\cdot\|_m$ if it admits an exponential dichotomy with respect to the norms $\|\cdot\|_m$ with projections $P_m = \mathrm{Id}$ for all $n$. The following result is a direct consequence of Theorems 6 and 7 and their proofs.

**Theorem 9.** *The sequence $(A_m)_{m\ge 0}$ is exponentially stable with respect to the norms $\|\cdot\|_m$ if and only if the operator $T$ is Fredholm and $\operatorname{ind} T = d$.*

One can now proceed in a similar manner to that in the proof of Theorem 8 to obtain the robustness property of the notion of nonuniform exponential stability (which is a particular case of the notion of nonuniform exponential dichotomy).

## 4. Two-sided exponential dichotomies

Now we consider exponential dichotomies on $\mathbb{Z}$. We always assume in this section that each norm $\|\cdot\|_n$ is induced by a scalar product $\langle\cdot,\cdot\rangle_n$. We note that for the purposes of our work there is no loss of generality in this hypothesis. Indeed, it is easy to show that one can always consider norms $\|\cdot\|_n$ in Proposition 4 that are induced by scalar products. Then there exist invertible $d\times d$ matrices $D_n$, for $n\in\mathbb{Z}$, such that

$$\langle x, y\rangle_n = \langle D_n x, y\rangle \quad \text{for } n\in\mathbb{Z} \text{ and } x, y\in\mathbb{R}^d.$$

In this section we replace the space $l^\infty$ in (8) by

$$l^\infty_{\mathbb{Z}} = \left\{\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^d : \|\mathbf{x}\|_\infty := \sup_{m\in\mathbb{Z}}\|x_m\|_m < +\infty\right\}.$$

Again $(l^\infty_{\mathbb{Z}}, \|\cdot\|_\infty)$ is a Banach space. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d\times d$ matrices, we consider the linear operator $S : \mathcal{D}(S) \to l^\infty_{\mathbb{Z}}$ defined by

$$(S\mathbf{x})_m = x_m - A_{m-1}x_{m-1}, \quad m\in\mathbb{Z}, \tag{14}$$

in the domain $\mathcal{D}(S)$ composed of the sequence $\mathbf{x}\in l^\infty_{\mathbb{Z}}$ such that $S\mathbf{x}\in l^\infty_{\mathbb{Z}}$. Proceeding as in the proof of Proposition 5, one can show that $S$ is closed. Then we can introduce a graph norm as in (9) so that $S : (\mathcal{D}(S), \|\cdot\|_S) \to l^\infty_{\mathbb{Z}}$ becomes a bounded operator.

**Theorem 10.** *If the sequence $(A_m)_{m\in\mathbb{Z}}$ admits exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ with respect to some norms $\|\cdot\|_m$, then $S$ is a Fredholm operator.*

**Proof.** Under the assumptions of the theorem, it is shown in [2] that the operator $S$ is onto. On the other hand, $\operatorname{Ker} S$ consists of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^\infty_{\mathbb{Z}}$ such that $x_n = \mathcal{A}(n,0)x_0$ for $n\in\mathbb{Z}$. This implies that $x_0 \in \operatorname{Im} P_0^+ \cap \operatorname{Ker} P_0^-$, where $P_m^+$ are the projections associated to the exponential dichotomy on $\mathbb{Z}_0^+$ and $P_m^-$ are the projections associated to the exponential dichotomy on $\mathbb{Z}_0^-$. Therefore, $\dim\operatorname{Ker} S < +\infty$. □

Now let $c_0$ be as in (1) and define

$$R = S|_E : E \to c_0, \quad \text{where} \quad E = S^{-1}c_0.$$

We say that $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ has *bounded support* if there exists $n\in\mathbb{N}$ such that $x_m = 0$ for $|m| > n$.

**Theorem 11.** *If $S$ is a Fredholm operator, then the sequence $(A_m)_{m\in\mathbb{Z}}$ admits exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ with respect to some norms $\|\cdot\|_m$. Moreover, if $R$ is injective, then the sequence admits an exponential dichotomy on $\mathbb{Z}$ with respect to some norms $\|\cdot\|_m$.*

**Proof.** We begin with an auxiliary result. □

**Lemma 6.** *Let $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ be a sequence with bounded support. Then there exists a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ with bounded support such that $S\mathbf{x} = \mathbf{y}$ if and only if*

$$\sum_{k\in\mathbb{Z}} \mathcal{A}(0,k)y_k = 0. \tag{15}$$

**Proof of the lemma.** Assume that $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ satisfies $S\mathbf{x} = \mathbf{y}$. Since $\mathbf{y}$ has bounded support, for all sufficiently large $n > 0$ we have

$$x_n = \mathcal{A}(n,0)\left(x_0 + \sum_{k=1}^{\infty}\mathcal{A}(0,k)y_k\right). \tag{16}$$

Similarly, for all sufficiently small $n < 0$ we have

$$x_n = \mathcal{A}(n,0)\left(x_0 - \sum_{k=-\infty}^{0}\mathcal{A}(0,k)y_k\right). \tag{17}$$

It follows from (16) and (17) that $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ has bounded support if and only if

$$x_0 + \sum_{k=1}^{\infty}\mathcal{A}(0,k)y_k = x_0 - \sum_{k=-\infty}^{0}\mathcal{A}(0,k)y_k = 0,$$

that is, if and only if (15) holds. Now assume that (15) holds and define

$$x_m = \begin{cases} \mathcal{A}(m,0)x_0 + \displaystyle\sum_{k=1}^{m}\mathcal{A}(m,k)y_k & \text{if } m \ge 0, \\ \mathcal{A}(m,0)x_0 - \displaystyle\sum_{k=m+1}^{0}\mathcal{A}(m,k)y_k & \text{if } m < 0, \end{cases}$$

where $x_0 = -\sum_{k=1}^{\infty} \mathcal{A}(0,k)y_k$. Then $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ has bounded support and $S\mathbf{x} = \mathbf{y}$. □

**Lemma 7.** *$\operatorname{Ker} R$ consists of all $\alpha \in c_0^*$ for which there exists a sequence $(y_n)_{n\in\mathbb{Z}} \subset \mathbb{R}^d$ such that:*

1. $y_{n+1} = (A_n^*)^{-1} y_n$ *for* $n \in \mathbb{Z}$ *and*
$$\sum_{n\in\mathbb{Z}} \|D_n^{-1} y_n\|_n < +\infty; \tag{18}$$
2. 
$$\alpha(\mathbf{x}) = \sum_{n\in\mathbb{Z}} \langle D_n^{-1} y_n, x_n\rangle_n \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0. \tag{19}$$

**Proof of the lemma.** Take $\alpha \in \operatorname{Ker} R$. Let $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}^d$ be a sequence with bounded support and let $(\phi_n)_{n\in\mathbb{Z}}$ be a sequence of positive real numbers such that $\sum_{n\in\mathbb{Z}} \phi_n = 1$ and $\phi_n = 0$ for $|n|$ sufficiently large. We define

$$\tilde{x}_n = x_n - \phi_n \sum_{k\in\mathbb{Z}} \mathcal{A}(n,k)x_k \tag{20}$$

for $n \in \mathbb{Z}$. Then $\tilde{\mathbf{x}} = (\tilde{x}_n)_{n\in\mathbb{Z}}$ is a sequence with bounded support and

$$\sum_{n\in\mathbb{Z}} \mathcal{A}(0,n)\tilde{x}_n = \sum_{n\in\mathbb{Z}} \mathcal{A}(0,n)x_n - \sum_{n\in\mathbb{Z}} \phi_n \sum_{k\in\mathbb{Z}} \mathcal{A}(0,k)x_k = 0.$$

It follows from Lemma 6 that $\tilde{\mathbf{x}} \in \operatorname{Im} S$ and thus $\tilde{\mathbf{x}} \in \operatorname{Im} R$. Therefore, $\tilde{\mathbf{x}} = R\mathbf{z}$ for some $\mathbf{z} \in E$ and

$$\alpha(\tilde{\mathbf{x}}) = \alpha(R\mathbf{z}) = (R^*\alpha)\mathbf{z} = 0.$$

Hence, it follows from (20) that

$$\alpha(\mathbf{x}) = \alpha\left(\left(\phi_n \sum_{k\in\mathbb{Z}} \mathcal{A}(n,k)x_k\right)_{n\in\mathbb{Z}}\right).$$

Let

$$y_n = \sum_{i=1}^{d} \alpha\left(\left(\phi_j \mathcal{A}(j,0)e_i\right)_{j\in\mathbb{Z}}\right)\mathcal{A}(0,n)^* e_i,$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$. We note that the sequence $(y_n)_n$ satisfies $y_{n+1} = (A_n^*)^{-1} y_n$ for $n \in \mathbb{Z}$. Then

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \langle D_n^{-1} y_n, x_n\rangle_n &= \sum_{n\in\mathbb{Z}} \langle y_n, x_n\rangle \\
&= \sum_{n\in\mathbb{Z}} \left\langle \sum_{i=1}^{d} \alpha\left(\left(\phi_j \mathcal{A}(j,0)e_i\right)_{j\in\mathbb{Z}}\right)\mathcal{A}(0,n)^* e_i, x_n \right\rangle \\
&= \sum_{n\in\mathbb{Z}} \left\langle \sum_{i=1}^{d} \alpha\left(\left(\phi_j \mathcal{A}(j,0)e_i\right)_{j\in\mathbb{Z}}\right) e_i, \mathcal{A}(0,n)x_n \right\rangle \\
&= \left\langle \sum_{i=1}^{d} \alpha\left(\left(\phi_j \mathcal{A}(j,0)e_i\right)_{j\in\mathbb{Z}}\right) e_i, \sum_{n\in\mathbb{Z}} \mathcal{A}(0,n)x_n \right\rangle \\
&= \sum_{i=1}^{d} \alpha\left(\left(\phi_j \mathcal{A}(j,0)e_i\right)_{j\in\mathbb{Z}}\right) e_i^* \sum_{n\in\mathbb{Z}} \mathcal{A}(0,n)x_n \\
&= \alpha\left(\left(\phi_j \sum_{n\in\mathbb{Z}} \mathcal{A}(j,n)x_n\right)_{j\in\mathbb{Z}}\right) = \alpha(\mathbf{x})
\end{aligned} \tag{21}$$

for all sequences $\mathbf{x}$ with bounded support. In particular,

$$\left|\sum_{n\in\mathbb{Z}} \langle D_n^{-1} y_n, x_n\rangle_n\right| = |\alpha(\mathbf{x})| \le \|\alpha\| \cdot \|\mathbf{x}\| \tag{22}$$

for all $\mathbf{x}$ with bounded support. Take $K \in \mathbb{N}$ and define $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ by

$$x_n = \begin{cases} \dfrac{D_n^{-1} y_n}{\|D_n^{-1} y_n\|_n} & \text{if } n \in [-K, K], \\ 0 & \text{if } n \notin [-K, K]. \end{cases}$$

By (22), we obtain

$$\sum_{n=-K}^{K} \|D_n^{-1} y_n\|_n \le \|\alpha\|$$

and since $K$ is arbitrary, we conclude that (18) holds.

Now we define $\beta : c_0 \to \mathbb{R}$ by

$$\beta(\mathbf{x}) = \sum_{n\in\mathbb{Z}} \langle D_n^{-1} y_n, x_n\rangle_n \quad \text{for} \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0.$$

It follows from (18) that $\beta$ is a bounded linear functional on $c_0$ and by (21), $\beta$ and $\alpha$ coincide on the dense set of the sequences with bounded support. Hence, $\alpha = \beta$ and (19) holds.

Now assume that $\alpha$ is given by (19) with $(y_n)_{n\in\mathbb{Z}}$ as in Lemma 7. For each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0$ we have

$$\begin{aligned}
(R^*\alpha)(\mathbf{x}) &= \alpha(R\mathbf{x}) \\
&= \sum_{n\in\mathbb{Z}} \langle D_n^{-1} y_n, x_n - A_{n-1}x_{n-1}\rangle_n \\
&= \sum_{n\in\mathbb{Z}} \left(\langle D_n^{-1} y_n, x_n\rangle_n - \langle D_n^{-1} y_n, A_{n-1}x_{n-1}\rangle_n\right) \\
&= \sum_{n\in\mathbb{Z}} \left(\langle D_n^{-1} y_n, x_n\rangle_n - \langle D_{n-1}^{-1} y_{n-1}, x_{n-1}\rangle_{n-1}\right) \\
&\quad + \sum_{n\in\mathbb{Z}} \left(\langle D_{n-1}^{-1} y_{n-1}, x_{n-1}\rangle_{n-1} - \langle D_n^{-1} y_n, A_{n-1}x_{n-1}\rangle_n\right) \\
&= \lim_{n\to+\infty} \langle D_n^{-1} y_n, x_n\rangle_n - \lim_{n\to-\infty} \langle D_n^{-1} y_n, x_n\rangle_n \\
&\quad + \sum_{n\in\mathbb{Z}} \left(\langle y_{n-1}, x_{n-1}\rangle - \langle y_n, A_{n-1}x_{n-1}\rangle\right) \\
&= \sum_{n\in\mathbb{Z}} \langle y_{n-1} - A_{n-1}^* y_n, x_{n-1}\rangle = 0.
\end{aligned}$$

Therefore $\alpha \in \operatorname{Ker} R^*$ and the proof of the lemma is complete. □

We continue with the proof of the theorem. We first observe that

$$\operatorname{Im} R = \left\{\mathbf{x} \in c_0 : \alpha(\mathbf{x}) = 0 \text{ for } \alpha \in \operatorname{Ker} R^*\right\}. \tag{23}$$

Let $\mathbf{y}^i = (y_n^i)_{n\in\mathbb{Z}}$, for $i = 1, \ldots, m$, be a basis of the space of solutions $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ of the equation

$$y_{n+1} = (A_n^*)^{-1} y_n, \quad n \in \mathbb{Z},$$

such that $\sum_{n\in\mathbb{Z}} \|D_n^{-1} y_n\|_n < +\infty$. For each $i = 1, \ldots, m$, define $\alpha_i \in c_0^*$ by

$$\alpha_i(\mathbf{x}) = \sum_{n=-\infty}^{-1} \langle D_n^{-1} y_n^i, x_n\rangle_n \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0.$$

Moreover, for each $j = 1, \dots, d$, we define $\beta_j \in c_0^*$ by

$$\beta_j(\mathbf{x}) = x_0^j \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0.$$

**Lemma 8.** *The set $\{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_d\}$ is linearly independent.*

**Proof of the lemma.** Assume that $\sum_{i=1}^m \lambda_i\alpha_i = \sum_{j=1}^d \mu_j\beta_j$ for some constants $\lambda_i, \mu_j \in \mathbb{R}$. This implies that

$$\sum_{n=-\infty}^{-1} \langle D_n^{-1} y_n, x_n\rangle_n = \sum_{j=1}^{d} \mu_j x_0^j \tag{24}$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0$, where

$$\mathbf{y} = (y_n)_{n\in\mathbb{Z}} = \sum_{i=1}^m \lambda_i \mathbf{y}^i.$$

Assume that $y_{n_0} \neq 0$ for some $n_0 < 0$. Applying (24) to the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ given by

$$x_n = \begin{cases} D_{n_0}^{-1} y_{n_0}/\|D_{n_0}^{-1} y_{n_0}\| & \text{if } n = n_0,\\ 0 & \text{if } \neq n_0 \end{cases}$$

we reach a contradiction. Hence, $\mathbf{y} = 0$, which implies that $\lambda_1 = \cdots = \lambda_m = 0$. On the other hand, the functionals $\beta_1, \dots, \beta_d$ are clearly independent and so $\mu_1 = \cdots = \mu_d = 0$.

Now take $\mathbf{z} = (z_n)_{n\in\mathbb{Z}} \in c_0$ with $z_0 = 0$. By Lemma 8, we can choose $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in c_0$ such that

$$\alpha_i(\mathbf{x}) = -\sum_{n=0}^{\infty} \langle D_n^{-1} y_n^i, z_n\rangle_n \quad \text{for } i = 1, \dots, m, \tag{25}$$

and

$$\beta_j(\mathbf{x}) = x_0^i = 0 \quad \text{for } j = 1, \dots, d,$$

that is $x_0 = 0$. We define $\tilde{\mathbf{x}} = (\tilde{x}_n)_{n\in\mathbb{Z}}$ by $\tilde{x}_n = x_n$ for $n \leq 0$ and $\tilde{x}_n = z_n$ for $n \geq 0$. Clearly, $\tilde{\mathbf{x}} \in c_0$. It follows from (25) that

$$\sum_{n\in\mathbb{Z}} \langle D_n^{-1} y_n^i, \tilde{x}_n\rangle_n = 0 \quad \text{for } i = 1, \dots, m$$

and thus, it follows from Lemma 7 that $\alpha(\tilde{\mathbf{x}}) = 0$ for $\alpha \in \operatorname{Ker} R^*$. It follows from (23) that $\tilde{\mathbf{x}} \in \operatorname{Im} R$. Therefore, there exists a sequence $\mathbf{w} = (w_n)_{n\in\mathbb{Z}} \in l_{\mathbb{Z}}^\infty$ such that $R\mathbf{w} = \tilde{\mathbf{x}}$ and hence

$$w_{n+1} - A_n w_n = \tilde{x}_{n+1} \quad \text{for } n \geq 0.$$

It follows now from results in [3] that the sequence $(A_n)_n$ admits an exponential dichotomy on $\mathbb{Z}_0^+$ with respect to some norms $\|\cdot\|_m$. One establishes similarly the existence of an exponential dichotomy on $\mathbb{Z}_0^-$.

Assume now that $R$ is injective. Then $R$ is bijective and it follows from the results in [1] that the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy on $\mathbb{Z}$ with respect to some norms $\|\cdot\|_m$. $\square$

We also establish a version of Theorem 8 for exponential dichotomies on $\mathbb{Z}$.

**Theorem 12.** *Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be sequences of invertible $d \times d$ matrices such that $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$ and*

$$\lim_{n\to\pm\infty} \left(\|A_n - B_n\| e^{\varepsilon|n|}\right) = 0. \tag{26}$$

*Then the sequence $(B_m)_{m\in\mathbb{Z}}$ admits nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$. Moreover, if there exists no nonzero sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ such that $x_{n+1} = B_n x_n$ for $n \in \mathbb{Z}$ and $\sup_{n\in\mathbb{Z}} \|x_n\|_n < +\infty$, where*

$$\|x\|_m = \sup_{n\geq m}\left(\|\mathcal{A}(n,m)P_m x\| e^{\lambda(n-m)}\right) + \sup_{n\leq m}\left(\|\mathcal{A}(n,m)Q_m x\| e^{\lambda(m-n)}\right), \tag{27}$$

*then $(B_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$.*

**Proof.** It follows from (3) and (4) that

$$\|x\| \leq \|x\|_m \leq 2De^{\varepsilon|m|}\|x\| \quad \text{for } x \in \mathbb{R}^d \text{ and } m \in \mathbb{Z}, \tag{28}$$

for the norms $\|\cdot\|_m$ in (27). Let $l_{\mathbb{Z}}^\infty$ be defined with respect to those norms. By Theorem 10, the operator $S$ defined by (14) is Fredholm. Now we consider the operator $C : l_{\mathbb{Z}}^\infty \to l_{\mathbb{Z}}^\infty$ defined by

$$(C\mathbf{x})_{m+1} = (A_m - B_m)x_m, \quad m \in \mathbb{Z}.$$

Using (26) and (28), it is easy to verify that $C$ is a well-defined compact operator. We also define an operator $U : \mathcal{D}(S) \to l_{\mathbb{Z}}^\infty$ by

$$(U\mathbf{x})_{m+1} = x_{m+1} - B_m x_m, \quad m \in \mathbb{Z}.$$

Then $U = S + C$ is a Fredholm operator. It follows from Theorem 11 and Proposition 4 that the sequence $(B_m)_{m\in\mathbb{Z}}$ admits nonuniform exponential dichotomies on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$. In view of Theorem 10 this also implies that $U$ is onto.

Finally, under the assumption that there exists no nonzero sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \neq 0$ as in the statement of the theorem, the operator $U$ is injective and hence $U$ it is also bijective. $\square$

## References


[1] Barreira L, Dragičević D, Valls C. Nonuniform hyperbolicity and admissibility. Adv Nonlinear Stud 2014;14:791–811.
[2] Barreira L, Dragičević D, Valls C. From one-sided dichotomies to two-sided dichotomies. Discret Contin Dyn Syst 2015;35:2817–44.
[3] Barreira L., Dragičević D., Valls C. Nonuniform hyperbolicity and one-sided admissibility, Rend. Lincei Mat. Appl., to appear.
[4] Barreira L, Pesin Ya. Nonuniform hyperbolicity, encyclopedia of mathematics and its applications 115. Cambridge University Press; 2007.
[5] Barreira L, Valls C. Stability of nonautonomous differential equations. Lecture notes in mathematics, vol. 1926. Springer; 2008.
[6] Blázquez C. Transverse homoclinic orbits in periodically perturbed parabolic equations. Nonlinear Anal 1986;10:1277–91.
[7] Chow SN, Leiva H. Unbounded perturbation of the exponential dichotomy for evolution equations. J Differ Equ 1996;129:509–31.
[8] Coppel W. Dichotomies and reducibility. J Differ Equ 1967;3:500–21.
[9] Dalec'kiĭ Ju, Kreĭn M. Stability of solutions of differential equations in Banach space. Transl Math Monogr Am Math Soc 1974;43.
[10] Dragičević D, Slijepčević S. Characterization of hyperbolicity and generalized shadowing lemma. Dyn Syst 2011;24:483–502.
[11] Lin XB. Exponential dichotomies and homoclinic orbits in functional-differential equations. J Differ Equ 1986;63:227–54.
[12] Massera J, Schäffer J. Linear differential equations and functional analysis. Ann Math I 1958;2(67):517–73.
[13] Naulin R, Pinto M. Admissible perturbations of exponential dichotomy roughness. Nonlinear Anal 1998;31:559–71.
[14] Palmer K. Exponential dichotomies and transversal homoclinic points. J Differ Equ 1984;55:225–56.
[15] Palmer K. Exponential dichotomies and Fredholm operators. Proc Am Math Soc 1988;104:149–56.
[16] Pesin Ya. Families of invariant manifolds corresponding to nonzero characteristic exponents. Math USSR Izv 1976;10:1261–305.
[17] Pliss V, Sell G. Robustness of exponential dichotomies in infinite-dimensional dynamical systems. J Dyn Differ Equ 1999;11:471–513.

[18] Popescu L. Exponential dichotomy roughness on Banach spaces. J Math Anal Appl 2006;314:436–54.
[19] Rodrigues H, Ruas-Filho J. Evolution equations: dichotomies and the Fredholm alternative for bounded solutions. J Differ Equ 1995;119:263–83.
[20] Rodrigues H, Silveira M. Properties of bounded solutions of linear and nonlinear evolution equations: Homoclinics of a beam equation. J Differ Equ 1987;70:403–40.
[21] Sacker R, Sell G. Dichotomies for linear evolutionary equations in Banach spaces. J Differ Equ 1994;113:17–67.
[22] Taylor A, Lay D. Introduction to functional analysis. John Wiley & Sons; 1980.
[23] Zeng W. Transversality of homoclinic orbits and exponential dichotomies for parabolic equations. J Math Anal Appl 1997;216:466–80.
[24] Zhang W. The Fredholm alternative and exponential dichotomies for parabolic equations. J Math Anal Appl 1985;191:180–201.