UNCOUNTABLY MANY PLANAR EMBEDDINGS OF UNIMODAL INVERSE LIMIT SPACES

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ABSTRACT. For point x in the inverse limit space X with a single unimodal bonding map we construct, with the use of symbolic dynamics, a planar embedding such that x is accessible. It follows that there are uncountably many non-equivalent planar embeddings of X.

1. INTRODUCTION

Inverse limit spaces can be often used as a model to construct attractors of some plane diffeomorphisms, see for example [14, 15, 3]. One of the simplest examples is the Knaster continuum (bucket handle), which is the attractor of Smale's horseshoe map and can be modeled as the inverse limit space of the full tent map $T_2(x) := \min\{2x, 2(1-x)\}$ for $x \in [0,1]$, see [1].

Inverse limit spaces with unimodal bonding maps $T : [0,1] \to [0,1]$ (from now on denoted by X) are chainable. The study of embeddings of chainable continua dates back to 1951 when Bing proved in [4] that every chainable continuum can be embedded in the plane. However, his proof does not offer any insight what such embeddings look like. The first explicit class of embeddings of X was given by Brucks & Diamond in [5]. Later, Bruin [6] extended this result showing that the embedding can be made such that the shift-homeomorphism extends to a Lipschitz map on $\mathbb{R}^2$. Both mentioned results are using symbolic dynamics as the main tool in the description of X.

Locally, inverse limit spaces of unimodal maps roughly resemble Cantor sets of arcs. However, this is not true in general. In [2] Barge, Brucks & Diamond proved that in the tent family T_s for a dense G_δ set of slopes $s \in [\sqrt{2}, 2]$, every open set of the inverse limit space $\varprojlim([s(1 - \frac{s}{2}), \frac{s}{2}], T_s)$ not only contains a homeomorphic copy of the space itself but also homeomorphic copies of every unimodal inverse limit space. Thus it would be interesting to see what kind of embeddings in the plane of complicated X are possible.

Philip Boyland posed the following questions on the Continuum Theory and Dynamical Systems Workshop in Vienna in July 2015:

> Can a complicated X be embedded in $\mathbb{R}^2$ in multiple ways? For example, do there exist embeddings in $\mathbb{R}^2$ of X that are non-equivalent to standard embeddings constructed in [5] and [6]?

For a special case of the full tent map these two questions were already answered in the affirmative by Mahavier [9], Schwartz [13] and Dębski & Tymchatyn [8]. A *composant* $\mathcal{U} \subset X$ of a point $x \in \mathcal{U}$ is a union of all proper subcontinua of X containing point x. There are two fixed points of the map T; 0 and r. Denote the components of $(\ldots, 0, 0)$ and $(\ldots, r, r)$ by $\mathcal{C}$ and $\mathcal{R}$ respectively. The *standard embeddings* given in [5] and [6] make the composants $\mathcal{C}$ and $\mathcal{R}$ respectively accessible.

Definition 1. *A point $a \in X \subset \mathbb{R}^2$ is* accessible *(i.e., from the complement of X) if there exists an arc $A = [x, y] \subset \mathbb{R}^2$ such that $a = x$ and $A \cap X = \{a\}$. We say that a composant $\mathcal{U} \subset X$ is* accessible, *if $\mathcal{U}$ contains an accessible point.*

In the special case of the full tent map, Mahavier showed in [9] that for every composant $\mathcal{U} \subset X$ there exists a homeomorphism $h : X \to \mathbb{R}^2$ such that each point of $h(\mathcal{U})$ is accessible. Schwartz extended Mahavier's result and proved that embeddings of X which do not make $\mathcal{C}$ or $\mathcal{R}$ accessible are non-equivalent to standard embeddings.

Definition 2. *Denote two planar embeddings of X by $g_1 \colon X \to E_1 \subset \mathbb{R}^2$ and $g_2 \colon X \to E_2 \subset \mathbb{R}^2$. We say that g_1 and g_2 are* equivalent embeddings *if there exists a homeomorphism $h \colon E_1 \to E_2$ which can be extended to the plane.*

In this paper we make use of symbolic dynamics description of X introduced in [5] and [6] and answer on the questions of Boyland in the affirmative. We construct embeddings of X by selecting (the itinerary of) the accessible point. For every unimodal map of positive topological entropy, we obtain uncountably many embeddings by making an arbitrary point accessible.

Theorem 1. *For every point $a \in X$ there exists an embedding of X in the plane such that a is accessible.*

Corollary 1. *There are uncountably many non-equivalent embeddings of X in the plane.*

The short outline of the paper is as follows. Section 2 provides the basic set-up as introduced in [5] and [6]. Next, we construct specific representations of X in the plane in Section 3. In Section 4 we prove that the representations given in Section 3 are indeed embeddings and prove the main results.

2. Preliminaries

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let $T : [0, 1] \to [0, 1]$ be a unimodal map fixing 0 and let c denote the *critical point* of T. We define the *inverse limit space with*

the bonding map T by

$$X := \{(\ldots, x_{-2}, x_{-1}, x_0) : T(x_{-(i+1)}) = x_{-i} \in [0,1], \ i \in \mathbb{N}_0\} \subset [0,1]^{-\mathbb{N}}$$

equipped with the product metric

$$d((\ldots, x_{-2}, x_{-1}, x_0), (\ldots, y_{-2}, y_{-1}, y_0)) := \sum_{i \leq 0} 2^i |x_i - y_i|.$$

This makes X a *continuum, i.e.,* a compact and connected metric space. Define the *shift homeomorphism* as $\sigma \colon X \to X$, $\sigma((\ldots, x_{-2}, x_{-1}, x_0)) := (\ldots, x_{-2}, x_{-1}, x_0, T(x_0))$ and the *projection map* as $\pi_n \colon X \to [0,1]$, $\pi_n((\ldots, x_{-2}, x_{-1}, x_0)) := x_{-n}$, for every $n \in \mathbb{N}_0$.

In the construction of planar embeddings of spaces X we recall a well-known symbolic description introduced in [5]. The space X will be represented by the quotient space $\Sigma_{adm}/\sim$, where $\Sigma_{adm} \subseteq \{0,1\}^{\mathbb{Z}}$ is equipped with the product topology. We first need to recall the kneading theory for unimodal maps. To every $x \in [0,1]$ we assign its *itinerary*:

$$I(x) := \nu_0(x)\nu_1(x)\ldots,$$

where

$$\nu_i(x) := \begin{cases} 0, & T^i(x) \in [0, c), \\ *, & T^i(x) = c, \\ 1, & T^i(x) \in (c, 1]. \end{cases}$$

Note that if $\nu_i(x) = *$ for some $i \in \mathbb{N}_0$, then $\nu_{i+1}(x)\nu_{i+2}(x)\ldots = I(T(c))$. The sequence $\nu := I(T(c))$ is called the *kneading sequence of T* and is denoted by $\nu = c_1 c_2 \ldots$, where $c_i := \nu_i(T(c)) \in \{0, *, 1\}$ for every $i \in \mathbb{N}$. Observe that if $*$ appears in the kneading sequence, then c is periodic under T, *i.e.,* there exists $n > 0$ such that $T^n(c) = c$ and the kneading sequence is of the form $\nu = (c_1 \ldots c_{n-1}*)^\infty$. In this case we adjust the kneading sequence by taking the smallest of $(c_1 \ldots c_{n-1}0)^\infty$ and $(c_1 \ldots c_{n-1}1)^\infty$ in the parity-lexicographical ordering defined below.

By $\#_1(a_1 \ldots a_n)$ we denote the number of ones in a finite word $a_1 \ldots a_n \in \{0,1\}^n$; it can be either even or odd.

Choose $t = t_1 t_2 \ldots \in \{0,1\}^{\mathbb{N}}$ and $s = s_1 s_2 \ldots \in \{0,1\}^{\mathbb{N}}$ such that $s \neq t$. Take the smallest $k \in \mathbb{N}$ such that $s_k \neq t_k$. Then the *parity-lexicographical ordering* is defined as

$$s \prec t \Leftrightarrow \begin{cases} s_k < t_k \text{ and } \#_1(s_1 \ldots s_{k-1}) \text{ is even, or} \\ s_k > t_k \text{ and } \#_1(s_1 \ldots s_{k-1}) \text{ is odd.} \end{cases}$$

This ordering is also well-defined on $\{0, *, 1\}^{\mathbb{N}}$ once we define $0 < * < 1$.
Thus if $(c_1 \ldots c_{n-1}0)^\infty \prec (c_1 \ldots c_{n-1}1)^\infty$ we modify $\nu = (c_1 \ldots c_{n-1}0)^\infty$, otherwise $\nu = (c_1 \ldots c_{n-1}1)^\infty$.

Example. If c is periodic of period 3 then the kneading sequence for T is $\nu = (10*)^\infty$. Since $101 \ldots < 100 \ldots$ in parity-lexicographical ordering, we modify $\nu = (101)^\infty$.

In the same way we modify itinerary of an arbitrary point $x \in [0,1]$. If $\nu_i(x) = *$ and i is the smallest positive integer with this property then we replace $\nu_{i+1}(x)\nu_{i+2}(x)\ldots$

with the modified kneading sequence. Thus $*$ can appear only once in the modified itinerary of an arbitrary point $x \in [0, 1]$.

From now onwards we assume that the itineraries of points from $[0, 1]$ are modified.

It is a well-known fact (see [10]) that a kneading sequence completely characterizes the dynamics of unimodal map in the sense of the following proposition:

Proposition 1. *If a sequence $s_0 s_1 \ldots \in \{0, *, 1\}^{\mathbb{N}}$ is the itinerary of some $x \in [T^2(c), T(c)]$, then*

(1) $$I(T^2(c)) \preceq s_k s_{k+1} \ldots \preceq \nu = I(T(c)), \text{ for every } k \in \mathbb{N}_0.$$

Conversely, assume $s_0 s_1 \ldots \in \{0, *, 1\}^{\mathbb{N}}$ satisfies (1). If there exists $j \in \mathbb{N}_0$ such that $s_{j+1} s_{j+2} \ldots = \nu$, and j is minimal with this property, assume additionally that $s_j = *$. Then $s_0 s_1 \ldots$ is realized as the itinerary of some $x \in [T^2(c), T(c)]$.

Definition 3. *We say that a sequence $y_0 y_1 \ldots \in \{0, *, 1\}^{\infty}$ is* admissible *if it is realized as the itinerary of some $x \in [0, 1]$.*

Remark 1. *Note that the Proposition 1 gives conditions on admissible sequences of points $x \in [T^2(c), T(c)]$. For points $y \in [0, T^2(c)) \cup (T(c), 1]$ it follows that the admissible sequences are exactly $0^{\mathbb{N}}$, $10^{\mathbb{N}}$, $0 \ldots 0 y_j y_{j+1} \ldots$, $10 \ldots 0 y_j y_{j+1} \ldots$ for sequences $y_j y_{j+1} \ldots$, $j \in \mathbb{N}$, that satisfy the conditions of Proposition 1.*

Next we show how to expand the above construction to X. Take $x = (\ldots, x_{-2}, x_{-1}, x_0) \in X$. Define the itinerary of x as a two-sided infinite sequence

$$\bar{I}(x) := \ldots \nu_{-2}(x) \nu_{-1}(x) . \nu_0(x) \nu_1(x) \ldots \in \{0, *, 1\}^{\mathbb{Z}},$$

where $\nu_0(x) \nu_1(x) \ldots = I(x_0)$ and

$$\nu_i(x) = \begin{cases} 0, & x_i \in [0, c), \\ *, & x_i = c, \\ 1, & x_i \in (c, 1], \end{cases}$$

for all $i < 0$.

We make the same modifications as above. If $*$ appears for the first time at $\nu_k(x)$ for some $k \in \mathbb{Z}$ then $\nu_{k+1}(x) \nu_{k+2}(x) \ldots = \nu$. If there is no such minimal k, then the kneading sequence is periodic, $\nu = (c_1 c_2 \ldots c_{n-1} *)^{\infty}$ and the itinerary is of the form $(c_1 \ldots c_{n-1} *)^{\mathbb{Z}}$. Replace $(c_1 \ldots c_{n-1} *)^{\mathbb{Z}}$ with the modified itinerary $(c_1 c_2 \ldots c_{n-1} c_n)^{\mathbb{Z}}$, where $\nu = (c_1 \ldots c_{n-1} c_n)^{\infty}$. In this way $*$ can appear at most once in every itinerary. Now we are ready to identify the inverse limit space with a quotient of a space of two-sided sequences consisting of two symbols.
Let $\Sigma = \{0, 1\}^{\mathbb{Z}}$ be the space of two-sided sequences equipped with the metric

$$d((s_i)_{i \in \mathbb{Z}}, (t_i)_{i \in \mathbb{Z}}) := \sum_{i \in \mathbb{Z}} \frac{|s_i - t_i|}{2^{|i|}},$$

and define the shift homeomorphism $\sigma\colon \Sigma \to \Sigma$ as

$$\sigma(\ldots s_{-2}s_{-1}.s_0 s_1 \ldots) := \ldots s_{-2}.s_{-1}s_0 s_1 \ldots$$

By $\Sigma_{adm} \subseteq \Sigma$ we denote all $s \in \Sigma$ such that either

- (a) $s_k s_{k+1} \ldots$ is admissible for every $k \in \mathbb{Z}$, or
- (b) there exists $k \in \mathbb{Z}$ such that $s_{k+1}s_{k+2} \ldots = \nu$ and $s_{k-i} \ldots s_{k-1} * s_{k+1}s_{k+2} \ldots$ is admissible for every $i \in \mathbb{N}$.

We abuse notation and call admissible also the two-sided sequences in Σ_{adm} that correspond to admissible points.

Let us define an equivalence relation on the space Σ_{adm}. For sequences $s = (s_i)_{i \in \mathbb{Z}}$, $t = (t_i)_{i \in \mathbb{Z}} \in \Sigma_{adm}$ we define a relation

$$s \sim t \Leftrightarrow \begin{cases} \text{either } s_i = t_i \text{ for every } i \in \mathbb{Z}, \\ \text{or if there exists } k \in \mathbb{Z} \text{ such that } s_i = t_i \text{ for all } i \neq k \text{ but } s_k \neq t_k \\ \text{and } s_{k+1}s_{k+2} \ldots = t_{k+1}t_{k+2} \ldots = \nu. \end{cases}$$

It is not difficult to see that this is indeed an equivalence relation on the space Σ_{adm}. Furthermore, every itinerary is identified with at most one different itinerary and that the quotient space $\Sigma_{adm}/\sim$ of Σ_{adm} is well defined. It was also shown that $\Sigma_{adm}/\sim$ is homeomorphic to X. So in order to embed X in the plane it is enough to embed $\Sigma_{adm}/\sim$ in the plane. For all observations in this paragraph we refer to the paper [5] of Brucks & Diamond (Lemmas 2.2-2.4 and Theorem 2.5).

An *arc* is a homeomorphic image of an interval $[a, b] \subset \mathbb{R}$. A key fact for constructing embeddings is that X is the union of *basic arcs* defined below. Let $\overleftarrow{s} = \ldots s_{-2}s_{-1}. \in \{0, 1\}^{-\mathbb{N}}$ be an admissible left-infinite sequence (*i.e.*, every finite subword is admissible). The subset

$$A(\overleftarrow{s}) := \{x \in X : \nu_i(x) = s_i, \forall i < 0\} \subset X$$

is called a *basic arc*. Note that $\pi_0 : A(\overleftarrow{s}) \to [0, 1]$ is injective. In [6, Lemma 1] it was observed that $A(\overleftarrow{s})$ is indeed an arc (possibly degenerate). For every basic arc we define two quantities as follows:

$$\begin{aligned} \tau_L(\overleftarrow{s}) &:= \sup\{n > 1 : s_{-(n-1)} \ldots s_{-1} = c_1 c_2 \ldots c_{n-1}, \#_1(c_1 \ldots c_{n-1}) \text{ odd}\}, \\ \tau_R(\overleftarrow{s}) &:= \sup\{n \geq 1 : s_{-(n-1)} \ldots s_{-1} = c_1 c_2 \ldots c_{n-1}, \#_1(c_1 \ldots c_{n-1}) \text{ even}\}. \end{aligned}$$

Remark 2. *For $n = 1$, $c_1 c_2 \ldots c_{n-1} = \emptyset$ and $\#_1(\emptyset)$ is even. Thus $\tau_R(\overleftarrow{s}) = 1$ if and only if $s_{-(n-1)} \ldots s_{-1} \neq c_1 c_2 \ldots c_{n-1}$ for all $n > 1$.*

These definitions first appeared in [6] in order to study the number of endpoints of inverse limit spaces X. We now adapt two lemmas from [6] that we will use later in the paper.

Lemma 1. *([6], Lemma 2) Let $\overleftarrow{s} \in \{0,1\}^{-\mathbb{N}}$ be admissible such that $\tau_L(\overleftarrow{s}), \tau_R(\overleftarrow{s}) < \infty$. Then*

$$\pi_0(A(\overleftarrow{s})) = [T^{\tau_L(\overleftarrow{s})}(c), T^{\tau_R(\overleftarrow{s})}(c)].$$

If $\overleftarrow{t} \in \{0,1\}^{-\mathbb{N}}$ is another admissible left-infinite sequence such that $s_i = t_i$ for all $i < 0$ except for $i = -\tau_R(\overleftarrow{s}) = -\tau_R(\overleftarrow{t})$ (or $i = -\tau_L(\overleftarrow{s}) = -\tau_L(\overleftarrow{t})$), then $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common boundary point.

Lemma 2. *([6], Lemma 3) If $\overleftarrow{s} \in \{0,1\}^{-\mathbb{N}}$ is admissible, then*

$$\sup \pi_0(A(\overleftarrow{s})) = \inf\{T^n(c) : s_{-(n-1)} \ldots s_{-1} = c_1 \ldots c_{n-1}, n \geq 1, \#_1(c_1 \ldots c_{n-1}) \text{ even}\},$$
$$\inf \pi_0(A(\overleftarrow{s})) = \sup\{T^n(c) : s_{-(n-1)} \ldots s_{-1} = c_1 \ldots c_{n-1}, n \geq 1, \#_1(c_1 \ldots c_{n-1}) \text{ odd}\}.$$

Example. Take the unimodal map with the kneading sequence $\nu = (101)^\infty$. Then $\overleftarrow{s} = (011)^\infty 010.$ and $\overleftarrow{t} = (011)^\infty 110.$ are admissible, $\tau_L(\overleftarrow{s}) = \tau_L(\overleftarrow{t}) = 3$, $\tau_R(\overleftarrow{s}) = \tau_R(\overleftarrow{t}) = 1$ and $s_i = t_i$ for all $i < 0$ except for $i = -3 = -\tau_L(\overleftarrow{s}) = -\tau_L(\overleftarrow{t})$. By Lemma 2, $\pi_0(A(\overleftarrow{s})) = \pi_0(A(\overleftarrow{t})) = [T^3(c), T(c)]$, and by Lemma 1, $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common boundary point which is projected to $T^3(c)$, see Figure 1. Note that in this example both τ_L and τ_R agree for $\overleftarrow{s}$ and $\overleftarrow{t}$, which need not be the case in general.
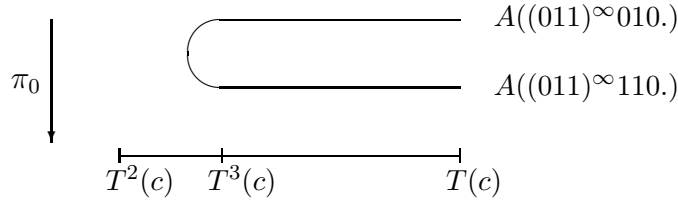

FIGURE 1. Example of two basic arcs having a boundary point in common.

3. REPRESENTATION IN THE PLANE

This section is the first step towards embedding X in the plane so that an arbitrary point $a \in X$ becomes accessible. Recall that we denote the symbolic representation of a by $\bar{I}(a) = \ldots l_{-2}l_{-1}.l_0 l_1 \ldots$, so $a \in A(\ldots l_{-2}l_{-1}.)$. We present the following ordering depending on some $L = \ldots l_{-2}l_{-1}.$ and we work with this ordering from now onwards.

Definition 4. *Let $\overleftarrow{s}, \overleftarrow{t} \in \{0,1\}^{-\mathbb{N}}$ and let $k \in \mathbb{N}$ be the smallest natural number such that $s_{-k} \neq t_{-k}$. Then*

$$\overleftarrow{s} \prec_L \overleftarrow{t} \Leftrightarrow \begin{cases} t_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \ldots s_{-1}) - \#_1(l_{-(k-1)} \ldots l_{-1}) \text{ even, or} \\ s_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \ldots s_{-1}) - \#_1(l_{-(k-1)} \ldots l_{-1}) \text{ odd.} \end{cases}$$

Note that such ordering is well-defined and the left infinite tail L is the largest sequence.

Lemma 3. *Assume $\overleftarrow{s} \prec_L \overleftarrow{u} \prec_L \overleftarrow{t}$ and assume that $s_{-n} \ldots s_{-1} = t_{-n} \ldots t_{-1}$. Then also $u_{-n} \ldots u_{-1} = s_{-n} \ldots s_{-1} = t_{-n} \ldots t_{-1}$.*

Proof. If $n = 1$ the statement follows easily so let us assume that $n \geq 2$. Assume that there exists $k < n$ such that $u_{-k} \neq s_{-k}$ and take k the smallest natural number with this property. Assume without loss of generality that $(-1)^{\#_1(s_{-(k-1)}\ldots s_{-1})} = (-1)^{\#_1(l_{-(k-1)}\ldots l_{-1})}$ (the proof follows similarly when the parities are different). Since $\overleftarrow{s} \prec_L \overleftarrow{u}$ it follows that $u_{-k} = l_{-k}$. Also, $\overleftarrow{u} \prec_L \overleftarrow{t}$ gives $t_{-k} = l_{-k}$. Since $u_{-k} \neq t_{-k}$, we get a contradiction. $\square$

Let $C \subset [0, 1]$ be the middle-third Cantor set,

$$C := [0, 1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right).$$

Points in C are coded by the left-infinite sequences of zeros and ones. We embed basic arcs in the plane as horizontal lines along the Cantor set and then join corresponding endpoints with semi-circles as in Figure 1. The ordering has to be defined in a way that semi-circles neither cross horizontal lines nor each other.

Example. For $L = 1^\infty.$, points in C are coded as in Figure 2 (a). Note that this is the same ordering as in the paper by Bruin [6]. The ordering obtained by $L = 0^\infty 1.$ is the ordering from the paper by Brucks and Diamond [5]. In Figure 2 (b) points in Cantor set are coded with respect to $L = \ldots 101.$

From now onwards, we assume that $\overleftarrow{s} \in \{0, 1\}^{\mathbb{N}}$ is an admissible left-infinite sequence. Define $\psi_L : \{0, 1\}^{-\mathbb{N}} \to C$ as

$$\psi_L(\overleftarrow{s}) := \sum_{i=1}^{\infty} (-1)^{\#_1(l_{-i}\ldots l_{-1}) - \#_1(x_{-i}\ldots x_{-1})} 3^{-i} + \frac{1}{2},$$

and we let $C_{adm} = \{\psi_L(\overleftarrow{s}) : \overleftarrow{s} \text{ admissible left-infinite sequence}\}$ be the subset of "admissible vertical coordinates". Note that $\psi_L(L) = 1$ is the largest point in C_{adm}.

From now onwards let d_e denote the Euclidean distance in $\mathbb{R}^2$.

Remark 3. *Note that if $\overleftarrow{s}, \overleftarrow{t} \in \{0, 1\}^{-\mathbb{N}}$ are such that $s_{-n} \ldots s_{-1} = t_{-n} \ldots t_{-1}$, then $d_e(\psi_L(\overleftarrow{s}), \psi_L(\overleftarrow{t})) \leq 3^{-n}$. If $s_{-n} \neq t_{-n}$, then $d_e(\psi_L(\overleftarrow{s}), \psi_L(\overleftarrow{t})) \geq 3^{-n}$.*

Now we represent X as the quotient space of the subset of $I \times C_{adm}$ for $I := [0, 1]$. To every point $x = (\ldots, x_{-2}, x_{-1}, x_0) \in X$ we will assign either a point or two points in $I \times C_{adm}$ by rule (2) below. From now on, write $\ldots s_{-3}s_{-2}s_{-1} = \ldots \nu_{-3}(x)\nu_{-2}(x)\nu_{-1}(x)$. Let $\varphi : X \to I \times C_{adm}$ be defined in the following way:

$$(2) \qquad \varphi(x) := \begin{cases} (x_0, \psi_L((s_i)_{i<0})), & \text{if } s_i \neq * \text{ for every } i < 0, \\ (x_0, p) \cup (x_0, q), & \text{if } s_i = * \text{ for some } i < 0, \end{cases}$$

FIGURE 2. Coding the Cantor set with respect to (a) $L = \ldots 111.$ and (b) $L = \ldots 101.$

where

$$
\begin{cases}
p = \psi_L(\ldots s_{-(i+1)} 0 s_{-(i-1)} \ldots s_{-1}.), \\
q = \psi_L(\ldots s_{-(i+1)} 1 s_{-(i-1)} \ldots s_{-1}.).
\end{cases}
$$

Set $Y := \varphi(X) \subset I \times C_{adm}$. The next step is to identify points in Y in the same way as they are identified in the symbolic representation of X. For $a, b \in Y$:

$$a \sim b \text{ if there exists } x \in X \text{ such that } a, b \in \varphi(x).$$

If $a \neq b \sim a$ we write $\tilde{a} := b$. If $\tilde{a} = b$ and $x \in X$ is such that $a, b \in \varphi(x)$ and $\nu_i(x) = *$ we say that a and b are *joined at level i*.

Note that $\varphi \colon X \to Y/\sim$ is a well-defined map. Equip Y with the Euclidean topology and $Y/\sim$ with the standard quotient topology. Let $\pi_C : I \times C \to C$ and $\pi_I : I \times C \to I$ denote the *natural projections*. The next proposition is an analogue of the Proposition 4 from [6]. We prove it here for the sake of completeness.

Proposition 2. *The map $\varphi \colon X \to Y/\sim$ is a homeomorphism.*

Proof. We first prove that $Y/\sim$ is a Hausdorff space and because X is compact it is enough to check that φ is a continuous bijection to obtain a homeomorphism between X and $Y/\sim$, see *e.g.* Theorem 26.6. in [11].

Take $x \neq y \in Y$ such that $x \neq \tilde{y}$. First assume that $|\pi_I(x) - \pi_I(y)| = 0$. Let $\delta := \min\{|\pi_C(x) - \pi_C(y)|, |\pi_C(\tilde{x}) - \pi_C(y)|\}$. Then take $\{z : |\pi_C(x) - \pi_C(z)| \text{ or } |\pi_C(\tilde{x}) -$

$\pi_C(z)| < \delta/3\}$ and $\{z : |\pi_C(y) - \pi_C(z)|$ or $|\pi_C(\tilde{y}) - \pi_C(z)| < \delta/3\}$ for open neighbourhoods of x and y respectively and they are disjoint. Now assume that $\varepsilon := |\pi_I(x) - \pi_I(y)| > 0$. Then $\{z : |\pi_I(x) - \pi_I(z)| < \varepsilon/3\}$ and $\{z : |\pi_I(y) - \pi_I(z)| < \varepsilon/3\}$ are disjoint open neighbourhoods for x and y respectively, so $Y/\sim$ is indeed a Hausdorff space.

Now we prove that φ is continuous. It is enough to prove that for $a \in X$ and a sequence $(x^n)_{n \in \mathbb{N}} \subset X$ such that $\lim_{n\to\infty} x^n = a$ it holds that $\lim_{n\to\infty} \varphi(x^n) = \varphi(a)$. Assume that $\lim_{n\to\infty} x^n = a$. Thus for every $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ it follows that $\nu_{-M}(x^n) \ldots \nu_M(x^n) = \nu_{-M}(a) \ldots \nu_M(a)$. We need to show that for every open $\varphi(a) \in U \subset Y/\sim$ there exists $N' \in \mathbb{N}$ such that for every $n \geq N'$ it holds that $\varphi(x^n) \in U$. Let us fix an open set $U \ni \varphi(a)$. If for $x = (\ldots, x_{-1}, x_0) \in X$ there exists $i \in \mathbb{N}$ such that $\nu_{-i}(x) = *$ then we set $\varphi(x) = \varphi'(x) \cup \varphi''(x)$ where $\varphi'(x) := (x_0, \psi_L(\ldots \nu_{-(i+1)}(x)0\nu_{-(i-1)}(x) \ldots \nu_{-2}(x)\nu_{-1}(x).))$ and $\varphi''(x) := (x_0, \psi_L(\ldots \nu_{-(i+1)}(x)1\nu_{-(i-1)}(x) \ldots \nu_{-2}(x)\nu_{-1}(x).))$.

Case I: For every $i \in \mathbb{N}$, $\nu_{-i}(a) \neq *$. If there exists $K \in \mathbb{N}$ such that for every $n \geq K$ it follows that $\nu_{-j}(x^n) \neq *$ for every $j \in \mathbb{N}$, then there is $N' \geq K$ such that $\varphi(x^n) \in U$ for every $n \geq N'$.
Now assume that there exists an increasing sequence $(n_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that $\nu_{-j}(x^{n_i}) = *$ for some $j \in \mathbb{N}$. Then there exist open sets $U_1^{n_i}, U_2^{n_i} \subset Y$ such that $\varphi'(x^{n_i}) \in U_1^{n_i}$ and $\varphi''(x^{n_i}) \in U_2^{n_i}$ and $\varphi^{-1}(U) = U_1^{n_i} \cup U_2^{n_i}$ for every $i \in \mathbb{N}$. Because $x^n \to a$ as $n \to \infty$ and by the definition of φ it follows that $\varphi'(x^{n_i}) \to \varphi(a)$ and $\varphi''(x^{n_i}) \to \varphi(a)$ as $i \to \infty$. Thus we again conclude that there exists $N' \in \mathbb{N}$ such that for every $n \geq N'$ it follows that $\varphi(x^n) \in U$.

Case II: Let $K \in \mathbb{N}$ be such that $\nu_K(a) = *$ and thus $\varphi(a) = \varphi'(a) \cup \varphi''(a)$. Take $M > K$ so that $\nu_{-M}(a) \ldots \nu_M(a) = \nu_{-M}(x^n) \ldots \nu_{-K}(x^n) \ldots \nu_K(x^n) \ldots \nu_M(x^n)$ for every $n \geq N$, and so $\varphi(x^n) = \varphi'(x^n) \cup \varphi''(x^n)$. Thus there exist open sets $U_1, U_2 \subset Y$ such that $\varphi'(a) \in U_1$, $\varphi''(a) \in U_2$ and $\varphi^{-1}(U) = U_1 \cup U_2$. It follows that there exists $N' > N$ such that for every $n > N'$ it holds that $\varphi'(x^n) \in U_1$ and $\varphi''(x^n) \in U_2$ and thus $\varphi(x^n) \in U$. $\qquad\square$

Now we are ready to represent X in the plane. This is still not an embedding but it is the first step towards it. Connect identified points in $I \times C_{adm}$ with semi-circles. Suppose $a \neq b \in Y$ are joined at level n. By Lemma 1, points a and b are both endpoints of basic arcs in $I \times C_{adm}$ and are both right or left endpoints. If $\#_1(c_1 \ldots c_{n-1})$ is even (odd), a and b are right (left) endpoints and we join them with a semi-circle on the right (left), see Figure 1.

Proposition 3. *Every semi-circle defined above crosses neither Y nor another semi-circle.*

Proof. **Case I:** Assume that there is a semi-circle oriented to the right which intersects an arc A in Y. (See Figure 3.)
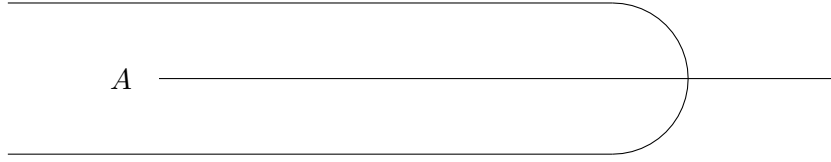
FIGURE 3. Case I in the proof of Proposition 3.

Translated to symbolics, this means that there exist $n \in \mathbb{N}$ and $\overleftarrow{s} \prec_L \overleftarrow{u} \prec_L \overleftarrow{t}$ such that $s_{-(n-1)} \ldots s_{-1} = t_{-(n-1)} \ldots t_{-1} = c_1 \ldots c_{n-1}$, $s_{-n} \neq t_{-n}$ and $\#_1(c_1 \ldots c_{n-1})$ is even. By Lemma 3, $u_{-(n-1)} \ldots u_{-1} = c_1 \ldots c_{n-1}$. By Lemma 2 it follows that $\sup\{\pi_I(A)\} \leq T^n(c)$, and thus an intersection between the arc A and a semi-circle cannot occur.

Case II: Assume that we have a crossing of two semi-circles which project to the same point in I. (See Figure 4.)

$$\overleftarrow{v} = \ldots v_{n+1} 1 c_1 \ldots c_{n-1}.$$
$$\overleftarrow{t} = \ldots t_{n+1} 1 c_1 \ldots c_{n-1}.$$
$$\overleftarrow{u} = \ldots u_{n+1} 0 c_1 \ldots c_{n-1}.$$
$$\overleftarrow{s} = \ldots s_{n+1} 0 c_1 \ldots c_{n-1}.$$
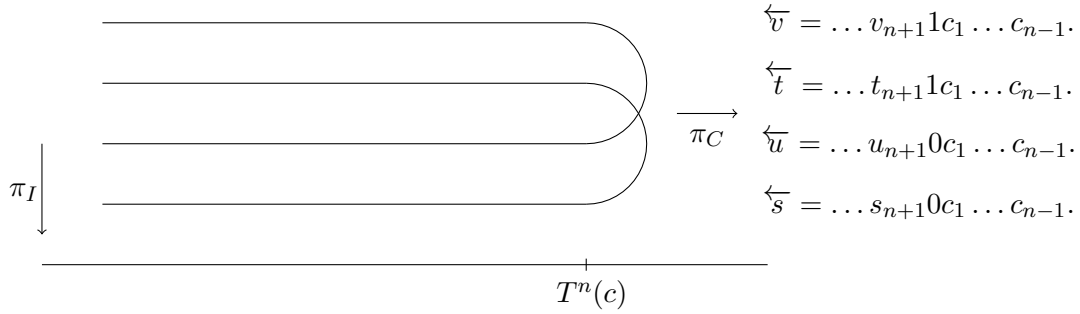
FIGURE 4. Case II in the proof of Proposition 3.

Assume that there exist $n \in \mathbb{N}$ and $\overleftarrow{s} \prec_L \overleftarrow{u} \prec_L \overleftarrow{t} \prec_L \overleftarrow{v}$ such that $s_i = t_i$ for all $i < 0$ except for $i = -n$ and $s_{-(n-1)} \ldots s_{-1} = t_{-(n-1)} \ldots t_{-1} = c_1 \ldots c_{n-1}$ and $u_i = v_i$ for all $i < 0$ except for $i = -n$ and $u_{-(n-1)} \ldots u_{-1} = v_{-(n-1)} \ldots v_{-1} = c_1 \ldots c_{n-1}$. If $s_{-n} = v_{-n}$, then by Lemma 3 also $t_{-n} = u_{-n} = s_{-n} = v_{-n}$ which contradicts the assumption. It follows that $s_{-n} \neq v_{-n}$, because $\overleftarrow{v}, \overleftarrow{u}$ and $\overleftarrow{t}, \overleftarrow{s}$ are respectively connected by a right semi-circle. Assume without loss of generality that $v_{-n} = 1$ and $s_{-n} = 0$. This gives $t_{-n} = 1$ and $u_{-n} = 0$.

Now take the smallest integer $m > n$ such that $v_{-m} \neq t_{-m}$; this m is also the smallest integer such that $u_{-m} \neq s_{-m}$. By the previous paragraph, if $(-1)^{\#_1(s_{-(m-1)} \ldots s_{-1})} = (-1)^{\#_1(u_{-(m-1)} \ldots u_{-1})} \neq (-1)^{\#_1(t_{-(m-1)} \ldots t_{-1})} = (-1)^{\#_1(v_{-(m-1)} \ldots v_{-1})}$, the possibilities for $s_{-m}, u_{-m}, t_{-m}, v_{-m}$ are (depending on the parities of ones): (1) $s_{-m} = 0, u_{-m} = 1, t_{-m} = 1, v_{-m} = 0$, or (2) $s_{-m} = 1, u_{-m} = 0, t_{-m} = 0, v_{-m} = 1$. Both cases lead to a contradiction with $s_{-m} = t_{-m}$ and $u_{-m} = v_{-m}$. □

Thus our ordering gives a representation $Y \cup \{\text{semi-circles}\}$ of X in the plane. Figure 5 and Figure 6 give two examples of these planar representations.
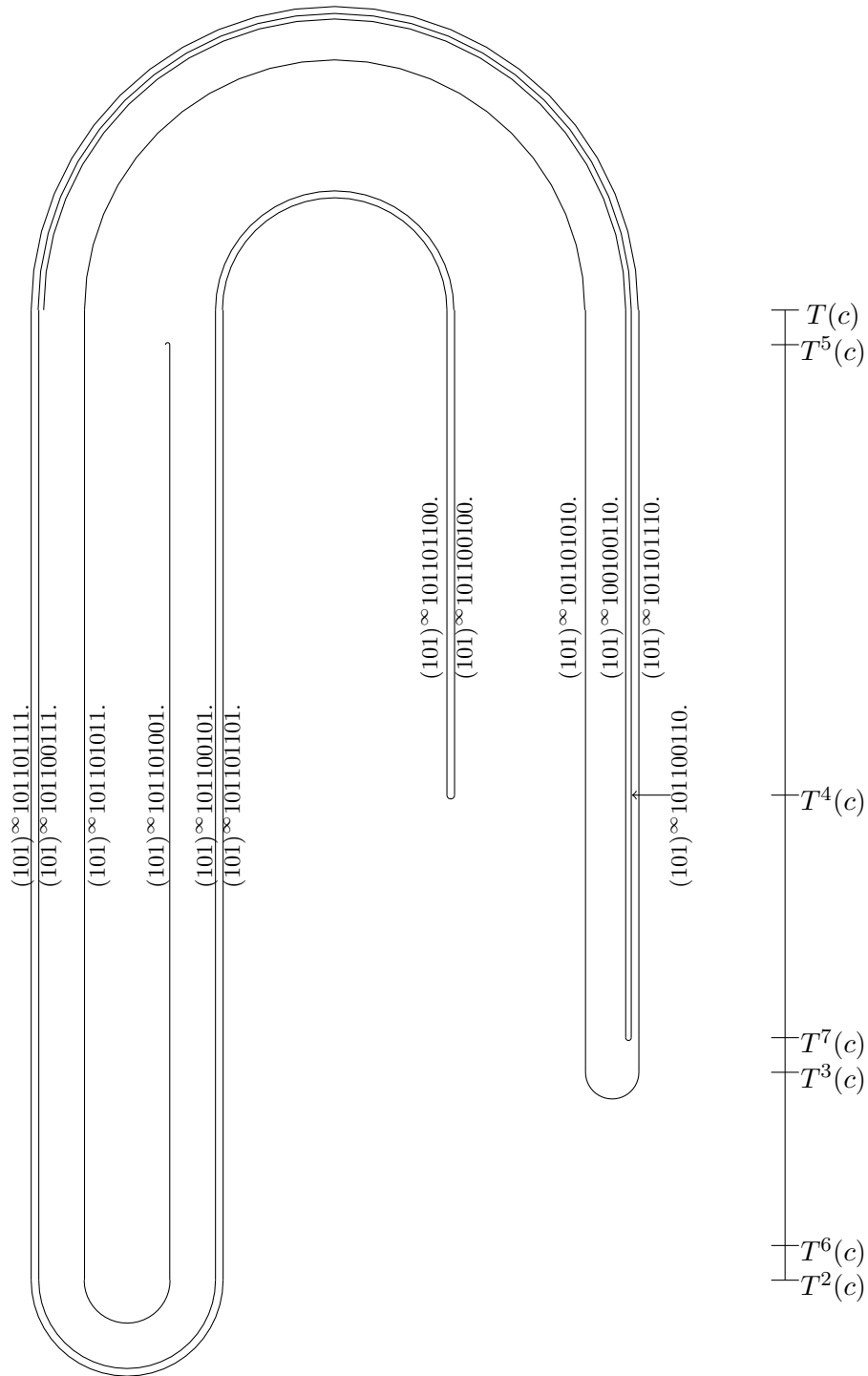
FIGURE 5. The planar representation of an arc in X with the corresponding kneading sequence $\nu = 100110010\ldots$. The ordering on basic arcs is such that the basic arc coded by $L = 1^\infty$. is the largest.
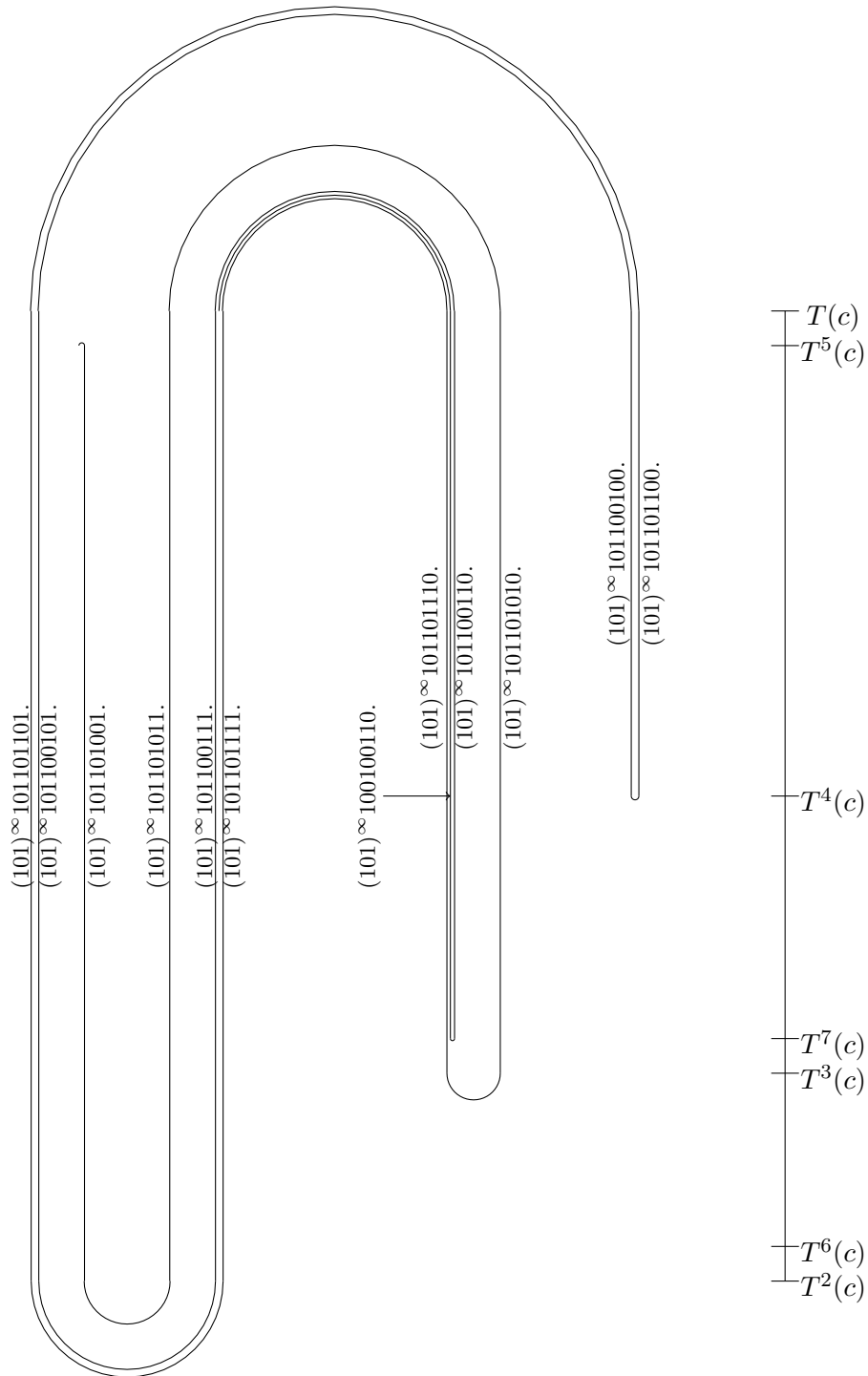
FIGURE 6. The planar representation of the same arc as in Figure 5 in X with the corresponding kneading sequence $\nu = 100110010\ldots$. The ordering on basic arcs is such that the basic arc coded by $L = (101)^{\infty}$. is the largest.

4. Embeddings

In this section we show that representations of X constructed in the previous section are indeed embeddings.

Lemma 4. *Let $U \subset \mathbb{R}^2$ be homeomorphic to the open unit disk, and let $W \subset \mathbb{R}$ be a closed set such that $W \times J \subset U$ for some closed interval J. There exists a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\{w\} \times J)$ is a point for every $w \in W$, $f(\{w\} \times J) \neq f(\{w'\} \times J)$ for every $w \neq w' \in W$, $f|_{U \setminus W \times J}$ is injective and $f|_{\mathbb{R}^2 \setminus U}$ is the identity.*

Proof. Without loss of generality we can take $U := (-1, 2) \times (-1, 1)$, $J := [-1/2, 1/2]$ and $\min(W) = 0$, $\max(W) = 1$. See Figure 7.
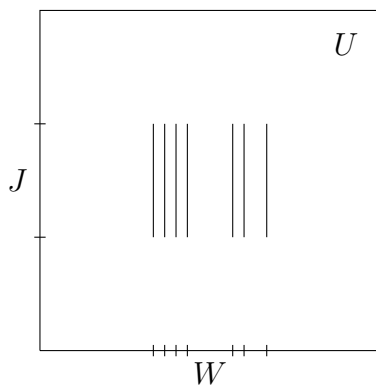

FIGURE 7. Set-up in Lemma 4.

For every $a \in [0, 2]$ we define a continuous function $g(a, \cdot) : [-1, 1] \to [-1, 1]$ as

$$g(a, x) := \begin{cases} (2 - a)x + 1 - a, & x \in [-1, -1/2] \\ ax, & x \in [-1/2, 1/2] \\ (2 - a)x + a - 1, & x \in [1/2, 1]. \end{cases}$$

Note that $g(a, \cdot)$ is injective for every $a \in [0, 2]$, $g(0, x) = 0$ for all $x \in [-1/2, 1/2]$, and $g(1, x) = x$ for all $x \in [-1, 1]$.

Define $\hat{f} : [-1, 2] \times [-1, 1] \to [-1, 2] \times [-1, 1]$ as

$$\hat{f}(x, y) := (x, g(d_e(x, W), y)),$$

where $d_e(x, W) = \inf_{w \in W}\{d_e(x, w)\}$. Note that $x \mapsto d_e(x, W)$ is continuous, so $\hat{f}$ is continuous. Also, $\hat{f}(w, y) = (w, g(0, y)) = (w, 0)$ for $(w, y) \in W \times J$ and $\hat{f}$ is injective otherwise. Also note that $\hat{f}$ is the identity on the boundary of $[-1, 2] \times [-1, 1]$, so $\hat{f}$ can be extended continuously to the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f|_{\mathbb{R}^2 \setminus U}$ is the identity. $\square$

Define $W_n \subset \mathbb{R}^2$ to be the set consisting of all semi-circles that join pairs of points at level n. Note that there exists a set $W \subset \mathbb{R}$ such that W_n is homeomorphic to $W \times J$.

Observe that W is closed. Indeed, if for a sequence $(\overleftarrow{s}^k)_{k\in\mathbb{N}} \subset \{0,1\}^{\mathbb{N}}$ there exists $m \in \mathbb{N}$ such that $\tau_R(\overleftarrow{s}^k) = m$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} \overleftarrow{s}^k = \overleftarrow{s}$, then $\tau_R(\overleftarrow{s}) = m$. The analogous argument holds for τ_L.

Lemma 5. *There exist open sets $U_n \subset \mathbb{R}^2$ such that $W_n \subset U_n$ and for every $n \neq m \in \mathbb{N}$, $U_n \cap U_m = \emptyset$ and $\operatorname{diam}(U_n) \to 0$ as $n \to \infty$.*

Proof. We define the set $G_n := \{\psi_L(\overleftarrow{s}) : \overleftarrow{s} \in \{0,1\}^{-\mathbb{N}} \text{ admissible}, \tau_R(\overleftarrow{s}) = n \text{ or } \tau_L(\overleftarrow{s}) = n\}$ for every $n \in \mathbb{N}$ and let A_n be the smallest interval in $[0,1]$ containing G_n. Note that A_n is closed and $\operatorname{diam}(A_n) \leq 3^{-n}$.

Let M_n denote the midpoint of A_n. If $\#_1(c_1 \ldots c_n)$ is odd, let

$$V_n' = \left\{ (x,y) \in \mathbb{R}^2 : (x - T^n(c))^2 + (y - M_n)^2 \leq \left(\frac{\operatorname{diam}(A_n)}{2}\right)^2, \ x \leq T^n(c) \right\}$$

be the closed left semi-disc centered around $(T^n(c), M_n)$. Similarly, if $\#_1(c_1 \ldots c_n)$ is even, let

$$V_n' = \left\{ (x,y) \in \mathbb{R}^2 : (x - T^n(c))^2 + (y - M_n)^2 \leq \left(\frac{\operatorname{diam}(A_n)}{2}\right)^2, \ x \geq T^n(c) \right\}.$$

be the closed right semi-disc centered around $(T^n(c), M_n)$. Note that $W_n \subset V_n'$, $\operatorname{diam}(V_n') \leq 3^{-n}$ and that $d_e(A_n, \psi_L(\overleftarrow{t})) > 3^{-n}$ for all $\psi_L(\overleftarrow{t}) \notin A_n$. Let V_n be the $\frac{\operatorname{diam}(A_n)}{2\cdot 3}$-neighbourhood of V_n', that is,

$$V_n = \left\{ x \in \mathbb{R}^2 : \text{ there exists } y \in V_n' \text{ such that } d_e(x,y) < \frac{\operatorname{diam}(A_n)}{2\cdot 3} \right\},$$

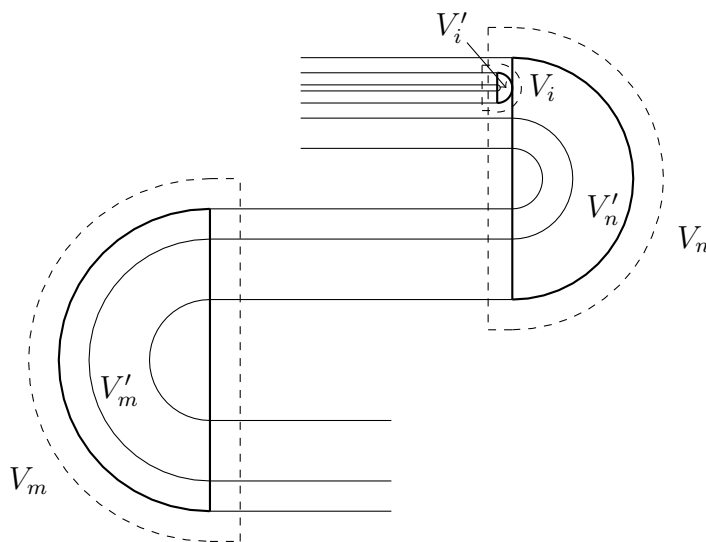see Figure 8. For every $n \in \mathbb{N}$, the open set

Figure 8. Sets constructed in the proof of Lemma 4.

$$U_n := V_n \setminus \overline{\cup_{i>n} V_i}$$

contains W_n, because otherwise there exists an increasing sequence $(i_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ so that points $x^{i_k} \in \{T^{i_k}(c)\} \times G_{i_k}$ and $x := \lim_k x^{i_k} \in W_n$. Since $x^{i_k} \in \{T^{i_k}(c)\} \times G_{i_k}$, the corresponding itinerary satisfies $\tau_R(\overleftarrow{s}^{i_k}) = i_k$, but because $i_k \to \infty$ as $k \to \infty$ this implies that $\tau_R(\overleftarrow{s}) = \infty$, a contradiction.

Note that $\operatorname{diam}(U_n) \leq \operatorname{diam}(V_n) \to 0$ as $n \to \infty$. $\qquad\square$

Now define a continuous function $f_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ as in Lemma 4 replacing U with U_n and W with W_n. Let $F_n : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $F_n := f_n \circ \ldots \circ f_1$ for every $n \in \mathbb{N}$. We need to show that $F := \lim_{n\to\infty} F_n$ exists and is continuous. It is enough to show the following:

Lemma 6. *Sequence $(F_n)_{n\in\mathbb{N}}$ is uniformly Cauchy.*

Proof. Take $n < m \in \mathbb{N}$ and note that $\sup_{x\in\mathbb{R}^2} d_e(F_m(x), F_n(x)) = \sup_{x\in\mathbb{R}^2} d_e(f_m \circ \ldots \circ f_{n+1} \circ F_n(x), F_n(x)) < \max\{\operatorname{diam}(U_{n+1}), \ldots, \operatorname{diam}(U_m)\} \to 0$ as $n, m \to \infty$. $\qquad\square$

Denote by $Z := Y \cup \{\text{semi-circles}\} \subset \mathbb{R}^2$. We want to argue that $F(Z) \subset \mathbb{R}^2$ is homeomorphic to $Y/\sim$. Since $F \colon Z \to F(Z)$ is continuous, it follows from [12, Theorem 3.21], that $\{F^{-1}(y) : y \in F(Z)\}$ is a decomposition of Z homeomorphic to $F(Z)$. Note that this decomposition is exactly $Y/\sim$.

Proof of Theorem 1. Assume that the symbolic representation of $a = (\ldots, a_{-2}, a_{-1}, a_0) \in X$ is given by $\bar{I}(a) = \ldots l_{-2}l_{-1}.l_0 l_1 \ldots$. Consider the planar representation Z of X obtained by the ordering on C making $L = \ldots l_{-2}l_{-1}$ the largest. The point a is represented as $(a_0, 1)$. Take the arc $A = \{(a_0, t+1), t \in [0,1]\}$ which is a vertical interval in the plane (see Figure 9). Note that $A \cap Z = \{a\}$. Then $F(A)$ is an arc such that $F(A) \cap F(Z) = \{F(a)\}$ which concludes the proof. $\qquad\square$
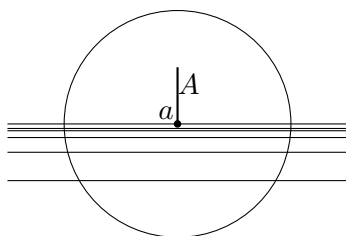

FIGURE 9. Point $a = (a_0, \psi_L(L))$ is accessible.

Proof of Corollary 1. Assume that $g_1 \colon X \to E_1$ and $g_2 \colon X \to E_2$ are equivalent embeddings. It was proven in [7] that every homeomorphism $h \colon X \to X$ is isotopic to σ^R, where $R \in \mathbb{Z}$. Thus if $p \in E_1$ is an accessible point, then $g_2 \circ \sigma^R \circ g_1^{-1}(p)$ is also accessible. Note that if the topological entropy $h_{top}(T) > 0$, then X allows uncountably many admissible itineraries which are not shifts of one another. $\qquad\square$

Remark 4. *Let $\mathcal{U} \subset X$ denote a composant such that $(\ldots, 0, 0), (\ldots, r, r) \notin \mathcal{U}$. Then an embedding making a point $a \in \mathcal{U}$ accessible as described above is not equivalent to the standard embeddings constructed in* [6] *and* [5].

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