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Strong nonuniform spectrum for arbitrary growth rates

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We consider the notion of *strong* nonuniform spectrum for a nonautonomous dynamics with discrete time obtained from a sequence of matrices, which is defined in terms of the existence of strong nonuniform exponential dichotomies with an arbitrarily small nonuniform part. The latter exponential dichotomies are ubiquitous in the context of ergodic theory and correspond to have *both* lower and upper bounds along the stable and unstable directions, besides possibly a nonuniform conditional stability although with an arbitrarily small exponential dependence on the initial time. Moreover, we consider arbitrary growth rates instead of only the usual exponential rates. We give a complete characterization of the possible strong nonuniform spectra and for a Lyapunov regular trajectory, we show that the spectrum is the set of Lyapunov exponents. In addition, we provide explicit examples of nonautonomous dynamics for all possible strong nonuniform spectra. A remarkable consequence of our results is that for a sequence of matrices A_m, either $e^{-a}A_m$ does not admit a strong exponential dichotomy for any $a \in \mathbb{R}$, or if $e^{-a}A_m$ admits an exponential dichotomy for some $a \in \mathbb{R}$, then it also admits a strong exponential dichotomy for that a. We emphasize that this result is not in the literature even in the special case of uniform exponential dichotomies.

Keywords: Exponential dichotomies; robustness; spectrum.

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1. Introduction

1.1. *Uniform exponential behavior*

The spectrum introduced by Sacker and Sell in [11] can be seen as a generalization of the spectrum of a matrix (the set of its eigenvalues). In order to explain why this is so, let $(A_m)_{m\in\mathbb{Z}}$ be a two-sided sequence of invertible $d\times d$ matrices and consider the dynamics

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{Z}$$

on $\mathbb{R}^d$. When $A_m = A$ for $m \in \mathbb{Z}$ and some invertible matrix A, for each given $a \in \mathbb{R}$ the following properties are equivalent:

(1) $a = -\log|\mu|$ for some eigenvalue μ of A;
(2) the sequence $(e^{-a}A)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy.

We recall briefly the notion of a uniform exponential dichotomy, which corresponds to have exponential bounds along certain (stable and unstable) subspaces. For each $m, n \in \mathbb{Z}$, let

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n, & m > n, \\ \mathrm{Id}, & m = n, \\ A_m^{-1}\cdots A_{n-1}^{-1}, & m < n. \end{cases} \tag{1}$$

We say that sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *uniform exponential dichotomy* if there exist projections P_m for $m \in \mathbb{Z}$ (that is, matrices with $P_m^2 = P_m$) satisfying

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n \quad \text{for } m,n \in \mathbb{Z} \tag{2}$$

and constants $D, \lambda > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)} \quad \text{for } m \ge n \tag{3}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)} \quad \text{for } m \le n, \tag{4}$$

where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$. The families of stable and unstable subspaces are respectively $P_m(\mathbb{R}^d)$ and $Q_m(\mathbb{R}^d)$. In the particular case of a constant sequence $A_m = A$ this is equivalent to require that there exist a projection P satisfying $PA = AP$ and constants $D, \lambda > 0$ such that

$$\|(e^{-a}A)^m P\| \le De^{-\lambda m} \quad \text{and} \quad \|(e^{-a}A)^{-m}Q\| \le De^{-\lambda m}$$

for $m \ge 0$, where $Q = \mathrm{Id} - P$ is also a projection. The equivalence of properties (1) and (2) motivates defining a *spectrum* associated to an arbitrary sequence $(A_m)_{m\in\mathbb{Z}}$

as the set of all numbers $a \in \mathbb{R}$ such that $(e^{-a}A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy. This is precisely the notion introduced in [11], in fact in the more general case of linear cocycles or, equivalently, linear skew product flows (which essentially corresponds to consider families of sequences of matrices instead of a single sequence A_m), although with restrictive assumptions on the base.

1.2. *Nonuniform exponential behavior*

In this paper we consider a *nonuniform* version of the spectrum introduced by Sacker and Sell, in the sense that the estimates in (3) and (4) can be spoiled with arbitrarily small exponentials, thus leading to a nonuniform stability along the stable and unstable subspaces. Before proceeding, we recall that a sequence $(A_m)_{m\in\mathbb{Z}}$ is said to admit a *nonuniform exponential dichotomy with an arbitrarily small nonuniform part* if there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \ge n \tag{5}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \le n, \tag{6}$$

where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$. The extra terms $e^{\varepsilon|n|}$ in (5) and (6) correspond to the "nonuniform part" of the exponential dichotomy. We emphasize that all the requirements in the notion of a nonuniform exponential dichotomy with an arbitrarily small nonuniform part are ubiquitous in the context of ergodic theory. More precisely, let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism preserving a probability measure μ on $\mathbb{R}^d$. This means that

$$\mu(f^{-1}A) = \mu(A)$$

for every measurable set $A \subset \mathbb{R}^d$. For example, any time-1 map of a Hamiltonian flow preserves the Liouville measure on each energy level, and so there are plenty examples already in the somewhat classical context of mechanical systems. We also consider the Lyapunov exponents

$$\lambda(x,v) = \limsup_{m\to\infty} \frac{1}{m} \log\|d_x f^m v\|$$

for $x, v \in \mathbb{R}^d$ with $v \neq 0$. If $\log^+\|df\| = \max\{0, \log\|df\|\}$ is μ-integrable (for example, if the measure μ has compact support, such as the Liouville measure on any compact energy level), then for μ-almost every x with $\lambda(x,v) \neq 0$ for all $v \neq 0$ the sequence of matrices

$$A_m = d_{f^m(x)}f, \quad m \in \mathbb{Z} \tag{7}$$

admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part (we refer the reader to [3, 6] for details and references).

1.3. *Strong nonuniform exponential behavior*

In fact we consider an even stronger version of exponential dichotomy that is also ubiquitous in the context of ergodic theory. Roughly speaking, besides requiring contraction and expansion respectively along the stable and unstable subspaces, one requires lower bounds in the stable direction for positive time and in the unstable direction for negative time. Namely, we say that a sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *strong nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *strong dichotomy* if there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), constants $\mu \geq \lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that in addition to the inequalities in (5) and (6), we have

$$\|\mathcal{A}(m,n)Q_n\| \leq De^{\mu(m-n)+\varepsilon|n|} \quad \text{for } m \geq n \tag{8}$$

and

$$\|\mathcal{A}(m,n)P_n\| \leq De^{\mu(n-m)+\varepsilon|n|} \quad \text{for } m \leq n.$$

Example 1. The sequence of real numbers

$$A_m = e^{(m+1)^3-m^3}, \quad m \in \mathbb{Z}$$

admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part (with $P_m = 0$ for each m), but not a strong dichotomy (that is, a strong nonuniform exponential dichotomy with an arbitrarily small nonuniform). Indeed,

$$\mathcal{A}(m,n) = e^{m^3-n^3} = e^{(m-n)(m^2+mn+n^2)}$$

and since the quadratic form $x^2 + xy + y^2$ is positive definite, there exists $\lambda > 0$ such that $\mathcal{A}(m,n) \geq e^{\lambda(m-n)}$ for $m \geq n$. This yields inequalities (5) and (6) taking $P_m = 0$ for each $m \in \mathbb{Z}$, $D = 1$ and ε arbitrary. On the other hand, inequality (8), that is,

$$e^{(m-n)(m^2+mn+n^2)} \leq De^{\mu(m-n)+\varepsilon|n|}$$

cannot hold for any μ: for a given n let $m \to +\infty$.

Again, let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism preserving a probability measure μ. If $\log^+\|df\|$ is μ-integrable, then for μ-almost every point with nonzero Lyapunov exponents the sequence of matrices in (7) admits a strong dichotomy (see [3, 6] for details).

1.4. *The strong nonuniform spectrum*

Now we introduce the notion of strong nonuniform spectrum. While the original notion introduced in [11] was defined in terms of uniform exponential dichotomies, we define the spectrum in terms of strong dichotomies. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d \times d$ matrices, its *strong nonuniform spectrum* is the set Σ of all numbers $a \in \mathbb{R}$ such that the sequence $(e^{-a}A_m)_{m\in\mathbb{Z}}$ does not admit a strong dichotomy. In particular, we describe the structure of the strong nonuniform spectrum and how it relates to certain subspaces $W_i(n)$ (see Theorem 4). Namely, the spectrum can be either the whole $\mathbb{R}$ or a finite union $\bigcup_{i=1}^k [a_i, b_i]$, for some real numbers

$$a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$$

and some integer $k \le d$. Moreover, we show that each nonzero vector in the subspace $W_i(n)$ has (lower and upper) Lyapunov exponents

$$\liminf_{m\to\pm\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \quad \text{and} \quad \limsup_{m\to\pm\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\|$$

inside the same interval $[a_i, b_i]$ of the spectrum. We emphasize that in strong contrast to what happens in work of Aulbach and Siegmund in [2] for the spectrum defined in terms of uniform exponential dichotomies (following closely work of Siegmund in [13] in the case of continuous time), the strong nonuniform spectrum is never empty and it is either compact or the whole line. The construction of the associated invariant subspaces follows a simple yet powerful idea apparently used first by Oseledets in [10] in his proof of the multiplicative ergodic theorem (see [4]).

Moreover, we obtain related results for continuous time (see Sec. 7). The proofs are analogous to those for discrete time and so we omit them. For related work in the case of uniform exponential dichotomies (that are not necessarily strong) we refer the reader to [1, 7–9, 12] (in particular, [1] considers noninvertible systems of difference equations, [9] describes the relation to ergodic theory and [8, 12] study infinite-dimensional systems).

1.5. *Application to strong dichotomies*

This characterization of the strong nonuniform spectrum turns out to imply the following remarkable property of exponential dichotomies that to the best of our knowledge is not in the literature even in the special case of uniform exponential behavior. In a certain sense this is the main contribution of our work (see Sec. 5).

Theorem 1. *If $\Sigma \ne \mathbb{R}$ and the sequence of matrices $e^{-a}A_m$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part, then it also admits a strong nonuniform exponential dichotomy with an arbitrarily small nonuniform part (that is, a strong dichotomy).*

In other words, we have the following alternative:

(1) either no sequence of matrices $e^{-a}A_m$ admits a strong dichotomy (for any $a \in \mathbb{R}$), in which case $\Sigma = \mathbb{R}$;
(2) or any sequence of matrices $e^{-a}A_m$ that admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part also admits a strong dichotomy, in which case $\Sigma \ne \mathbb{R}$.

Further properties are described at the end of Sec. 5.

1.6. *Lyapunov regular sequences*

In addition, in the particular case of a Lyapunov regular sequence of matrices, we show that the strong nonuniform spectrum is the set of the Lyapunov exponents (see Theorems 6 and 7). We recall that a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d\times d$ matrices is said to be *Lyapunov regular* if there exist a decomposition $\mathbb{R}^d = \bigoplus_{i=1}^s E_i$ and real numbers $\lambda_1 < \cdots < \lambda_s$ for some integer $s \le d$ such that

$$\lim_{m\to\pm\infty} \frac{1}{m} \log\|\mathcal{A}(m,0)v\| = \lambda_i \tag{9}$$

for $i = 1,\ldots,s$ and $v \in E_i\setminus\{0\}$, and

$$\lim_{m\to\pm\infty} \frac{1}{m} \log|\det \mathcal{A}(m,0)| = \sum_{i=1}^s \lambda_i \dim E_i. \tag{10}$$

Essentially, this corresponds to control not only the asymptotic behavior into the future and into the past, but also the angles between the various directions, which cannot approach with exponential speed.

Example 2. The sequence of matrices

$$A_m = \begin{pmatrix} 1 & 0 \\ -2^{m+1} & 4 \end{pmatrix}, \quad m \in \mathbb{Z}$$

satisfies property (9) but not property (10). Indeed, we have

$$\mathcal{A}(m,0) = \begin{pmatrix} 1 & 0 \\ 2^m & 4^m \end{pmatrix}, \quad m \in \mathbb{Z},$$

which shows that $\lambda_1 = \log 2$ with E_1 generated by $e_1 = (1,0)$, and $\lambda_2 = \log 4$ with E_2 generated by $e_2 = (0,1)$. However, $\det \mathcal{A}(m,0) = 4^m$ and so the left-hand and right-hand sides of (10) are respectively $\log 4$ and $\log 2 + \log 4$. One can easily verify that the angle between $\mathcal{A}(m,0)e_1$ and $\mathcal{A}(m,0)e_2$ goes to zero exponentially when $m \to \infty$.

On the other hand, any constant or periodic sequence of matrices is Lyapunov regular. Further examples are given in Sec. 6. The discussion of the technical details of the former interpretation of the notion of Lyapunov regularity is out of the scope of our work (again we refer the reader to [3, 6] for full details). Nevertheless, we remark that the notion of Lyapunov regularity is also ubiquitous in the context of ergodic theory: if $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism preserving a probability measure μ such that $\log^+\|df\|$ is μ-integrable, then for μ-almost every x the sequence of matrices in (7) is Lyapunov regular (see [3]).

1.7. *Arbitrary growth rates*

We note that in all the results of the paper we allow the usual exponential rate to be replaced by an arbitrary growth rate, that is, an increasing function $\rho\colon \mathbb{Z} \to \mathbb{R}$

satisfying

$$\lim_{n\to-\infty}\rho(n)=-\infty \quad \text{and} \quad \lim_{n\to+\infty}\rho(n)=+\infty. \tag{11}$$

For example, for a sequence of matrices $(A_m)_{m\in\mathbb{Z}}$ the notion of a *ρ-uniform exponential dichotomy* is obtained from that of a uniform exponential dichotomy by replacing inequalities (3) and (4) respectively by

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(\rho(m)-\rho(n))} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(\rho(n)-\rho(m))} \quad \text{for } m \le n.$$

When $\rho(n)=n$ we recover the usual notion of a uniform exponential dichotomy. This generalization to arbitrary growth rates corresponds to consider dynamics that may have zero or infinite Lyapunov exponents with respect to the usual rate $\rho(n)=n$ but not with respect to other rates.

2. Preliminaries

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible $d\times d$ matrices. For each $m,n\in\mathbb{Z}$ we define $\mathcal{A}(m,n)$ as in (1). Now let $\rho\colon\mathbb{Z}\to\mathbb{R}$ be an increasing function satisfying (11). We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *ρ-strong nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *ρ-strong dichotomy* if there exist projections P_m for $m\in\mathbb{Z}$ satisfying (2), constants $\mu\ge\lambda>0$ and for each $\varepsilon>0$ a constant $D=D(\varepsilon)>0$ such that

$$\begin{aligned}\|\mathcal{A}(m,n)P_n\| &\le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},\\ \|\mathcal{A}(m,n)Q_n\| &\le De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}\end{aligned} \tag{12}$$

for $m\ge n$ and

$$\begin{aligned}\|\mathcal{A}(m,n)Q_n\| &\le De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},\\ \|\mathcal{A}(m,n)P_n\| &\le De^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}\end{aligned} \tag{13}$$

for $m\le n$, where $Q_m=\mathrm{Id}-P_m$ for each $m\in\mathbb{Z}$. The families of *stable* and *unstable subspaces* are respectively $P_m(\mathbb{R}^d)$ and $Q_m(\mathbb{R}^d)$.

We first show that the images of the projections P_m and Q_m are uniquely determined.

Proposition 2. *For each $n\in\mathbb{Z}$, we have*

$$P_n(\mathbb{R}^d)=\left\{v\in\mathbb{R}^d : \limsup_{m\to+\infty}\frac{1}{\rho(m)}\log\|\mathcal{A}(m,n)v\|<0\right\}$$

and

$$Q_n(\mathbb{R}^d) = \left\{v \in \mathbb{R}^d : \limsup_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|\mathcal{A}(m,n)v\| < 0\right\}.$$

Proof. It follows from (12) that

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| < 0 \tag{14}$$

for $v \in P_n(\mathbb{R}^d)$. On the other hand, if $v \in \mathbb{R}^d$ satisfies (14), then it follows again from (12) that

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)Q_n v\| < 0. \tag{15}$$

By (13), for $m \geq n$ we have

$$\|Q_n v\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(m)|}\|\mathcal{A}(m,n)Q_n v\|,$$

that is,

$$\frac{1}{D}e^{\lambda(\rho(m)-\rho(n))-\varepsilon|\rho(m)|}\|Q_n v\| \leq \|\mathcal{A}(m,n)Q_n v\|.$$

Whenever $Q_n v \neq 0$, this yields that

$$0 < \lambda - \varepsilon \leq \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)Q_n v\|$$

for any sufficiently small $\varepsilon > 0$, which contradicts to inequality (15). Therefore, $Q_n v = 0$ and $v \in P_n(\mathbb{R}^d)$. The proof of the second assertion of the lemma is completely analogous. □

For a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d \times d$ matrices, its *strong nonuniform spectrum* is the set Σ of all $a \in \mathbb{R}$ such that the sequence $(B_m)_{m\in\mathbb{Z}}$, where

$$B_m = e^{-a(\rho(m+1)-\rho(m))}A_m \quad \text{for } m \in \mathbb{Z}, \tag{16}$$

does not admit a ρ-strong dichotomy. For each $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, let

$$S_a(n) = \left\{v \in \mathbb{R}^d : \limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| < a\right\}$$

and

$$U_a(n) = \left\{v \in \mathbb{R}^d : \limsup_{m\to-\infty} \frac{1}{|\rho(m)|} \log\|\mathcal{A}(m,n)v\| < -a\right\}.$$

It follows from Proposition 2 that if $a \in \mathbb{R}\setminus\Sigma$, then

$$\mathbb{R}^d = S_a(n) \oplus U_a(n) \quad \text{for } n \in \mathbb{Z}, \tag{17}$$

with the projections P_n and Q_n associated to the sequence $(B_m)_{m\in\mathbb{Z}}$ satisfying $P_n(\mathbb{R}^d) = S_a(n)$ and $Q_n(\mathbb{R}^d) = U_a(n)$ for $n \in \mathbb{Z}$. For each $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$A_n S_a(n) = S_a(n+1) \quad \text{and} \quad A_n U_a(n) = U_a(n+1). \tag{18}$$

In particular, this implies that the dimensions $\dim S_a(n)$ and $\dim U_a(n)$ are independent of n. We shall denote the common values simply by $\dim S_a$ and $\dim U_a$ respectively. Moreover, if $a < a'$, then

$$S_a(n) \subset S_{a'}(n) \quad \text{and} \quad U_{a'}(n) \subset U_a(n).$$

3. Structure of the Spectrum

In this section we describe completely the structure of the strong nonuniform spectrum as well as its associated invariant subspaces.

We start with a preliminary result. Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible $d \times d$ matrices and let ρ be an increasing function satisfying (11).

Proposition 3. *The set $\Sigma \subset \mathbb{R}$ is closed. Moreover, for each $a \in \mathbb{R}\backslash\Sigma$ we have $S_a(n) = S_b(n)$ and $U_a(n) = U_b(n)$ for all $n \in \mathbb{Z}$ and all b in some open neighborhood of a.*

Proof. Given $a \in \mathbb{R}\backslash\Sigma$, there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2), constants $\lambda, \mu > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},$$
$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},$$
$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. Therefore, for each $b \in \mathbb{R}$ we have

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{-(\lambda-a+b)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},$$
$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{(\mu+a-b)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{-(\lambda+a-b)(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},$$
$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{(\mu-a+b)(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. In particular, whenever

$$|a-b| < \min\{\lambda, \mu\},$$

we find that $b \in \mathbb{R}\backslash\Sigma$, $S_b(n) = S_a(n)$ and $U_b(n) = U_a(n)$ for $n \in \mathbb{Z}$. □

Our main result is the following. It describes the structure of the strong nonuniform spectrum and how it relates to certain invariant subspaces. We define the *angle* between two subspaces $E, F \subset \mathbb{R}^d$ by

$$\angle(E,F) = \inf\{\|x-y\| : x \in E, y \in F, \|x\| = \|y\| = 1\}.$$

Theorem 4. *For a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $d\times d$ matrices*:

(1) *either $\Sigma=\mathbb{R}$ or $\Sigma=[a_1,b_1]\cup\cdots\cup[a_k,b_k]$, for some finite numbers*

$$a_1\le b_1<a_2\le b_2<\cdots<a_k\le b_k \quad \textit{and} \quad k\le d; \tag{19}$$

(2) *in the second case, taking numbers $c_0<a_1, c_k>b_k$ and*

$$c_i\in(b_i,a_{i+1}) \quad \textit{for } i=1,\ldots,k-1$$

for each $n\in\mathbb{Z}$ the subspaces $W_i(n)=U_{c_{i-1}}(n)\cap S_{c_i}(n)$ satisfy

$$A_nW_i(n)=W_i(n+1) \quad \textit{for } i=1,\ldots,k \tag{20}$$

and form the direct sum

$$\mathbb{R}^d=\bigoplus_{i=1}^k W_i(n); \tag{21}$$

(3) *the subspaces $W_i(n)$ are independent of the numbers $c_0,\ldots,c_k$;*
(4) *for each $i=1,\ldots,k$ and $v\in W_i(n)\backslash\{0\}$ we have*

$$a_i\le\liminf_{m\to\pm\infty}\frac{1}{\rho(m)}\log\|v_m\|\le\limsup_{m\to\pm\infty}\frac{1}{\rho(m)}\log\|v_m\|\le b_i, \tag{22}$$

where $v_m=\mathcal{A}(m,n)v$;
(5) *for each $i,j=1,\ldots,k$ with $i\ne j$ we have*

$$\lim_{n\to\pm\infty}\frac{1}{\rho(n)}\log\angle(W_i(n),W_j(n))=0. \tag{23}$$

Proof. Assume that $\Sigma\ne\mathbb{R}$ and take $a\in\mathbb{R}\backslash\Sigma$. Then, there exist projections P_n for $n\in\mathbb{Z}$ satisfying (2), constants $\lambda,\mu>0$ and for each $\varepsilon>0$ a constant $D=D(\varepsilon)>0$ such that

$$\begin{aligned}\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\|&\le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},\\ \|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\|&\le De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}\end{aligned} \tag{24}$$

for $m\ge n$ and

$$\begin{aligned}\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\|&\le De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},\\ \|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\|&\le De^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}\end{aligned} \tag{25}$$

for $m\le n$. It follows from (24) and (25) that

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)\|\le 2De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \tag{26}$$

for $m\ge n$ and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)\|\le 2De^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \tag{27}$$

for $m\le n$. Now take $b>\mu+a$. Then

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)\|\le 2De^{(\mu+a-b)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-b(\rho(m)-\rho(n))}\mathcal{A}(m,n)\| \leq 2De^{(\mu-a+b)(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. Since $\mu - a + b > -\mu - a + b > 0$, this shows that $(\mu + a, +\infty) \subset \mathbb{R}\backslash\Sigma$ and $S_b(n) = \mathbb{R}^d$ for $b \in (\mu + a, \infty)$. One can show in a similar manner that $(-\infty, \mu - a) \subset \mathbb{R}\backslash\Sigma$ and $S_b(n) = \{0\}$ for $b \in (-\infty, \mu - a)$. In particular, $\Sigma \subset [\mu - a, \mu + a]$ and so Σ is compact.

In order to show that Σ is nonempty, let

$$c = \inf\{\rho \in \mathbb{R}\backslash\Sigma : S_\rho(n) = \mathbb{R}^d\}.$$

Clearly, $\mu - a \leq c \leq \mu + a$. Now assume that $c \notin \Sigma$. Then:

(1) If $S_c(n) = \mathbb{R}^d$, then by Proposition 3 we have $S_{\rho'}(n) = \mathbb{R}^d$ and $\rho' \in \mathbb{R}\backslash\Sigma$ for all $\rho' \in (c - \varepsilon, c]$ and some $\varepsilon > 0$. But this contradicts to the definition of c.
(2) If $S_c(n) \neq \mathbb{R}^d$, then by Proposition 3 we have $S_{\rho'}(n) \neq \mathbb{R}^d$ and $\rho' \in \mathbb{R}\backslash\Sigma$ for all $\rho' \in [c, c + \varepsilon)$ and some $\varepsilon > 0$, which again contradicts to the definition of c.

Therefore, $c \in \Sigma$ and so $\Sigma \neq \emptyset$.

We continue with an auxiliary result.

Lemma 1. *Take $a_1, a_2 \in \mathbb{R}\backslash\Sigma$ such that $a_1 < a_2, S_{a_1}(n) = S_{a_2}(n)$ and $U_{a_1}(n) = U_{a_2}(n)$ for some $n \in \mathbb{Z}$. Then $[a_1, a_2] \subset \mathbb{R}\backslash\Sigma$, and $S_a(n) = S_{a_1}(n)$ and $U_a(n) = U_{a_1}(n)$ for every $a \in [a_1, a_2]$ and $n \in \mathbb{Z}$.*

Proof of the Lemma. It follows from the hypothesis that there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2), constants $\lambda_i, \mu_i > 0$ for $i = 1, 2$ and for each $\varepsilon > 0$ constants $D_i = D_i(\varepsilon) > 0$ for $i = 1, 2$ such that

$$\|e^{-a_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq D_i e^{-\lambda_i(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},$$
$$\|e^{-a_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq D_i e^{\mu_i(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-a_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq D_i e^{-\lambda_i(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},$$
$$\|e^{-a_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq D_i e^{\mu_i(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. For each $a \in [a_1, a_2]$, we have

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq D_1 e^{-\lambda_1(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},$$
$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq D_1 e^{\mu_1(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq D_2 e^{-\lambda_2(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},$$
$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq D_2 e^{\mu_2(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. Taking the constants $\lambda = \min\{\lambda_1, \lambda_2\}$, $\mu = \max\{\mu_1, \mu_2\}$ and $D = \max\{D_1, D_2\}$ yields that $[a_1, a_2] \subset \mathbb{R}\backslash\Sigma$. The last statement in the lemma follows now readily from Proposition 2. ◻

Now assume that Σ is composed of $d+1$ disjoint closed intervals and take $b_1, \ldots, b_d \in \mathbb{R}\backslash\Sigma$ such that all intervals

$$(-\infty, b_1), (b_1, b_2), \ldots, (b_{d-1}, b_d), (b_d, +\infty)$$

intersect Σ. By Lemma 1, we have

$$0 \leq \dim S_{b_1} < \dim S_{b_2} < \cdots < \dim S_{b_d} \leq d. \tag{28}$$

Since $S_c(n) = \mathbb{R}^d$ for $c > \mu + a$, it follows from Lemma 1 that $\dim S_{b_d} < d$. Moreover, since $S_c(n) = \{0\}$ for $c < \mu - a$, it follows again from Lemma 1 that $\dim S_{b_1} > 0$. Hence, property (28) cannot hold and so Σ is composed of at most d closed intervals.

Property (20) follows readily from (18). Moreover, for $i < j$ we have

$$W_i(n) \subset S_{c_i}(n) \subset S_{c_{j-1}}(n)$$

and $W_j(n) \subset U_{c_{j-1}}(n)$, which implies that

$$W_i(n) \cap W_j(n) = \{0\} \quad \text{for } n \in \mathbb{Z}.$$

Now observe that since

$$(A + B) \cap C = A + (B \cap C)$$

whenever A, B and C are subspaces with $A \subset C$, it follows from (17) that

$$\begin{aligned}
\mathbb{R}^d &= S_{c_k}(n) \\
&= (S_{c_{k-1}}(n) \oplus U_{c_{k-1}}(n)) \cap S_{c_k}(n) \\
&= S_{c_{k-1}}(n) \oplus (S_{c_k}(n) \cap U_{c_{k-1}}(n)) \\
&= S_{c_{k-1}}(n) \oplus W_k(n)
\end{aligned}$$

for each $n \in \mathbb{Z}$. Proceeding inductively, we obtain the direct sum in (21).

For the third property with need another auxiliary result.

Lemma 2. *If* $a_1, a_2 \in \mathbb{R}\backslash\Sigma$ *with* $a_1 < a_2$ *are such that* $\dim S_{a_1}(n) < \dim S_{a_2}(n)$ *for some* $n \in \mathbb{Z}$, *then* $(a_1, a_2) \cap \Sigma \neq \emptyset$.

Proof of the Lemma. Let

$$b = \inf\{a \in \mathbb{R}\backslash\Sigma : S_a(n) = S_{a_2}(n) \text{ for some } n \in \mathbb{Z}\}.$$

Since $S_{a_1}(n) \neq S_{a_2}(n)$, it follows from Proposition 3 that $a_1 < b < a_2$. Now assume that $b \notin \Sigma$. Then:

(1) If $S_b(n) = S_{a_2}(n)$, then by Proposition 3 we have $S_{b'}(n) = S_{a_2}(n)$ and $b' \in \mathbb{R}\backslash\Sigma$ for all $b' \in (b - \varepsilon, b]$ and some $\varepsilon > 0$. But this contradicts to the definition of b.
(2) If $S_b(n) \neq S_{a_2}(n)$, then by Proposition 3 we have $S_{b'}(n) \neq S_{a_2}(n)$ and $b' \in \mathbb{R}\backslash\Sigma$ for all $b' \in [b, b + \varepsilon)$ and some $\varepsilon > 0$, which again contradicts to the definition of b.

Therefore $(a_1, a_2) \cap \Sigma \neq \emptyset$. □

The lemma implies that each $W_i(n)$ is independent of the choice of numbers $c_0, \ldots, c_k$. Indeed, it follows from Lemma 2 that if $d_i \in (b_i, a_{i+1})$ for $i = 1, \ldots, k-1$, then $U_{d_i} = U_{c_i}(n)$ and $S_{d_i}(n) = S_{c_i}(n)$ for $i = 1, \ldots, k-1$. Therefore, each $W_i(n)$ is independent of the choice of numbers $c_0, \ldots, c_k$.

For the fourth property, take $v \in W_i(n)\backslash\{0\}$. Since $c_i \notin \Sigma$, there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-c_i(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \geq n. \tag{29}$$

Since $v \in S_{c_i}(n)$, we have

$$\|\mathcal{A}(m,n)v\| \leq De^{-(\lambda-c_i)(\rho(m)-\rho(n))}$$

for $m \geq n$ and so,

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| \leq c_i - \lambda < c_i.$$

Letting $c_i \searrow b_i$ yields that

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| \leq b_i.$$

Similarly, since $c_{i-1} \notin \Sigma$, there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-c_{i-1}(\rho(m)-\rho(n))}\mathcal{A}(m,n)(\mathrm{Id} - P_n)\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \leq n.$$

Since $v \in U_{c_{i-1}}(n)$, we have

$$\frac{1}{D}e^{(\lambda+c_{i-1})(\rho(m)-\rho(n))-\varepsilon|\rho(m)|}\|v\| \leq \|\mathcal{A}(m,n)v\|$$

for $m \geq n$ and so,

$$\liminf_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| > c_{i-1}.$$

Letting $c_{i-1} \nearrow a_i$ yields that

$$\liminf_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,n)v\| \geq a_i.$$

Therefore,

$$a_i \leq \liminf_{n\to+\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,n)v\| \leq \limsup_{n\to+\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,n)v\| \leq b_i$$

for $v \in W_i(n)\backslash\{0\}$. One can show in a similar manner that

$$a_i \leq \liminf_{n\to-\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,n)v\| \leq \limsup_{n\to-\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,n)v\| \leq b_i$$

for $v \in W_i(n)\backslash\{0\}$.

For the last property, without loss of generality take $j > i$ and note that

$$\angle\left(\bigoplus_{l=1}^{i} W_l(n), \bigoplus_{l=i+1}^{k} W_l(n)\right) \le \angle(W_i(n), W_j(n)) \le 2.$$

By construction we have

$$\bigoplus_{l=1}^{i} W_l(n) = S_{c_i}(n) \quad \text{and} \quad \bigoplus_{l=i+1}^{k} W_l(n) = U_{c_i}(n).$$

Applying for example Proposition 2.4 in [6], for the projections P_n in (29), we obtain

$$\angle(S_{c_i}(n), U_{c_i}(n)) \ge \frac{1}{\|P_n\|},$$

since

$$S_{c_i}(n) = P_n(\mathbb{R}^d) \quad \text{and} \quad U_{c_i}(n) = Q_n(\mathbb{R}^d).$$

Hence,

$$\frac{1}{\|P_n\|} \le \angle(W_i(n), W_j(n)) \le 2.$$

Taking $m = n$ in (29) gives $\|P_n\| \le De^{\varepsilon|\rho(n)|}$ and the arbitrariness of ε yields property (23). This completes the proof of the theorem.

4. Examples

In this section we provide explicit examples of all possible forms of the strong nonuniform spectrum Σ given by Theorem 4. Let ρ be an *arbitrary* increasing function satisfying (11).

Example 3 (Case of $\Sigma = \mathbb{R}$). This is an elaboration of Example 1. Consider the sequence of matrices

$$A_m = e^{\rho(m+1)^3 - \rho(m)^3}\mathrm{Id}, \quad m \in \mathbb{Z}.$$

Then

$$\mathcal{A}(m,n) = e^{\rho(m)^3 - \rho(n)^3}\mathrm{Id} = e^{(\rho(m)-\rho(n))(\rho(m)^2+\rho(m)\rho(n)+\rho(n)^2)}\mathrm{Id}.$$

Now take $a \in \mathbb{R}$ and consider the sequence B_m in (16). Proceeding as in Example 1 we find that the first inequalities in (12) and (13) hold taking $P_m = 0$ for $m \in \mathbb{Z}$, $D = 1$ and ε arbitrary, for some constant $\lambda = \lambda(a) > 0$; in fact one can take $\lambda(a) = \lambda(0) - a$. On the other hand, the second inequality in (12), that is,

$$e^{(\rho(m)-\rho(n))(\rho(m)^2+\rho(m)\rho(n)+\rho(n)^2)} \le De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

cannot hold for any μ. Hence, the sequence B_m does not admit a ρ-strong dichotomy and so $\Sigma = \mathbb{R}$.

Example 4 (Case of $\Sigma = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$). Take numbers as in (19) and take positive integers $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k = d$. For each $n \in \mathbb{Z}$, consider the block matrix

$$A_n = \operatorname{diag}(A_n^1 \mathrm{Id}_{n_1}, \ldots, A_n^k \mathrm{Id}_{n_k}), \tag{30}$$

where Id_{n_i} is the identity on $\mathbb{R}^{n_i}$ and

$$A_n^j = \begin{cases} e^{b_j(\rho(n+1)-\rho(n))+\sqrt{|\rho(n+1)|}\cos\rho(n+1)-\sqrt{|\rho(n)|}\cos\rho(n)}, & n \geq 0, \\ e^{a_j(\rho(n+1)-\rho(n))+\sqrt{|\rho(n+1)|}\cos\rho(n+1)-\sqrt{|\rho(n)|}\cos\rho(n)}, & n < 0 \end{cases}$$

for $j = 1, \ldots, k$. For each $j = 1, \ldots, k$ and $m, n \in \mathbb{Z}$, let

$$\mathcal{A}^j(m,n) = \begin{cases} A_{m-1}^j \cdots A_n^j, & m > n, \\ 1, & m = n, \\ (A_m^j)^{-1} \cdots (A_{n-1}^j)^{-1}, & m < n. \end{cases}$$

Then

$$\mathcal{A}^j(m,n) = \begin{cases} e^{b_j(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}, & m, n \geq 0, \\ e^{b_j\rho(m)-a_j\rho(n)+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}, & m \geq 0, \quad n < 0, \\ e^{a_j(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}, & m, n < 0. \end{cases}$$

Clearly,

$$\mathcal{A}^j(m,n) \leq e^{b_j(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}$$

for $m \geq n$ and

$$\mathcal{A}^j(m,n) \leq e^{a_j(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}$$

for $m \leq n$, since $a_j \leq b_j$. Now take $a > b_j$. We have

$$e^{-a(\rho(m)-\rho(n))}\mathcal{A}^j(m,n) \leq e^{-(a-b_j)(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}+\sqrt{|\rho(n)|}} \tag{31}$$

for $m \geq n$. On the other hand, given $\delta > 0$, there exists $D = D(\delta) > 0$ such that

$$e^{\sqrt{|\rho(n)|}} \leq De^{\delta|\rho(n)|} \quad \text{for } n \in \mathbb{Z}. \tag{32}$$

Hence, it follows from (31) that

$$\begin{aligned} e^{-a(\rho(m)-\rho(n))}\mathcal{A}^j(m,n) &\leq D^2 e^{-(a-b_j)(\rho(m)-\rho(n))+\delta|\rho(m)|+\delta|\rho(n)|} \\ &\leq D^2 e^{-(a-b_j-\delta)(\rho(m)-\rho(n))+2\delta|\rho(n)|} \end{aligned}$$

for $m \geq n$. We also have

$$\begin{aligned} e^{-a(\rho(m)-\rho(n))}\mathcal{A}^j(m,n) &\leq e^{-(a-a_j)(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}+\sqrt{|\rho(n)|}} \\ &\leq D^2 e^{-(a-a_j)(\rho(m)-\rho(n))+2\delta|\rho(n)|} \end{aligned}$$

for $m \leq n$. Since $a - a_j \geq a - b_j > 0$ and δ is arbitrary, this shows that the sequence

$$B_m^j = e^{-a(\rho(m+1)-\rho(m))} A_m^j \tag{33}$$

admits a ρ-strong contraction, that is, a ρ-strong dichotomy with projections $P_m = \mathrm{Id}$ for $m \in \mathbb{Z}$.

Similarly, one can show that for $a < a_j$ the sequence B^j_m in (33) admits a ρ-strong expansion, that is, a ρ-strong dichotomy with projections $P_m = 0$ for $m \in \mathbb{Z}$. Indeed, by (32), for $m \le n$ we obtain

$$\begin{aligned} e^{-a(\rho(n)-\rho(m))}\mathcal{A}^j(n,m) &\ge e^{-(a-a_j)(\rho(n)-\rho(m))-\sqrt{|\rho(m)|}-\sqrt{|\rho(n)|}} \\ &\ge D^{-2}e^{-(a-a_j+\delta)(\rho(n)-\rho(m))-2\delta|\rho(n)|} \end{aligned}$$

and so,

$$e^{-a(\rho(m)-\rho(n))}\mathcal{A}^j(m,n) \le D^2 e^{-(a-a_j+\delta)(\rho(m)-\rho(n))+2\delta|\rho(n)|}.$$

Moreover, for $m \ge n$, we have

$$\begin{aligned} e^{-a(\rho(n)-\rho(m))}\mathcal{A}^j(n,m) &\ge e^{-(a-b_j)(\rho(n)-\rho(m))-\sqrt{|\rho(m)|}-\sqrt{|\rho(n)|}} \\ &\ge D^{-2}e^{-(a-b_j)(\rho(n)-\rho(m))-2\delta|\rho(n)|}. \end{aligned}$$

Since $a-b_j \le a-a_j < 0$ and δ is arbitrary, this shows that B^j_m admits a ρ-strong expansion.

Now take $a \in [a_j, b_j]$. We note that $e^{-a(\rho(m)-\rho(n))}\mathcal{A}^j(m,n)$ is given by

$$\begin{cases} e^{(b_j-a)(\rho(m)-\rho(n))+\sqrt{\rho(m)}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}, & m,n \ge 0, \\ e^{(b_j-a)\rho(m)-(a_j-a)\rho(n)+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}, & m \ge 0, n < 0, \\ e^{(a_j-a)(\rho(m)-\rho(n))+\sqrt{|\rho(m)|}\cos\rho(m)-\sqrt{|\rho(n)|}\cos\rho(n)}, & m,n < 0. \end{cases}$$

Since $b_j - a \ge 0$, the first branch precludes B^j_m from admitting a ρ-strong contraction and since $a_j - a \le 0$, the third branch precludes B^j_m from admitting a ρ-strong expansion. Hence, it follows from the former discussion that B^j_m admits a ρ-strong dichotomy if and only if $a \in \mathbb{R}\setminus[a_j, b_j]$.

Finally, we determine the strong nonuniform spectrum of the sequence $(A_m)_{m\in\mathbb{Z}}$. Take $a \in \mathbb{R}\setminus\bigcup_{j=1}^k [a_j, b_j]$. When $b_j < a < a_{j+1}$ for some j, let

$$P_m(x_1,\dots,x_d) = (x_1,\dots,x_{n_1+\cdots+n_j},0,\dots,0).$$

Otherwise, when $a > b_k$ let $P_m = \mathrm{Id}$ and when $a < a_1$ let $P_m = 0$. Then the sequence B_m in (16) admits a ρ-strong dichotomy with these projections, since for each j the sequence B^j_m admits a ρ-strong contraction or a ρ-strong expansion. More precisely, assume that $b_j < a < a_{j+1}$. For $i \le j$ we have

$$\begin{aligned} e^{-a(\rho(m)-\rho(n))}\mathcal{A}^i(m,n) &\le D^2 e^{-(a-b_i-\delta)(\rho(m)-\rho(n))+2\delta|\rho(n)|} \\ &\le D^2 e^{-(a-b_j-\delta)(\rho(m)-\rho(n))+2\delta|\rho(n)|} \end{aligned}$$

for $m \ge n$ and

$$\begin{aligned} e^{-a(\rho(m)-\rho(n))}\mathcal{A}^i(m,n) &\le D^2 e^{-(a-a_i)(\rho(m)-\rho(n))+2\delta|\rho(n)|} \\ &\le D^2 e^{-(a-a_1)(\rho(m)-\rho(n))+2\delta|\rho(n)|} \end{aligned}$$

for $m \le n$.

Similarly, for $i > j$ we have

$$e^{-a(\rho(m)-\rho(n))}\mathcal{A}^i(m,n) \le D^2 e^{-(a-a_i+\delta)(\rho(m)-\rho(n))+2\delta|\rho(n)|}$$
$$\le D^2 e^{-(a-a_{j+1}+\delta)(\rho(m)-\rho(n))+2\delta|\rho(n)|}$$

for $m \le n$ and

$$e^{-a(\rho(m)-\rho(n))}\mathcal{A}^i(m,n) \le D^2 e^{-(a-b_i)(\rho(m)-\rho(n))+2\delta|\rho(n)|}$$
$$\le D^2 e^{-(a-b_k)(\rho(m)-\rho(n))+2\delta|\rho(n)|}$$

for $m \ge n$. The cases when $a > b_k$ or $a < a_1$ are analogous. This shows that $\Sigma \subset \bigcup_{j=1}^k [a_j, b_j]$.

For the reverse inclusion, we proceed by contradiction. Assume that for some j there exists $a \in [a_j, b_j]\backslash\Sigma$. In particular, the sequence B_m admits a ρ-strong dichotomy. The space

$$X_j = \{0\}^{n_1+\cdots+n_{j-1}} \times \mathbb{R}^{n_j} \times \{0\}^{n_{j+1}+\cdots+n_k}$$

is invariant under B_m^j, that is, $B_m^j X_j = X_j$ for all m, and so it must be contained either in the stable space or in the unstable space associated to B_m, since these are also invariant (and it is impossible that X_j has elements in both spaces since the blocks of the matrices A_m are multiples of the identity). On the other hand, as shown above, B_m^j admits neither a ρ-strong contraction nor a ρ-strong expansion, which implies that X_j cannot contain vectors with the bounds along the stable and unstable spaces associated to B_m. This contradiction shows that $\Sigma = \bigcup_{j=1}^k [a_j, b_j]$.

5. Relation to Nonstrong Exponential Dichotomies

In this section we relate the notions of a strong nonuniform exponential dichotomy and of a (nonstrong) nonuniform exponential dichotomy in an optimal manner. This is a principal part of our paper since the somewhat unexpected relation between the two is here detailed for the first time.

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible $d\times d$ matrices and let ρ be an increasing function satisfying (11). We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *ρ-nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *ρ-dichotomy* if there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},$$

for $m \ge n$ and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \le n$, where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$.

The following is our main result describing the relation between the two notions of exponential dichotomies. For each $a \in \mathbb{R}$ we denote by $\mathcal{B}_a$ the sequence of matrices B_m in (16).

Theorem 5. *If $\Sigma \neq \mathbb{R}$ and $\mathcal{B}_a$ admits a ρ-dichotomy, then $\mathcal{B}_a$ admits a ρ-strong dichotomy.*

Proof. Take $a \in \mathbb{R}$ such that the sequence $\mathcal{B}_a$ admits a ρ-dichotomy. Then there exist projections P_n for $n \in \mathbb{Z}$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. In particular,

$$\|P_n\| \leq De^{\varepsilon|\rho(n)|} \quad \text{and} \quad \|Q_n\| \leq De^{\varepsilon|\rho(n)|}$$

for $n \in \mathbb{Z}$. Since $\Sigma \neq \mathbb{R}$, it follows from (26) and (27) that there exist a constant $\mu > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)\| \leq De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|\mathcal{A}(m,n)\| \leq De^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. Hence,

$$\|\mathcal{A}(m,n)Q_n\| \leq D^2e^{\mu(\rho(m)-\rho(n))+2\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|\mathcal{A}(m,n)P_n\| \leq D^2e^{\mu(\rho(n)-\rho(m))+2\varepsilon|\rho(n)|}$$

for $m \leq n$. Therefore, there exists $\nu > 0$ such that

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq D^2e^{\nu(\rho(m)-\rho(n))+2\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq D^2e^{\nu(\rho(n)-\rho(m))+2\varepsilon|\rho(n)|}$$

for $m \leq n$, and so $\mathcal{B}_a$ admits a ρ-strong dichotomy. □

The following statements are simple consequences of Theorems 4 and 5 and reveal somewhat unexpected relations between the notions of a strong nonuniform exponential dichotomy and of a (nonstrong) nonuniform exponential dichotomy:

(1) either no $\mathcal{B}_a$ admits a ρ-strong dichotomy (for any $a \in \mathbb{R}$) or any $\mathcal{B}_a$ that admits a ρ-dichotomy also admits a ρ-strong dichotomy;
(2) if some $\mathcal{B}_a$ admits a ρ-dichotomy but not a ρ-strong dichotomy, then no $\mathcal{B}_b$ admits a ρ-strong dichotomy;

(3) if $\mathcal{B}_a$ admits a ρ-strong dichotomy for some a, then:
   (a) any $\mathcal{B}_b$ admitting a ρ-dichotomy admits a ρ-strong dichotomy;
   (b) some $\mathcal{B}_b$ does not admit a ρ-dichotomy;
   (c) some $\mathcal{B}_b$ does not admit a ρ-strong dichotomy;
   (d) $\mathcal{B}_b$ admits a ρ-strong dichotomy for any sufficiently large $|b|$.

6. Lyapunov Regularity

In this section we consider the concept of Lyapunov regularity for the Lyapunov exponents associated to a sequence $\rho(n)$. We say that a sequence $(A_n)_{n\in\mathbb{Z}}$ of invertible $d \times d$ matrices is *ρ-Lyapunov regular* if there exist a decomposition

$$\mathbb{R}^d = \bigoplus_{i=1}^{s} E_i \tag{34}$$

and real numbers $\lambda_1 < \cdots < \lambda_s$ for some integer $s \le d$ such that:

(1) if $i = 1, \ldots, s$ and $v \in E_i \setminus \{0\}$, then

$$\lim_{n\to\pm\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,0)v\| = \lambda_i; \tag{35}$$

(2)

$$\lim_{n\to\pm\infty} \frac{1}{\rho(n)} \log|\det \mathcal{A}(n,0)| = \sum_{i=1}^{s} \lambda_i \dim E_i. \tag{36}$$

Example 5. Consider the numbers

$$A_m = \begin{cases} e^{\rho(m+1)-\rho(m)}, & m \ge 0, \\ e^{-\rho(m+1)+\rho(m)}, & m < 0 \end{cases}$$

for $m \in \mathbb{Z}$. For $v \ne 0$ we have

$$\lim_{n\to+\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,0)v\| = 1 \quad \text{and} \quad \lim_{n\to-\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,0)v\| = -1.$$

Hence, the sequence $(A_m)_{m\in\mathbb{Z}}$ is not ρ-Lyapunov regular.

Example 6. Consider the matrices

$$A_m = \begin{pmatrix} e^{-\rho(m+1)+\rho(m)} & 0 \\ 0 & e^{\rho(m+1)-\rho(m)} \end{pmatrix}$$

for $m \in \mathbb{Z}$. Then (34) and (35) hold taking $\lambda_1 = -1$, $\lambda_2 = 1$,

$$E_1 = \{(x,0) : x \in \mathbb{R}\} \quad \text{and} \quad E_2 = \{(0,x) : x \in \mathbb{R}\}.$$

Since $\det A_m = 1$, the sequence $(A_m)_{m\in\mathbb{Z}}$ is ρ-Lyapunov regular.

For a Lyapunov regular sequence of matrices the strong nonuniform spectrum is simply the set of the Lyapunov exponents.

Theorem 6. *If the sequence* $(A_n)_{n\in\mathbb{Z}}$ *is* ρ*-Lyapunov regular, then*

$$\Sigma = \{\lambda_1, \ldots, \lambda_s\}. \tag{37}$$

Proof. Given $a \in \mathbb{R}\backslash\{\lambda_1, \ldots, \lambda_s\}$, the Lyapunov exponents associated to the sequence B_n in (16) are the numbers $-a + \lambda_i$ for $i = 1, \ldots, s$. Let P_0 and Q_0 be the projections associated to the decomposition

$$\mathbb{R}^d = \left(\bigoplus_{i:\lambda_i<a} E_i\right) \oplus \left(\bigoplus_{i:\lambda_i>a} E_i\right).$$

By results in [5], the sequence $(B_n)_{n\in\mathbb{Z}}$ admits a ρ-strong dichotomy on $\mathbb{Z}_0^+$ with projections

$$P_n = \mathcal{A}(n,0)P_0\mathcal{A}(0,n) \quad \text{for } n \geq 0.$$

Hence, there exists $\lambda, \mu > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that for $m, n \geq 0$ we have

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|},$$
$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}$$

for $m \geq n$ and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq De^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|},$$
$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq De^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

for $m \leq n$. Moreover, the sequence $(B_n)_{n\in\mathbb{Z}}$ admits a ρ-strong dichotomy on $\mathbb{Z}_0^-$ with projections

$$P_n = \mathcal{A}(n,0)P_0\mathcal{A}(0,n) \quad \text{for } n \leq 0$$

and without loss of generality with the same constants λ, μ and $D = D(\varepsilon)$ as above. For $n < 0 < m$, we have

$$\begin{aligned}
&\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \\
&\quad\leq \|e^{-a(\rho(m)-\rho(0))}\mathcal{A}(m,0)P_0\| \cdot \|e^{-a(\rho(0)-\rho(n))}\mathcal{A}(0,n)P_n\| \\
&\quad\leq D^2e^{\varepsilon|\rho(0)|}e^{-\lambda(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}
\end{aligned}$$

and

$$\begin{aligned}
&\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \\
&\quad\leq \|e^{-a(\rho(m)-\rho(0))}\mathcal{A}(m,0)Q_0\| \cdot \|e^{-a(\rho(0)-\rho(n))}\mathcal{A}(0,n)Q_n\| \\
&\quad\leq D^2e^{\varepsilon|\rho(0)|}e^{\mu(\rho(m)-\rho(n))+\varepsilon|\rho(n)|}.
\end{aligned}$$

Similarly, for $m < 0 < n$ we have

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)Q_n\| \leq D^2e^{\varepsilon|\rho(0)|}e^{-\lambda(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}$$

and

$$\|e^{-a(\rho(m)-\rho(n))}\mathcal{A}(m,n)P_n\| \leq D^2e^{\varepsilon|\rho(0)|}e^{\mu(\rho(n)-\rho(m))+\varepsilon|\rho(n)|}.$$

Therefore, the sequence $(B_n)_{n\in\mathbb{Z}}$ admits a ρ-strong dichotomy on $\mathbb{Z}$. This shows that $a \notin \Sigma$ and so $\Sigma \subset \{\lambda_1, \ldots, \lambda_s\}$.

For the reverse inclusion, take $i \in \{1, \ldots, s\}$ and assume that the sequence $(e^{-\lambda_i(\rho(n+1)-\rho(n))} A_n)_{n\in\mathbb{Z}}$ admits a ρ-strong dichotomy. Then there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-(\lambda-\lambda_i)(\rho(m)-\rho(n))+\varepsilon|\rho(n)|} \quad \text{for } m \ge n \tag{38}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-(\lambda+\lambda_i)(\rho(n)-\rho(m))+\varepsilon|\rho(n)|} \quad \text{for } m \le n. \tag{39}$$

For $v \in E_i\backslash\{0\}$, by (38), we have

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,0)P_0 v\| \le -\lambda + \lambda_i < \lambda_i.$$

Hence, by (35), $P_0 v \neq v$ and

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,0)Q_0 v\| = \lambda_i. \tag{40}$$

On the other hand, by (39),

$$\frac{1}{D} e^{(\lambda+\lambda_i-\varepsilon)\rho(m)}\|Q_0 v\| \le \|\mathcal{A}(m,0)(\mathrm{Id} - P_0)v\|$$

for $m \ge 0$. Since $P_0 v \neq v$, we obtain

$$\limsup_{m\to+\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(m,0)Q_0 v\| \ge \lambda + \lambda_i - \varepsilon > \lambda_i$$

for any sufficiently small $\varepsilon > 0$, which contradicts to (40). Therefore, $\lambda_i \in \Sigma$ and so $\{\lambda_1, \ldots, \lambda_s\} \subset \Sigma$. This completes the proof of the theorem. □

The following result is a partial converse of Theorem 6 and follows easily from Theorem 4.

Theorem 7. *If* (37) *holds for some numbers* $\lambda_1 < \cdots < \lambda_s$, *then* (35) *holds taking* $E_i = W_i(0)$ *in* (34). *If in addition*

$$\lim_{n\to\pm\infty} \frac{1}{\rho(n)} \log \angle(\mathcal{A}(n,0)v, \mathcal{A}(n,0)w) = 0 \tag{41}$$

for any nonzero vectors $v \neq w$ *in the same space* $W_i(0)$, *then* (36) *also holds.*

Proof. Taking $a_i = b_i = \lambda_i$ and $n = 0$ in (22), we obtain

$$\lambda_i \le \liminf_{m\to\pm\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(n,0)v\| \le \limsup_{m\to\pm\infty} \frac{1}{\rho(m)} \log\|\mathcal{A}(n,0)v\| \le \lambda_i$$

for $v \in W_i(0)\backslash\{0\}$, which yields property (35) taking $E_i = W_i(0)$.

For the second property, we first assume that $s = d$, in which case each space E_i is one-dimensional and so condition (41) is automatically satisfied. Consider an invertible matrix C whose ith column is a basis for E_i, for $i = 1, \dots, d$. Then

$$\begin{aligned}|\det \mathcal{A}(n,0)C| &= \prod_{i=1}^{d} \|\mathcal{A}(n,0)v_i\| \prod_{j=1}^{d-1} \sin\angle(\mathcal{A}(n,0)v_j, \mathcal{A}(n,0)v_{j+1}) \\ &= \prod_{i=1}^{d} \|\mathcal{A}(n,0)v_i\| \prod_{j=1}^{d-1} \sin\angle(W_j(n), W_{j+1}(n)).\end{aligned}$$

It follows now readily from (23) and (35) that

$$\begin{aligned}\lim_{n\to\pm\infty} \frac{1}{\rho(n)} \log|\det \mathcal{A}(n,0)| &= \sum_{i=1}^{d} \lim_{n\to\pm\infty} \frac{1}{\rho(n)} \log\|\mathcal{A}(n,0)v_i\| \\ &\quad + \sum_{j=1}^{d-1} \lim_{n\to\pm\infty} \frac{1}{\rho(n)} \log \sin\angle(W_j(n), W_{j+1}(n)) \\ &= \sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \lambda_i \dim E_i,\end{aligned}$$

and property (36) holds. When $s \neq d$, we consider an invertible matrix C whose columns are successively bases of $E_1, \dots, E_s$. Using condition (41), one can repeat the former argument to conclude that property (36) also holds. This completes the proof of the theorem. □

Applying Theorems 6 and 7, we can describe further examples of regular and nonregular sequences of matrices.

Example 7. Take numbers as in (19) and take positive integers $n_1, \dots, n_k$ such that $n_1 + \cdots + n_k = d$. By construction, the sequence of matrices A_m in (30) obtained from these numbers satisfies property (41). Hence, it follows from Theorems 6 and 7 that the sequence $(A_m)_{m\in\mathbb{Z}}$ is ρ-Lyapunov regular if and only if

$$a_i = b_i \quad \text{for } i = 1, \dots, k.$$

In particular, the sequence of matrices A_m in (30) with

$$A_n^j = \begin{cases} e^{a_j(\rho(n+1)-\rho(n)) + \sqrt{|\rho(n+1)|}\cos\rho(n+1) - \sqrt{|\rho(n)|}\cos\rho(n)}, & n \geq 0, \\ e^{a_j(\rho(n+1)-\rho(n)) + \sqrt{|\rho(n+1)|}\cos\rho(n+1) - \sqrt{|\rho(n)|}\cos\rho(n)}, & n < 0 \end{cases}$$

for $j = 1, \dots, k$ is ρ-Lyapunov regular.

7. Continuous Time

In this section we describe briefly versions of our results for continuous time. Consider a linear differential equation

$$x' = A(t)x \tag{42}$$

on $\mathbb{R}^d$, where $A(t)$ is a $d \times d$ matrix varying continuously with $t \in \mathbb{R}$. Let $T(t,s)$ be the evolution family associated to Eq. (42) and let $\rho\colon \mathbb{R} \to \mathbb{R}$ be an increasing function satisfying

$$\lim_{t\to-\infty} \rho(t) = -\infty \quad \text{and} \quad \lim_{t\to+\infty} \rho(t) = +\infty.$$

We say that the evolution family $T(t,s)$ admits a *ρ-strong nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *ρ-strong dichotomy* if there exist projections $P(t)$ for $t \in \mathbb{R}$ satisfying

$$P(t)T(t,s) = T(t,s)P(s)$$

for $t, s \in \mathbb{R}$, constants $\lambda, \mu > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that for every $t, s \in \mathbb{R}$ we have

$$\|T(t,s)P(s)\| \le De^{-\lambda(\rho(t)-\rho(s))+\varepsilon|\rho(s)|},$$
$$\|T(t,s)Q(s)\| \le De^{\mu(\rho(t)-\rho(s))+\varepsilon|\rho(s)|}$$

for $t \ge s$ and

$$\|T(t,s)Q(s)\| \le De^{-\lambda(\rho(s)-\rho(t))+\varepsilon|\rho(s)|},$$
$$\|T(t,s)P(s)\| \le De^{\mu(\rho(s)-\rho(t))+\varepsilon|\rho(s)|}$$

for $t \le s$, where $Q(t) = \mathrm{Id} - P(t)$ for each $t \in \mathbb{R}$.

The *strong nonuniform spectrum* of Eq. (42) is the set Σ of all $a \in \mathbb{R}$ for which the evolution family

$$T_a(t,s) = e^{-a(\rho(t)-\rho(s))}T(t,s) \tag{43}$$

does not admit a ρ-strong dichotomy. We note that when ρ is differentiable the evolution family associated to the equation

$$x' = (A(t) - a\rho'(t)\mathrm{Id})x$$

is precisely $T_a(t,s)$ in (43).

The following result is a version of Theorem 4 for continuous time. We emphasize that the proof is analogous to the proof of Theorem 4 and so we omit it.

Theorem 8. *For Eq.* (42) *the following properties hold*:

(1) *either* $\Sigma = \mathbb{R}$ *or* $\Sigma = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$, *for some finite numbers*

$$a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k \quad \textit{and} \quad k \le d;$$

(2) *in the second case, taking numbers* $c_0 < a_1, c_k > b_k$ *and*

$$c_i \in (b_i, a_{i+1}) \quad \textit{for } i = 1, \ldots, k-1,$$

for each $t, s \in \mathbb{R}$ *the subspaces* $W_i(t) = U_{c_{i-1}}(t) \cap S_{c_i}(t)$ *satisfy*

$$T(t,s)W_i(s) = W_i(t) \quad \textit{for } i = 1, \ldots, k$$

and form the direct sum

$$\mathbb{R}^d = \bigoplus_{i=1}^{k} W_i(t);$$

(3) *the subspaces* $W_i(t)$ *are independent of the numbers* $c_0, \ldots, c_k$;

(4) *for each* $i = 1, \ldots, k$ *and* $v \in W_i(s) \setminus \{0\}$ *we have*

$$a_i \le \liminf_{t\to\pm\infty} \frac{1}{\rho(t)} \log\|v(t)\| \le \limsup_{t\to\pm\infty} \frac{1}{\rho(t)} \log\|v(t)\| \le b_i,$$

where $v(t) = T(t, s)v$;

(5) *for each* $i, j = 1, \ldots, k$ *with* $i \ne j$ *we have*

$$\lim_{t\to\pm\infty} \frac{1}{\rho(t)} \log \angle(W_i(t), W_j(t)) = 0.$$

Finally, we consider also the concept of Lyapunov regularity. We say that Eq. (42) is *ρ-Lyapunov regular* if there exist a decomposition as in (34) and real numbers $\lambda_1 < \cdots < \lambda_s$ for some integer $s \le d$ such that:

(1) if $i = 1, \ldots, s$ and $v \in E_i \setminus \{0\}$, then

$$\lim_{t\to\pm\infty} \frac{1}{\rho(t)} \log\|T(t, 0)v\| = \lambda_i; \tag{44}$$

(2)

$$\lim_{t\to\pm\infty} \frac{1}{\rho(t)} \log|\det T(t, 0)| = \sum_{i=1}^{s} \lambda_i \dim E_i. \tag{45}$$

We also have the following version of Theorems 6 and 7 combined.

Theorem 9. *If Eq.* (42) *is ρ-Lyapunov regular, then*

$$\Sigma = \{\lambda_1, \ldots, \lambda_s\}. \tag{46}$$

Moreover, if (46) *holds for some numbers* $\lambda_1 < \cdots < \lambda_s$, *then* (44) *holds taking* $E_i = W_i(0)$ *in* (34), *and if in addition*

$$\lim_{t\to\pm\infty} \frac{1}{\rho(t)} \log \angle(T(t, 0)v, T(t, 0)w) = 0$$

for any nonzero vectors $v \ne w$ *in the same space* $W_i(0)$, *then* (45) *also holds.*

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