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Nonuniform exponential dichotomies and Fredholm operators for flows

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Abstract. For the flow determined by a nonautonomous linear differential equation, we characterize the existence of a strong nonuniform exponential dichotomy in terms of the Fredholm property of a certain linear operator. We consider both cases of one-sided and two-sided exponential dichotomies. Moreover, we use the characterizations to establish the robustness of the notion of a strong nonuniform exponential dichotomy in a simple manner.

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1. Introduction

For the flow determined by a nonautonomous linear differential equation $x' = A(t)x$ in a finite-dimensional space, we describe the relation between the existence of a strong nonuniform exponential dichotomy and the Fredholm property of the linear operator R defined by

$$(Rx)(t) = x'(t) - A(t)x(t), \quad t \geq 0,$$

between certain Banach spaces (see Sect. 3 for the details). In particular, we show that the equation $x' = A(t)x$ admits a strong nonuniform exponential dichotomy on $\mathbb{R}_0^+$ if and only if R is a Fredholm operator and the dynamics satisfies a certain bounded growth property (see (8)).

Related results were first obtained by Palmer [13,14]. Further results were obtained by Lin [10] for functional differential equations, by Blázquez [6], Rodrigues and Silveira [18], Zeng [20] and Zhang [21] for parabolic evolution

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equations, and by Chow and Leiva [7], Sacker and Sell [19] and Rodrigues and Ruas-Filho [17] for abstract evolution equations. All these works consider only uniform exponential dichotomies.

On the other hand, in the context of ergodic theory almost all linear variational equations with nonzero Lyapunov exponents of a measure-preserving flow admit a strong nonuniform exponential dichotomy (see [3]). The term "nonuniform" means that the stability or conditional stability need not be uniform, with a deviation from the uniform case that may grow exponentially with the initial time. The term "strong" refers to the fact that besides having upper bounds along the stable direction when the time increases and along the unstable direction when the time decreases, one has also lower bounds. As noted above, both situations are typical in the context of ergodic theory.

We use the characterization of the existence of a strong nonuniform exponential dichotomy, both for one-sided and two-sided exponential dichotomies, to establish the robustness of the notion. In the case of uniform exponential dichotomies, early related works are due to Massera and Schäffer [11], Coppel [8], and in the case of Banach spaces to Dalec′kiĭ and Kreĭn [9]. For more recent works we refer to [4,12,15,16] and the references therein.

2. Preliminaries

Let $I \in \{\mathbb{R}_0^+, \mathbb{R}_0^-, \mathbb{R}\}$, where

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\} \quad \text{and} \quad \mathbb{R}_0^- = \{x \in \mathbb{R} : x \leq 0\}.$$

We consider a linear nonautonomous differential equation

$$x' = A(t)x, \tag{1}$$

where $A\colon I \to M_d$ is a continuous function with values on the set M_d of all $d \times d$ matrices. Let $T(t,\tau)$, for $t,\tau \in I$, be the associated evolution family such that $x(t) = T(t,\tau)x(\tau)$ for any solution x of Eq. (1). Now we consider a family of norms $\|\cdot\|_t$, for $t \in I$, such that:

1. there exist constants $C > 0$ and $\epsilon \geq 0$ such that
$$\|x\| \leq \|x\|_\tau \leq Ce^{\epsilon|\tau|}\|x\|$$
for every $x \in X$ and $\tau \in I$;
2. the map $t \mapsto \|x\|_t$ is continuous on I for each $x \in X$.

We say that Eq. (1) admits a *strong exponential dichotomy* on I with respect to the family of norms $\|\cdot\|_t$ if there exist projections $P_t\colon \mathbb{R}^d \to \mathbb{R}^d$, for $t \in I$, satisfying

$$P_tT(t,\tau) = T(t,\tau)P_\tau \quad \text{for } t,\tau \in I \tag{2}$$

and constants

$$\underline{\lambda} \le \overline{\lambda} < 0 < \underline{\mu} \le \overline{\mu} \quad \text{and} \quad D > 0 \tag{3}$$

such that for each $x \in X$ we have

$$\begin{aligned} \|T(t,\tau)P_\tau x\|_t &\le De^{\overline{\lambda}(t-\tau)}\|x\|_\tau, \\ \|T(\tau,t)Q_t x\|_\tau &\le De^{-\underline{\mu}(t-\tau)}\|x\|_t \end{aligned} \tag{4}$$

for $t \ge \tau$ and

$$\begin{aligned} \|T(t,\tau)P_\tau x\|_t &\le De^{\underline{\lambda}(t-\tau)}\|x\|_\tau, \\ \|T(\tau,t)Q_t x\|_\tau &\le De^{-\overline{\mu}(t-\tau)}\|x\|_t \end{aligned} \tag{5}$$

for $t \le \tau$, where $Q_\tau = \mathrm{Id} - P_\tau$. Moreover, we say that Eq. (1) admits a *strong nonuniform exponential dichotomy* on I if:

1. there exist projections $P_t \colon \mathbb{R}^d \to \mathbb{R}^d$, for $t \in I$, satisfying (2);
2. there exist constants as in (3) and $\epsilon \ge 0$ such that

$$\begin{aligned} \|T(t,\tau)P_\tau\| &\le De^{\overline{\lambda}(t-\tau)+\epsilon|\tau|}, \\ \|T(\tau,t)Q_t\| &\le De^{-\underline{\mu}(t-\tau)+\epsilon|t|} \end{aligned}$$

   for $t \ge \tau$ and

$$\begin{aligned} \|T(t,\tau)P_\tau\| &\le De^{\underline{\lambda}(t-\tau)+\epsilon|\tau|}, \\ \|T(\tau,t)Q_t\| &\le De^{-\overline{\mu}(t-\tau)+\epsilon|t|} \end{aligned}$$

   for $t \le \tau$, where $Q_\tau = \mathrm{Id} - P_\tau$.

The two notions are related as follows.

Proposition 1. *Equation (1) admits a strong nonuniform exponential dichotomy on I if and only if it admits a strong exponential dichotomy on I with respect to a family of norms $\|\cdot\|_t$ satisfying conditions 1 and 2.*

3. One-sided exponential dichotomies

In this section we consider exponential dichotomies on $\mathbb{R}_0^+$. Let Y be the set of all continuous functions $x \colon \mathbb{R}_0^+ \to \mathbb{R}^d$ such that

$$\|x\|_\infty := \sup_{t\ge 0}\|x(t)\|_t < +\infty.$$

It is easy to verify that Y is a Banach space with the norm $\|\cdot\|_\infty$.

Given a continuous function $A \colon \mathbb{R}_0^+ \to M_d$, let R be the linear operator defined by

$$(Rx)(t) = x'(t) - A(t)x(t), \quad t \ge 0,$$

in the domain $\mathcal{D}(R)$ formed by all $x \in Y$ such that $Rx \in Y$ (this includes the requirement that $x'(t)$ exists and is continuous).

Proposition 2. *The operator $R\colon \mathcal{D}(R) \to Y$ is closed.*

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(R)$ converging to $x \in Y$ such that $y_k = Rx_k$ converges to $y \in Y$. For each $\tau \geq 0$, we have

$$\begin{aligned} x(t) - x(\tau) &= \lim_{k\to\infty} (x_k(t) - x_k(\tau)) \\ &= \lim_{k\to\infty} \int_\tau^t x_k'(s)\,ds \\ &= \lim_{k\to\infty} \int_\tau^t \big(y_k(s) + A(s)x_k(s)\big)\,ds \end{aligned}$$

for $t \geq \tau$. Moreover, we have

$$\begin{aligned} \left\| \int_\tau^t y_k(s)\,ds - \int_\tau^t y(s)\,ds \right\| &\leq \int_\tau^t \|y_k(s) - y(s)\|\,ds \\ &\leq \int_\tau^t \|y_k(s) - y(s)\|_s\,ds \\ &\leq (t-\tau)\|y_k - y\|_\infty. \end{aligned}$$

Since $y_k \to y$ in Y, we obtain

$$\lim_{k\to\infty} \int_\tau^t y_k(s)\,ds = \int_\tau^t y(s)\,ds.$$

Similarly,

$$\begin{aligned} \left\| \int_\tau^t A(s)x_k(s)\,ds - \int_\tau^t A(s)x(s)\,ds \right\| &\leq M \int_\tau^t \|x_k(s) - x(s)\| ds \\ &\leq M(t-\tau)\|y_k - y\|_\infty, \end{aligned}$$

where

$$M = \sup\big\{ \|A(s)\| : s \in [\tau, t] \big\} < +\infty.$$

Since $y_k \to y$ in Y, we obtain

$$\lim_{k\to\infty} \int_\tau^t A(s)x_k(s)\,ds = \int_\tau^t A(s)x(s)\,ds.$$

Therefore,

$$x(t) - x(\tau) = \int_\tau^t \big(A(s)x(s) + y(s)\big)\,ds,$$

which implies that $Rx = y$ and $x \in \mathcal{D}(R)$. □

For $x \in \mathcal{D}(R)$ we consider the graph norm

$$\|x\|_R = \|x\|_\infty + \|Rx\|_\infty. \tag{6}$$

Since R is closed, $(\mathcal{D}(R), \|\cdot\|_R)$ is a Banach space. Moreover, the operator

$$R\colon (\mathcal{D}(R), \|\cdot\|_R) \to Y \tag{7}$$

is bounded and from now on we denote it simply by R.

Theorem 3. *If equation* (1) *admits a strong exponential dichotomy on* $\mathbb{R}_0^+$ *with respect to a family of norms* $\|\cdot\|_t$*, then* R *is a Fredholm operator with*

$$\operatorname{ind} R = \dim \operatorname{Im} P_t \quad \text{for } t \geq 0$$

and there exist $a, K > 0$ *such that*

$$\|T(t,s)x\|_t \leq Ke^{a|t-s|}\|x\|_s \quad \text{for } t,s \geq 0 \text{ and } x \in \mathbb{R}^d. \tag{8}$$

Proof. We first show that R is onto. Take $y \in Y$ and extend it to a function $y\colon \mathbb{R} \to X$ by letting $y(t) = 0$ for $t < 0$. For $t \geq 0$, we define

$$x(t) = \int_0^t T(t,\tau)P(\tau)y(\tau)\,d\tau - \int_t^{+\infty} T(t,\tau)Q(\tau)y(\tau)\,d\tau.$$

It follows from (4) that

$$\begin{aligned}
&\int_0^t \|T(t,\tau)P(\tau)y(\tau)\|_t\,d\tau + \int_t^{+\infty} \|T(t,\tau)Q(\tau)y(\tau)\|_t\,d\tau \\
&\quad\leq D\|y\|_\infty \left(\int_0^t e^{\overline{\lambda}(t-\tau)}d\tau + \int_t^{+\infty} e^{-\underline{\mu}(\tau-t)}d\tau\right) \\
&\quad= D\left(-\frac{1}{\overline{\lambda}} + \frac{1}{\underline{\mu}}\right)\|y\|_\infty
\end{aligned}$$

for $t \geq 0$ and thus, $x(t)$ is well defined and $x \in Y$. Moreover, for each $t_0 \geq 0$ we have

$$\begin{aligned}
x(t) &= \int_{t_0}^t T(t,\tau)y(\tau)\,d\tau - \int_{t_0}^t T(t,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_{t_0}^t T(t,\tau)Q(\tau)y(\tau)\,d\tau + \int_0^t T(t,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_t^{+\infty} T(t,\tau)Q(\tau)y(\tau)\,d\tau \\
&= \int_{t_0}^t T(t,\tau)y(\tau)\,d\tau + \int_0^{t_0} T(t,\tau)P(\tau)y(\tau)\,d\tau \\
&\quad - \int_{t_0}^{+\infty} T(t,\tau)Q(\tau)y(\tau)\,d\tau \\
&= \int_{t_0}^t T(t,\tau)y(\tau)\,d\tau + T(t,t_0)x(t_0)
\end{aligned}$$

and hence,

$$x(t) = T(t,t_0)x(t_0) + \int_{t_0}^t T(t,\tau)y(\tau)\,d\tau \tag{9}$$

for $t \geq 0$. It follows from (9) that the function $x: \mathbb{R} \to X$ is differentiable and that $Rx = y$. On the other hand, one can verify that $S: \operatorname{Ker} R \to \mathbb{R}^d$ defined by $Sx = x(0)$ is a bijection onto $\operatorname{Im} P_0$ and thus $\dim \operatorname{Ker} R < +\infty$. Finally, property (8) is a direct consequence of (4) and (5). □

Now we establish the converse of Theorem 3.

Theorem 4. *If R is a Fredholm operator and there exist $a, K > 0$ satisfying (8), then Eq. (1) admits a strong exponential dichotomy on $\mathbb{R}_0^+$ with respect to a family of norms $\|\cdot\|_t$.*

Proof. We start with an auxiliary result. □

Lemma 1. *Given a continuous function $y: \mathbb{R}_0^+ \to \mathbb{R}^d$ with bounded support, there exists a continuous function $x: \mathbb{R}_0^+ \to \mathbb{R}^d$ with bounded support such that $Rx = y$.*

Proof of the lemma. Take $t_0 > 0$ such that $y(t) = 0$ for $t \geq t_0$ and let

$$v = -\int_0^{t_0} T(0, s) y(s)\, ds.$$

We define

$$x(t) = \begin{cases} T(t, 0)v + \int_0^t T(t, s) y(s)\, ds & \text{if } 0 \leq t \leq t_0, \\ 0 & \text{if } t > t_0. \end{cases}$$

The function x satisfies the desired properties. □

Now let Y_0 be the closed subspace of all functions $x \in Y$ such that $\lim_{t \to +\infty} \|x(t)\|_t = 0$ and let $E = R^{-1} Y_0$. We write $S = R|_E: E \to Y_0$ and we denote by $S^*: Y_0^* \to E^*$ the adjoint operator.

Lemma 2. *We have* $\operatorname{Ker} S^* = \{0\}$.

Proof of the lemma. Take $\phi \in \operatorname{Ker} S^*$ and $y \in Y_0$ with bounded support. By Lemma 1, there exists $x \in Y$ such that $Rx = y$ and thus $Sx = y$. We have

$$\phi(y) = \phi(Sx) = (S^* \phi)x = 0$$

for $y \in Y_0$ with bounded support. Since the continuous functions with bounded support are dense in Y_0, we conclude that $\phi = 0$. □

Lemma 3. *For each $y \in Y_0$, there exists $x \in Y$ such that $Rx = y$.*

Proof of the lemma. Since R is a Fredholm operator, its image and so also $\operatorname{Im} S$ are closed. On the other hand, we have

$$\{\operatorname{Im} S\}^0 := \left\{ \phi \in Y_0^* : \phi(x) = 0 \text{ for } x \in Y_0 \right\} = \operatorname{Ker} S^*.$$

It follows from Lemma 2 that $\{\operatorname{Im} S\}^0 = \{0\}$ and hence, by the Hahn–Banach theorem we have $\operatorname{Im} S = Y_0$, which yields the statement in the lemma. □

Lemma 4. *There exists a subspace Z of $\mathbb{R}^d$ such that for each $y \in Y_0$, there exists a unique $x \in Y$ with $x(0) \in Z$ and $Rx = y$.*

Proof of the lemma. Let Z be the subspace of $\mathbb{R}^d$ consisting of all vectors $x \in \mathbb{R}^d$ such that

$$\sup_{t \geq 0} \|T(t,0)x\|_t = 0.$$

Moreover, let Z' be any subspace of $\mathbb{R}^d$ such that $\mathbb{R}^d = Z \oplus Z'$. Given $y \in Y_0$, by Lemma 3 there exists $x \in Y$ such that $Rx = y$. Write $x_0 = y_0 + z_0$, where $y_0 \in Z$ and $z_0 \in Z'$. Now we consider the function

$$x^*(t) = x(t) - T(t,0)y(0).$$

Then $x \in Y$, $x(0)^* \in Z'$ and $Rx^* = y$. Now assume that for some $y \in Y_0$ there exist functions $x^i \in Y$ satisfying $Rx^i = y$ and $x^i(0) \in Z'$ for $i = 1, 2$. We have

$$x^1(t) - x^2(t) = T(t,0)(x^1(0) - x^2(0))$$

for $t \geq 0$ and $x^1(0) - x^2(0) \in Z$. Hence, $x^1(0) - x^2(0) \in Z \cap Z'$ and so $x^1(0) = x^2(0)$. Therefore, $x^1(t) = x^2(t)$ for $t \geq 0$, that is, $x^1 = x^2$. □

It follows from Lemma 4 and results in [2] that Eq. (1) admits an exponential dichotomy on $\mathbb{R}_0^+$ with respect to a family of norms $\|\cdot\|_t$ (although [2] considers a certain space Y_1 instead of Y_0, all goes through for the new space exactly with the same proof). In view of condition (8) this is in fact a strong exponential dichotomy. □

As an application of Theorems 3 and 4, we establish the robustness of one-sided exponential dichotomies.

Theorem 5. *Assume that Eq. (1) admits a strong nonuniform exponential dichotomy on $\mathbb{R}_0^+$ and that $B\colon \mathbb{R}_0^+ \to M_d$ is a continuous function satisfying*

$$\|B(t) - A(t)\| \leq \delta e^{-\epsilon t} \tag{10}$$

for all $t \geq 0$ and some $\delta > 0$. If δ is sufficiently small, then the equation

$$x' = B(t)x \tag{11}$$

admits a strong nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projections P'_t such that $\dim P'_t = \dim P_t$ for $t \geq 0$.

Proof. By Proposition 1, there exists a family of norms $\|\cdot\|_t$, for $t \geq 0$, satisfying conditions 1 and 2 such that Eq. (1) admits a strong exponential dichotomy on $\mathbb{R}_0^+$ with respect to this family. We consider the operator R defined by (7). By Theorem 3, R is Fredholm and property (8) holds.

For the evolution facility $S(t,s)$ associated to Eq. (11), it follows from (8) and (10) that

$$\begin{aligned}\|S(t,s)x\|_t &= \left\|T(t,s)x + \int_s^t T(t,\tau)[B(\tau)-A(\tau)]S(\tau,s)x\,d\tau\right\|_t \\ &\le Ke^{a(t-s)}\|x\|_s + K\int_s^t e^{a(t-\tau)}\big\|[B(\tau)-A(\tau)]S(\tau,s)x\big\|_\tau\,d\tau \\ &\le Ke^{a(t-s)}\|x\|_s + \delta CK\int_s^t e^{a(t-\tau)}\|S(\tau,s)x\|_\tau\,d\tau\end{aligned}$$

for $t \ge s$. This shows that the function $\phi(t) = e^{-at}\|S(t,s)x\|_t$ satisfies

$$\phi(t) \le K\phi(s) + \delta CK\int_s^t \phi(\tau)\,d\tau$$

for $t \ge s$. Using Gronwall's lemma, we obtain

$$\phi(t) \le K\phi(s)e^{\delta CK(t-s)}$$

or, equivalently,

$$\|S(t,s)x\|_t \le Ke^{(a+\delta CK)(t-s)}\|x\|_s$$

for $t \ge s$. A similar argument applies to the case when $t \le s$ and so,

$$\|S(t,s)x\|_t \le Ke^{(a+\delta CK)|t-s|}\|x\|_s \quad \text{for } t,s \ge 0.$$

Now let U be the linear operator associated to Eq. (11), defined by

$$(Ux)(t) = x'(t) - B(t)x(t), \quad t \ge 0,$$

in the domain $\mathcal{D}(U)$ formed by all $x \in Y$ for which $Ux \in Y$. Moreover, let $P\colon Y \to Y$ be the linear operator defined by

$$(Px)(t) = (B(t) - A(t))x(t).$$

By (10) we have

$$\begin{aligned}\|(B(t)-A(t))x(t)\|_t &\le Ce^{\epsilon|t|}\|(B(t)-A(t))x(t)\| \\ &\le \delta C\|x(t)\| \le \delta C\|x(t)\|_t\end{aligned}$$

for $t \ge 0$. Hence,

$$\sup_{t\ge0}\|(Px)(t)\|_t \le \delta C\|x\|_\infty \le \delta C\|x\|_R$$

for $t \ge 0$ and P is bounded. Since a sufficiently small perturbation of a Fredholm operator is a Fredholm operator with the same index, if δ is sufficiently small, then, by Theorem 4, Eq. (11) admits a strong exponential dichotomy on $\mathbb{R}_0^+$ with respect to a family of norms $\|\cdot\|_t$, with projections P'_t such that $\dim P'_t = \dim P_t$. Hence, by Proposition 1, the equation admits a strong nonuniform exponential dichotomy on $\mathbb{R}_0^+$. $\square$

4. Two-sided exponential dichotomies

In this section we consider exponential dichotomies on $\mathbb{R}$. We shall always assume that each norm $\|\cdot\|_t$ is induced by a scalar product $\langle\cdot,\cdot\rangle_t$. We emphasize that there is no loss of generality in this assumption, since one can always consider norms in Proposition 1 that are induced by scalar products. Then there exists invertible $d \times d$ matrices D_t, for $t \in \mathbb{R}$, such that

$$\langle x, y\rangle_t = \langle D_t x, y\rangle \quad \text{for } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^d.$$

Let Y be the set of all continuous functions $x\colon \mathbb{R} \to \mathbb{R}^d$ such that

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_t < +\infty.$$

It is easy to verify that Y is a Banach space with the norm $\|\cdot\|_\infty$. Moreover, let R be the linear operator defined by

$$(Rx)(t) = x'(t) - A(t)x(t), \quad t \in \mathbb{R},$$

in the domain $\mathcal{D}(R)$ formed by all $x \in Y$ such that $Rx \in Y$. Proceeding as in the proof of Proposition 2, we find that R is a closed operator. We can now define a norm $\|\cdot\|_R$ as in (6) and from now on we denote by R the operator in (7). We also consider the adjoint equation

$$x' = -A(t)^* x. \tag{12}$$

One can easily verify that its evolution family is $T'(t, s) = T(s, t)^*$.

Theorem 6. *If Eq. (1) admits a strong exponential dichotomy on $\mathbb{R}$ with respect to a family of norms $\|\cdot\|_t$, then R is a Fredholm operator and there exist $a, K > 0$ such that*

$$\|T(t, s)x\|_t \le K e^{a|t-s|} \|x\|_s \quad \text{for } t, s \in \mathbb{R} \text{ and } x \in \mathbb{R}^d. \tag{13}$$

Proof. Let P_t^+ and P_t^- be, respectively, the projections associated to the exponential dichotomies on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$. Under the assumption in the theorem, it is proved in [1] that the operator R is onto. Moreover, $\operatorname{Ker} R$ consists of all functions $x \in Y$ such that $x(t) = R(t, 0)x(0)$ for $t \in \mathbb{R}$. This implies that $x(0) \in \operatorname{Im} P_0^+ \cap \operatorname{Ker} P_0^-$ and so $\dim \operatorname{Ker} R < +\infty$. Property (13) is a direct consequence of (4) and (5). □

The following is a partial converse to Theorem 6 (see also Theorem 8).

Theorem 7. *If R is a Fredholm operator and there exist $a, K > 0$ satisfying (13), then Eq. (1) admits strong exponential dichotomies on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ with respect to a family of norms $\|\cdot\|_t$.*

Proof. We begin with the following statement. □

Lemma 5. *Given $y \in Y$ with bounded support, there exists $x \in Y$ with bounded support such that $Rx = y$ if and only if*

$$\int_{-\infty}^{\infty} T(0,s)y(s)\,ds = 0. \tag{14}$$

Proof of the lemma. Assume that $x \in Y$ satisfies $Rx = y$. Since y has bounded support, for all sufficiently large t we have

$$x(t) = T(t,0)\biggl(x(0) + \int_0^{\infty} T(0,s)y(s)\,ds\biggr). \tag{15}$$

Similarly, for sufficiently small t we have

$$x(t) = T(t,0)\biggl(x(0) - \int_{-\infty}^{0} T(0,s)y(s)\,ds\biggr). \tag{16}$$

It follows from (15) and (16) that x has bounded support if and only if

$$x(0) + \int_0^{\infty} T(0,s)y(s)\,ds = x(0) - \int_{-\infty}^{0} T(0,s)y(s)\,ds = 0,$$

which implies that (14) holds. Now assume that (14) holds and define

$$x(t) = \begin{cases} T(t,0)x_0 + \int_0^t T(t,s)y(s)\,ds & \text{if } t \ge 0, \\ T(t,0)x_0 - \int_t^0 T(t,s)y(s)\,ds & \text{if } t < 0, \end{cases}$$

where

$$x_0 = -\int_0^{\infty} T(0,s)y(s)\,ds.$$

Then the function x has bounded support and $Rx = y$. □

Now let

$$Y_0 = \Bigl\{x \in Y : \lim_{|t| \to +\infty} \|x(t)\|_t = 0\Bigr\}$$

and $E = R^{-1}c_0$. We consider the operator $S = R|_E \colon E \to Y_0$ and its adjoint $S^* \colon Y_0^* \to E^*$.

Lemma 6. Ker S^* *consists of all $\alpha \in Y_0^*$ for which there exists a function y solving* (12) *such that:*

1. $$\int_{-\infty}^{\infty} \|D_t^{-1}y(t)\|_t\,dt < +\infty \quad \textit{and} \quad \sup_{t \in \mathbb{R}} \|D_t^{-1}y(t)\|_t < +\infty; \tag{17}$$
2. $$\alpha(x) = \int_{-\infty}^{\infty} \langle D_t^{-1}y(t), x(t)\rangle_t\,dt \quad \textit{for } x \in Y_0. \tag{18}$$

Proof of the lemma. Take $\alpha \in \operatorname{Ker} S$ and $x \in Y$ with bounded support. Let $\phi\colon \mathbb{R} \to \mathbb{R}^+$ be a continuous function with bounded support such that $\int_{-\infty}^{\infty} \phi(t)\, dt = 1$. Finally, let

$$\tilde{x}(t) = x(t) - \phi(t) \int_{-\infty}^{\infty} T(t,s)x(s)\, ds. \tag{19}$$

Then $\tilde{x}$ is a continuous function with bounded support and

$$\begin{aligned}\int_{-\infty}^{\infty} T(0,s)\tilde{x}(s)\, ds = {}& \int_{-\infty}^{\infty} T(0,s)x(s)\, ds \\ & - \int_{-\infty}^{\infty} \phi(t)\, dt \int_{-\infty}^{\infty} T(0,s)x(s)\, ds = 0.\end{aligned}$$

It follows from Lemma 5 that $\tilde{x} \in \operatorname{Im} R$ and so $\tilde{x} \in \operatorname{Im} S$. Therefore, $\tilde{x} = Sz$ for some $z \in E$ and

$$\alpha(\tilde{x}) = \alpha(Sz) = (S^*\alpha)z = 0.$$

Hence, it follows from (19) that $\alpha(x) = \alpha(\psi)$, where

$$\psi(t) = \int_{-\infty}^{\infty} \phi(t)\, dt \int_{-\infty}^{\infty} T(0,s)x(s)\, ds.$$

Let

$$y(t) = \sum_{i=1}^{d} \alpha(\phi_i) T(0,t)^* e_i,$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$ and $\phi_i(t) = \phi(t)T(t,0)e_i$. We note that y is a solution of (12). We have

$$\begin{aligned}\int_{-\infty}^{\infty} \langle D_t^{-1} y(t), x(t)\rangle_t\, dt &= \int_{-\infty}^{\infty} \langle y(t), x(t)\rangle\, dt \\ &= \int_{-\infty}^{\infty} \left\langle \sum_{i=1}^{d} \alpha(\phi_i) T(0,t)^* e_i, x(t) \right\rangle dt \\ &= \int_{-\infty}^{\infty} \left\langle \sum_{i=1}^{d} \alpha(\phi_i) e_i, T(0,t)x(t) \right\rangle dt \\ &= \left\langle \sum_{i=1}^{d} \alpha(\phi_i) e_i, \int_{-\infty}^{\infty} T(0,t)x(t)\, dt \right\rangle \\ &= \sum_{i=1}^{d} \alpha(\phi_i) e_i^* \int_{-\infty}^{\infty} T(0,t)x(t)\, dt \\ &= \alpha(\psi) = \alpha(x).\end{aligned} \tag{20}$$

In particular,

$$\left| \int_{-\infty}^{\infty} \langle D_t^{-1} y(t), x(t) \rangle_t \, dt \right| = |\alpha(x)| \le \|\alpha\| \cdot \|x\| \tag{21}$$

for any x with bounded support. Now take $T > 0$ and consider a continuous function $\phi \colon \mathbb{R} \to \mathbb{R}_0^+$ with bounded support such that $\phi(t) = 1$ for $t \in [-T, T]$ and $\phi(t) \le 2$ for $t \in \mathbb{R}$. Taking

$$x(t) = \phi(t) \frac{D_t^{-1} y(t)}{\|D_t^{-1} y(t)\|},$$

it follows from (21) that

$$\int_{-T}^{T} \|D_t^{-1} y(t)\| \, dt \le 2\|\alpha\|.$$

Letting $T \to +\infty$ yields the first inequality in (17).

Now take $s \in \mathbb{R}$ and for each $h > 0$, consider a continuous function $g \colon \mathbb{R} \to \mathbb{R}_0^+$ vanishing outside $[s-h, s+h]$ such that $g(t) = 1$ for $t \in [s, s+h/2]$ and $\int_{-\infty}^{\infty} g(t) \, dt = h$. Let

$$x(t) = g(t) \frac{D_t^{-1} y(t)}{\|D_t^{-1} y(t)\|_t} \quad \text{for } t \in \mathbb{R}.$$

Clearly, $x \in Y_0$. Applying (21) yields the inequality

$$\frac{2}{h} \int_{s}^{s+h/2} \|D_t^{-1} y(t)\|_t \, dt \le 2\|\alpha\|.$$

Letting $h \to 0$, we obtain $\|D_s^{-1} y(s)\|_s \le 2\|\alpha\|$, which yields the second inequality in (17).

We define $\beta \colon Y_0 \to \mathbb{R}$ by

$$\beta(x) = \int_{-\infty}^{\infty} \langle D_t^{-1} y(t), x(t) \rangle_t \, dt \quad \text{for } x \in Y_0.$$

It follows from the first inequality in (17) that β is a bounded linear functional on Y_0. Moreover, by (20), β and α coincide on the dense set of functions with bounded support. Hence, $\alpha = \beta$ and (18) holds.

Now assume that α is given by (18) with y satisfying all the conditions in the lemma. For each $x \in Y_0$ we have

$$\begin{aligned}(S^*\alpha)(x) &= \alpha(Sx)\\ &= \int_{-\infty}^{\infty} \big\langle D_t^{-1} y(t), x'(t) - A(t)x(t)\big\rangle_t \, dt\\ &= \int_{-\infty}^{\infty} \big\langle y(t), x'(t) - A(t)x(t)\big\rangle \, dt\\ &= \langle y(t), x(t)\rangle|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \big\langle y'(t) + A(t)^* y(t), x(t)\big\rangle \, dt\\ &= \langle D_t^{-1} y(t), x(t)\rangle_t|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \big\langle y'(t) + A(t)^* y(t), x(t)\big\rangle \, dt = 0.\end{aligned}$$

Therefore, $\alpha \in \operatorname{Ker} S^*$ and the proof of the lemma is complete. □

We continue with the proof of the theorem. We have

$$\operatorname{Im} S = \big\{x \in Y_0 : \alpha(x) = 0 \text{ for } \alpha \in \operatorname{Ker} S^*\big\}.$$

Let $y_1, \dots, y_m$ be a basis of the space of solutions of (12) satisfying (17). For each $i = 1, \dots, m$, we define $\alpha_i \in Y_0^*$ by

$$\alpha_i(x) = \int_{-\infty}^{0} \langle D_t^{-1} y_i(t), x(t)\rangle_t \, dt \quad \text{for } x \in Y_0.$$

Moreover, for each $j = 1, \dots, d$, we define $\beta_j \in Y_0^*$ by

$$\beta_j(x) = x^j(0) \quad \text{for } x \in Y_0,$$

where $x = (x^1, \dots, x^d)$.

Lemma 7. *The set $\{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_d\}$ is linearly independent.*

Proof of the lemma. Assume that $\sum_{i=1}^m \lambda_i \alpha_i = \sum_{j=1}^d \mu_j \beta_j$ for some constants $\lambda_i, \mu_j \in \mathbb{R}$. This implies that

$$\int_{-\infty}^{0} \langle D_t^{-1} y(t), x(t)\rangle_t \, dt = \sum_{j=1}^{d} \mu_j x^j(0) \tag{22}$$

for each $x \in Y_0$, where $y = \sum_{i=1}^m \lambda_i y_i$. Assume that $y(t) \neq 0$ for some $t < 0$. Take $x \in Y_0$ of the form

$$x(t) = \phi(t) \frac{D_t^{-1} y(t)}{\|D_t^{-1} y(t)\|},$$

where $\phi\colon \mathbb{R} \to \mathbb{R}_0^+$ is a continuous function with $\phi(0) = 0$. Then the left-hand side of (22) is positive and $x(0) = 0$, thus leading to a contradiction. Hence, $y = 0$, which implies that $\lambda_1 = \cdots = \lambda_m = 0$. Moreover, the functionals $\beta_1, \dots, \beta_d$ are clearly independent and so $\mu_1 = \cdots = \mu_d = 0$. □

Now take $z \in Y_0$. By Lemma 7, there exists $x \in Y_0$ such that

$$\alpha_i(x) = -\int_0^\infty \langle D_t^{-1} y_i(t), z(t)\rangle_t \, dt \quad \text{for } i = 1, \ldots, m \tag{23}$$

and

$$\beta_j(x) = x^j(0) = z^j(0)$$

for $j = 1, \ldots, d$. We define a function $\tilde{x}$ by $\tilde{x}(t) = x(t)$ for $t \le 0$ and $\tilde{x}(t) = z(t)$ for $t \ge 0$. Clearly, $\tilde{x} \in Y_0$. It follows from (23) that

$$\int_{-\infty}^\infty \langle D_t^{-1} y_i(t), \tilde{x}(t)\rangle_t \, dt = 0 \quad \text{for } i = 1, \ldots, m$$

and thus, it follows from Lemma 6 that $\alpha(\tilde{x}) = 0$ for $\alpha \in \operatorname{Ker} S^*$. We conclude that $\tilde{x} \in \operatorname{Im} S$. Therefore, there exists a function $w \in Y$ such that $Sw = \tilde{x}$ and hence,

$$w'(t) - A(t)w(t) = \tilde{x}(t) = z(t)$$

for $t \ge 0$. It follows from results in [2] (see also [5]) that Eq. (1) admits an exponential dichotomy on $\mathbb{R}_0^+$ with respect to a family of norms $\|\cdot\|_t$. One establishes similarly the existence of an exponential dichotomy on $\mathbb{R}_0^-$ with respect to a family of norms $\|\cdot\|_t$. In view of condition (13) these are in fact strong exponential dichotomies. □

We also have the following stronger statement.

Theorem 8. *If R is a Fredholm operator and $\mathbb{R}^d = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$, then Eq. (1) admits a strong exponential dichotomy on $\mathbb{R}$ with respect to a family of norms $\|\cdot\|_t$.*

Proof. It follows from Theorem 7 that Eq. (1) admits strong exponential dichotomies on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ with respect to a family of norms $\|\cdot\|_t$. Since $\mathbb{R}^d = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$, it follows from results in [1] that Eq. (1) admits a strong exponential dichotomy on $\mathbb{R}$ with respect to a family of norms $\|\cdot\|_t$. □

The following result is a version of Theorem 5 for exponential dichotomies on $\mathbb{R}$.

Theorem 9. *Assume that Eq. (1) admits a strong nonuniform exponential dichotomy on $\mathbb{R}$ and that $B\colon \mathbb{R} \to M_d$ is a continuous function such that*

$$\|B(t) - A(t)\| \le \delta e^{-\epsilon|t|}$$

for all $t \in \mathbb{R}$ and some $\delta > 0$. If δ is sufficiently small, then either Eq. (11) admits a strong nonuniform exponential dichotomy on $\mathbb{R}$ or there exists a continuous nonzero function $x\colon \mathbb{R} \to \mathbb{R}^d$ such that $x'(t) = B(t)x(t)$ for $t \in \mathbb{R}$ and $\sup_{t\in\mathbb{R}} \|x(t)\|_t < +\infty$, where $\|\cdot\|_t$ are norms associated to the strong nonuniform exponential dichotomy of Eq. (1) as in Proposition 1.

Proof. It follows from Theorem 5 that Eq. (11) admits strong exponential dichotomies on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ with respect to a family of norms $\|\cdot\|_t$, say with projections, P_t^+ and P_t^-, respectively. Let $U(t,\tau)$ be the evolution family associated to Eq. (11). It is proved in [1] that

$$\operatorname{Im} P_t^+ = \left\{ v \in \mathbb{R}^d : \sup_{t \ge 0} \|U(t,0)v\|_t < +\infty \right\} \quad \text{for } t \ge 0$$

and

$$\operatorname{Ker} P_t^- = \left\{ v \in \mathbb{R}^d : \sup_{t \le 0} \|U(t,0)v\|_t < +\infty \right\} \quad \text{for } t \le 0.$$

Moreover,

$$\dim \operatorname{Im} P_t = \dim \operatorname{Im} P_t^+ \quad \text{for } t \ge 0$$

and

$$\dim \operatorname{Im} P_t = \dim \operatorname{Im} P_t^- \quad \text{for } t \le 0.$$

Now assume that there exists a nonzero vector $v \in \operatorname{Im} P_0^+ \cap \operatorname{Ker} P_0^-$ and define a function $x\colon \mathbb{R} \to \mathbb{R}^d$ by $x(t) = U(t,0)v$ for $t \in \mathbb{R}$. Clearly, x is continuous, solves (11) and $\sup_{t\in\mathbb{R}} \|x(t)\|_t < +\infty$. If $\operatorname{Im} P_0^+ \cap \operatorname{Ker} P_0^- = \{0\}$, then

$$\dim \operatorname{Im} P_0^+ + \dim \operatorname{Ker} P_0^- = \dim \operatorname{Im} P_0 + d - \dim \operatorname{Im} P_0 = d$$

and so $\mathbb{R}^d = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$. It follows from results in [1] that Eq. (11) admits a strong exponential dichotomy on $\mathbb{R}$ with respect to a family of norms $\|\cdot\|_t$. Hence, by Proposition 1, the equation admits a strong nonuniform exponential dichotomy on $\mathbb{R}$. □

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